Rev. Mat. Iberoamericana **19** (2003), 393–412

Proximity relations for real rank one valuations dominating a local regular ring

Ángel Granja and Cristina Rodríguez

Abstract

We study 0-dimensional real rank one valuations centered in a regular local ring of dimension $n \geq 2$ such that the associated valuation ring can be obtained from the regular ring by a sequence of quadratic transforms. We define two classical invariants associated to the valuation (the refined proximity matrix and the multiplicity sequence) and we show that are equivalent data of the valuation.

1. Introduction

Let R be a local noetherian regular ring of Krull dimension $n \geq 2$. Let v be a real rank one valuation of the quotient field $K(R)$ of R with V as associated valuation ring. Let us assume that V dominates R and that v is 0-dimensional (i.e. the field extension $R/M(R) \subset V/M(V)$ is algebraic, where $M(R)$ and $M(V)$ denotes the maximal ideals of R and V respectively).

In this situation, there exists one and only one sequence

$$(*) \quad R = R_0 \subset R_1 \subset \cdots \subset R_i \subset \cdots \subset V$$

such that R_{i+1} is the only quadratic transform of R_i which is dominated by V, $i \geq 0$. (Note that R_i has dimension n for all $i \geq 0$).

If $n = 2$ it is well-known (see [1]) that $\bigcup_{i\geq 0} R_i = V$. If $n \geq 3$, then $\bigcup_{i\geq 0} R_i = V$ if and only if the sequence $(*)$ switches strongly infinitely often. When v is a non-discrete valuation this is proved in [13]. In fact, if the sequence (*) switches strongly infinitely often the proof given in [13]

2000 Mathematics Subject Classification: Primary 13F30; Secondary 13H05.
Keywords: Valuation, real rank, regular ring, quadratic transform.

shows that $\bigcup_{i\geq 0} R_i = V$ regardless if v is discrete or non-discrete. If now $V = \bigcup_{i\geq 0} R_i$ is a discrete valuation ring, then we obtain that the sequence $(*)$ switches strongly infinitely often from domination and dimension of R_i. (See Theorem 6 below).

On the other hand, given the sequence $(*)$ one can define as usual when R_j is proximate to R_i for $j > i$. (See for example [8], [10], [11]...). This information is collected into the refined proximity matrix.

Also, other invariant associated to $(*)$ is the usual multiplicity sequence (see [11]). The aim of the paper is to prove that the refined proximity matrix and the multiplicity sequence are equivalent data associated to $(*)$, when the sequence switches strongly infinitely often. This is a classical well-known result for $n = 2$. In this case, there are many invariants associated to $(*)$ which are equivalent. (See [8], [9], [11]). All of them are invariants that characterize the equisingularity class of analytically irreducible plane curves, when the sequence $(*)$ is the resolution sequence of the corresponding singularity.

The paper is organized in six sections (including this introduction) as follows.

- Section 2 is devoted to remember some concepts, notations and properties about quadratic transforms.
- The main properties of rank one valuations centered on a local regular ring are given in section 3.
- In sections 4 and 5 we define and study the refined proximity matrix and the multiplicity sequence for a valuation centered on a local regular ring. Also we prove our main result (Theorem 18) that asserts that these invariants are equivalent data of valuation, when the corresponding sequence $(*)$ switches strongly infinitely often.
- The last section is devoted to obtain technical results that are used in the proof of main result.

2. Notations and preliminaries

The main part of the concepts and notations of this paper are equal or similar to some of [2], [3], [6] and [13].

For a noetherian local ring R, we denote by $M(R)$ the maximal ideal of R, by $\dim(R)$ the Krull dimension of R and by

$$\mathfrak{B}(R) = \{R_{\mathfrak{p}};\ \mathfrak{p} \text{ is a prime ideal of } R\}.$$

Also, for each non-zero principal ideal J of R we denote by $Ord_R(J)$ the usual multiplicity, that is the non-negative integer d such that $J \subset (M(R))^d$ and $J \not\subset (M(R))^{d+1}$.

A *quadratic transform of* R is a ring $R_1 = (R[z^{-1}M(R)])_{\mathfrak{q}}$, where z is a non-zero element of $M(R)$, $z \notin M(R)^2$ and $\mathfrak{q}$ is a prime ideal of $R[z^{-1}M(R)]$ such that $M(R)R[z^{-1}M(R)] \subset \mathfrak{q}$.

By a *hypersurface* we mean a pair (R, J), where R is a regular noetherian local ring and J is a non zero principal ideal of R. (Note that R/J might not be a reduced ring).

Let (R, J) be a hypersurface and R_1 a quadratic transform of R. The *strict transform* of (R, J) in R_1 is the hypersurface (R_1, J_1) where J_1 is the ideal such that $J_1 z^m R_1 = JR_1$ with $M(R)R_1 = zR_1$ and $m = Ord_R(J)$. The hypersurface (R_1, JR_1) is called the *total transform* of (R, J) in R_1.

Let

$$R = R_0 \subset R_1 \subset \cdots \subset R_N$$

be a sequence such that R_i is a quadratic transform of R_{i-1}, $1 \le i \le N$. The *strict transform* of a hypersurface (R, J) in R_i is the hypersurface (R_i, J_i) defined recursively as follows:

1. If $i = 0$, then $(R_0, J_0) = (R, J)$.
2. If $i > 0$, and (R_{i-1}, J_{i-1}) is the strict transform of (R, J) in R_{i-1}, then (R_i, J_i) is the strict transform of (R_{i-1}, J_{i-1}) in R_i.

Now we will prove two useful results.

Lemma 1 *Let R be a regular noetherian local ring with* $\dim(R) = n \ge 2$ *and let R' be a quadratic transform of R. Then there exists a basis $(y_1, \dots, y_n)$ of $M(R)$ such that $R' = (R[y_2/y_1, \dots, y_n/y_1])_Q$ with Q a prime ideal of $R[y_2/y_1, \dots, y_n/y_1]$ and $M(R)(R[y_2/y_1, \dots, y_n/y_1]) \subset Q$.*

Furthermore, $R \notin \mathfrak{B}(R')$ and $\mathfrak{B}(R) \cap \mathfrak{B}(R') = \{S' \in \mathfrak{B}(R')\,;\, y_1 \notin M(S')\}$. More precisely, if $\mathfrak{q}' \in Spec(R')$, $y_1 \notin M(R'_{\mathfrak{q}'})$, then $R'_{\mathfrak{q}'} = R_{\mathfrak{q}}$ where $\mathfrak{q} := \mathfrak{q}' \cap R$.

Proof: The first statement is an easy consequence of the definition of quadratic transformation. So we can assume $M(R) = (y_1, \dots, y_n)$, and

$$R' = (R[y_2/y_1, \dots, y_n/y_1])_Q\,.$$

If $R \in \mathfrak{B}(R')$ then $R = (R')_{\mathfrak{q}'}$ and $y_1 \in M(R) \subset \mathfrak{q}'R'$. As $y_2/y_1 \in R' \subset (R')_{\mathfrak{q}'} = R$ and $y_2 = (y_2/y_1)y_1$ then $(y_1, y_2, \dots, y_n)$ is not a regular system of parameters of R. Thus $R \notin \mathfrak{B}(R')$.

To finish the prove we will see

$$\{S' \in \mathfrak{B}(R')\,;\, y_1 \notin M(S')\} = \mathfrak{B}(R') \cap (\mathfrak{B}(R) - \{R\})\,.$$

Let us consider $S' \in \mathfrak{B}(R')$. We can write $S' = (R\,[y_2/y_1, \ldots, y_n/y_1])_{\mathfrak{q}'}$, with $\mathfrak{q}' = M(S') \cap R\,[y_2/y_1, \ldots, y_n/y_1]$.

If $y_1 \notin M(S')$ and $\mathfrak{q} = \mathfrak{q}' \cap R$ then $S' = (R\,[y_2/y_1, \ldots, y_n/y_1])_{\mathfrak{q}'} = R_{\mathfrak{q}}$.

Conversely, if $S' \in \mathfrak{B}(R') \cap (\mathfrak{B}(R) - \{R\})$ we have

$$S' = (R\,[y_2/y_1, \ldots, y_n/y_1])_{\mathfrak{q}'} = R_{\mathfrak{q}},$$

for some prime ideal $\mathfrak{q}$ of R.

Therefore, $y_2/y_1, \ldots, y_n/y_1 \in R_{\mathfrak{q}}$ and $y_1 \notin M(S')$. In fact, if $y_1 \in M(S')$ then $y_1 \in \mathfrak{q} = M(S') \cap R$. Thus we can write $y_2/y_1 = f/g$ with $f, g \in R$ without common factors and $g \notin \mathfrak{q}$. So $y_2 g = y_1 f$ and as $y_2 \notin y_1 R$ we have $g \in y_1 R \subset \mathfrak{q}$, which is a contradiction. Henceforth $y_1 \notin M(S')$. ■

Lemma 2 *Let R be a regular noetherian local ring with* $\dim(R) = n \geq 2$ *and let R_1 be a quadratic transform of R. Let us consider $f \in M(R)$ and let us write $(R_1, f_1 R_1)$ the strict transform of (R, fR) in R_1. Let us assume that $f_1 \in M(R_1)$ and $M(R) = (z_1, \ldots, z_n)$, with $M(R)R_1 = z_1 R_1$. Then we have the following statements:*

1. *$f_1 \notin z_1 R_1$.*
2. *If $f \notin z_j R_1$ and $z_j/z_1 \in M(R_1)$ for some $2 \leq j \leq n$, then $f_1 \notin (z_j/z_1)\,R_1$.*
3. *If f is an irreducible element of R, then f_1 is an irreducible element in R_1.*
4. *If f is an irreducible element of R and $f \notin z_1 R$, then $f_1 R_1 \cap R = fR$.*

Proof: Let us consider the ring $A := R[z_1^{-1} M(R)]$ and a prime ideal Q of A with $M(R)A \subset Q$ and $R_1 = R_Q$.

We point out that an element $r/z_1^{\beta} \in A,\ r \in M(R)^{\beta}$, is a unit in R_1 if and only if $Ord_R(rR) = \beta$. Note that, $Ord_R(rR) \geq \beta$ and if $Ord_R(rR) > \beta$ then $r/z_1^{\beta} = (r/z_1^{\beta+1})z_1 \in M(R_1)$.

We can write $f_1 R_1 = (f/z_1^d)R_1$, with $Ord_R(fR) = d$.

1) If $f_1 \in z_1 R_1$, let us write

$$\frac{f}{(z_1)^d} = \frac{\overline{h}}{\overline{r}}\, z_1,$$

with $\overline{h}, \overline{r} \in R_1$ and $\overline{r} \notin M(R_1)$.

In fact, $\overline{h} = h/z_1^{\alpha}$ and $\overline{r} = r/z_1^{\beta}$, with $h, r \in R$, $h, r \notin z_1R$, $Ord_R(hR) \geq \alpha$ and $Ord_R(rR) = \beta$.
Thus

$$fr = h(z_1)^{d+\beta-\alpha+1}.$$

If $d+\beta-\alpha+1 \geq 0$, then $Ord_R(fR) \geq d+1$ and if $d+\beta-\alpha+1 < 0$, then $h \in z_1R$. So in any case we get to a contradiction and necessarily $f_1 \notin z_1R_1$.

2) Let us assume $f \notin z_jR_1$ and $z_j/z_1 \in M(R_1)$ for some $2 \leq j \leq n$. If $f_1 \in (z_j/z_1)\, R_1$, we can write

$$\frac{f}{(z_1)^d} = \frac{\overline{h}}{\overline{r}} \frac{z_j}{z_1},$$

with $\overline{h}, \overline{r} \in R_1$ and $\overline{r} \notin M(R_1)$. As above, $\overline{h} = h/z_1^{\alpha}$ and $\overline{r} = r/z_1^{\beta}$, with $h, r \in R$, $h, r \notin z_1R$, $Ord_R(hR) \geq \alpha$ and $Ord_R(rR) = \beta$.

Thus

$$fr = hz_j(z_1)^{d+\beta-\alpha-1}.$$

As $f \notin z_jR$, then $r \in z_jR$. So $r = r'z_j$ and

$$\frac{r}{(z_1)^{\beta}} = \frac{r'}{(z_1)^{\beta-1}} \frac{z_j}{z_1} \in M(R_1)$$

which is a contradiction. So $f_1 \notin (z_j/z_1)\, R_1$.

3) First we note that $f \notin z_1R$, because in other case $fR = z_1R$ and $f_1 \notin M(R_1)$.

Let us assume, if it is possible, that

$$\frac{f}{(z_1)^d} = \frac{\overline{h}_1}{\overline{r}_1} \frac{\overline{h}_2}{\overline{r}_2},$$

with $\overline{h}_1, \overline{r}_1, \overline{h}_2, \overline{r}_2 \in R_1$ and $\overline{r}_1, \overline{r}_2 \notin M(R_1)$.

As always, $\overline{h}_1 = h_1/z_1^{\alpha_1}$, $\overline{h}_2 = h_2/z_1^{\alpha_2}$, $\overline{r}_1 = r_1/z_1^{\beta_1}$, $\overline{r}_2 = r_2/z_1^{\beta_2}$, with $h_1, h_2, r_1, r_2 \in R$ and $h_1, h_2, r_1, r_2 \notin z_1R$. Also note $Ord_R(h_iR) \geq \alpha_i$ and $Ord_R(r_iR) = \beta_i$, $1 \leq i \leq 2$.

Thus

$$fr_1r_2 = h_1h_2(z_1)^{d-\alpha_1-\alpha_2+\beta_1+\beta_2}.$$

Moreover, as $f, h_1, h_2, r_1, r_2 \notin z_1R$ we have $d - \alpha_1 - \alpha_2 + \beta_1 + \beta_2 = 0$.

Now, after removing out common factors, we can write

$$f = h_1'h_2'.$$

As f is an irreducible element of R then, either $h'_1 \notin M(R)$ or $h'_2 \notin M(R)$. Thus, either $\overline{h}_1/(\overline{r}_1\overline{r}_2) \notin M(R_1)$ or $\overline{h}_2/(\overline{r}_1\overline{r}_2) \notin M(R_1)$ and f_1 is an irreducible element of R_1.

4) Let us consider $g \in f_1R_1 \cap R$. We have

$$g = \frac{f}{(z_1)^d}\frac{\overline{h}}{\overline{r}}$$

with $\overline{h}, \overline{r} \in R_1$ and $\overline{r} \notin M(R_1)$. As above, $\overline{h} = h/z_1^{\alpha}$, $\overline{r} = r/z_1^{\beta}$, with $h, r \in R$ and $h, r \notin z_1R$. Also note $Ord_R(hR) \geq \alpha$ and $Ord_R(rR) = \beta$.

So we have

$$fh = gr(z_1)^{d+\alpha-\beta}.$$

As $fR \notin z_1R$, then $gr \in fR$. If $r \in fR$, then $r/z_1^{\beta} \in f_1R_1 \subset M(R_1)$ which is a contradiction. So $g \in fR$. ■

Remark 3 *Let $R = R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_n \subset \cdots$ be a sequence of regular noetherian local rings of the same dimension, such that R_i is a quadratic transform of R_{i-1}, for $i \geq 1$. If f is an irreducible element of R, (R_i, f_iR_i) is the strict transform of (R, fR) in R_i and $f_i \in M(R_i)$, then f_i is an irreducible element in R_i for $i \geq 0$.*

3. Valuations of real rank one

In this section we shall consider a local noetherian regular ring R with $\dim(R) = n \geq 2$ and let v be a valuation of the quotient field $K(R)$ of R with V as valuating ring. Let us assume that V dominates R (i.e. $R \subset V$ and $M(R) = R \cap M(V)$) and that v is 0-dimensional (i.e. the field extension $R/M(R) \subset V/M(V)$ is algebraic).

Also, we shall consider the sequence

$$(R_i) \equiv R = R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_n \subset \cdots$$

of regular noetherian local rings of the same dimension, such that R_i is the quadratic transform of R_{i-1}, along V for $i \geq 1$. Note that $\dim(R_i) = \dim(R_{i+1})$ implies that $R_{i+1}/M(R_{i+1})$ is a finite extension of $R_i/M(R_i)$ (because

$$\dim(R_{i+1}) = \dim(R_i) + \text{tr. deg.}((R_{i+1}/M(R_{i+1}) : R_i/M(R_i)))$$

(see [13], p. 296)).

Now, we point out the following definition. (See [13], p. 314).

Definition 4 *The quadratic sequence* (R_i) *switches strongly infinitely often if there does not exist an integer* j *and a height one prime ideal* $\mathfrak{p}$ *in* R_j *with the property that*

$$\bigcup_{i\geq 0} R_i \subset (R_j)_{\mathfrak{p}}\,.$$

Lemma 5 *With the above notations, let us assume that* (R_i) *switches strongly infinitely often. Let us consider* $f \in R = R_0$. *Then there exists a non-negative integer* $j_0 \geq 0$ *such that* $f_j R_j = R_j$ *for all* $j \geq j_0$, *where* $(R_j, f_j R_j)$ *is the strict transform of* (R, fR) *in* R_j *for all* $j \geq 0$.

Proof: If $f_j R_j \neq R_j$ for all $j \geq 0$, we can assume that f is an irreducible element of R. Thus, by Remark 3, f_j is an irreducible element of R_j for all $j \geq 0$.

On the other hand, let us write $M(R_j)R_{j+1} = z_j R_{j+1}$ with $z_j \in M(R_j)$ for $j \geq 0$.

Also by Lemma 2, $f_{j+1} \notin z_j R_{j+1}$, $f_{j+1}R_{j+1} \cap R_j = f_j R_j$, and by Lemma 1,

$$(R)_{fR} = (R_j)_{f_j R_j} \in \mathfrak{B}(R_{j+1}),$$

for all $j \geq 0$. So $R_{j+1} \subset (R_j)_{f_j R_j}$ and $\bigcup_{i=1}^{\infty} R_i \subset (R)_{fR}$ which is a contradiction. ■

Theorem 6 *With the above notations, let us assume that the valuation* v *has real rank one. Then the following statements are equivalent:*

1. $V = \bigcup_{i\geq 0} R_i$.
2. (R_i) *switches strongly infinitely often.*

Proof: When v is a non-discrete valuation this is proved in Proposition (4.18) of [13]. In fact, if the sequence (R_i) switches strongly infinitely often, the proof given in [13] shows that $V = \bigcup_{i\geq 0} R_i$ regardless if V is discrete or non-discrete, and if V is non-discrete and $V = \bigcup_{i\geq 0} R_i$ then (R_i) switches strongly infinitely often.

If now $V = \bigcup_{i\geq 0} R_i$ is a discrete valuation ring, and if there would exist a non-negative integer j and a height one prime ideal $\mathfrak{p}$ of R_j with $V \subset (R_j)_{\mathfrak{p}}$, then $V = (R_j)_{\mathfrak{p}}$. Hence $M(V) \cap R_j = \mathfrak{p}$, and V would not dominate R_j (since $\dim(R_j) = n \geq 2$). ■

Remark 7 *In the conditions of the above theorem, note that if* $V = \bigcup_{i\geq 0} R_i$ *then*

$$\frac{V}{M(V)} = \bigcup_{i\geq 0} \left(\frac{R_i}{M(R_i)} \right).$$

4. Proximity

In this section, let

$$(R_i) \equiv R = R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_n \subset \cdots$$

be a fixed sequence of regular noetherian local rings of the same dimension, such that R_i is a quadratic transform of R_{i-1}, for $i \geq 1$.

We will write

$$V = \bigcup_{i \geq 0} R_i.$$

Definition 8 *With the above notations, for $j > i$ we say that R_j is proximate to R_i if the valuation ring $V(R_i)$ of Ord_{R_i} contains R_j, where Ord_{R_i} is the usual valuation order of R_i.*

For $i \geq 0$, we will denote the set of proximate points of R_i by

$$\mathcal{P}(R_i) = \{R_j;\ R_j \text{ is proximate to } R_i\}.$$

Remark 9 *Note that $V(R_i) = (R_{i+1})_{M(R_i)R_{i+1}}$. Thus if $R_j \subset V(R_i)$ $(i < j)$ also $R_h \subset V(R_i)$ for $i < h \leq j$. So as a consequence of Lemma 1 we have*

$$V(R_i) = (R_h)_{\mathfrak{p}_h},$$

where $\mathfrak{p}_h$ is a height one prime ideal of R_h, $i < h \leq j$.

In this section we will use the following notations. We will denote by

$$E_i^{i+1} = (R_{i+1}, M(R_i)R_{i+1}) = (R_{i+1}, D_i^{i+1}),$$

that is, the *exceptional divisor* attached to R_i, $i \geq 0$.

Also, we will write

$$E_i^j = (R_j, D_i^j),$$

the strict transform of E_i^{i+1} in R_j, for $i < j$.

Lemma 10 *With the above notations, the following statements are equivalent for $i < j$:*

1) $R_j \in \mathcal{P}(R_i)$.

2) $D_i^j \neq R_j$.

Proof: It is an easy consequence of Remark 9 and Lemmas 1 and 2. ■

Lemma 11 *With the above notations, we have the following statements:*

1) $R_{i+1} \in \mathcal{P}(R_i)$*,* $i \geq 0$*.*

2) If $R_j \in \mathcal{P}(R_i)$*, then* $R_h \in \mathcal{P}(R_i)$*,* $i < h \leq j$*.*

3) Assume that V *is the valuation ring of a rank one valuation* v *of the quotient field of* $R = R_0$*. Then* $\mathcal{P}(R_i)$ *is a finite set,* $i \geq 0$*.*

Proof: 1) and 2) are an easy consequence of definition of proximity. 3) It is a consequence of Lemma 5, Lemma 10, and Theorem 6. ■

Now we will collect the proximity relations into the refined proximity matrix in the same way as in [11].

Definition 12 *With the notations as above, the refined proximity matrix associated to* (R_i) *is the infinite matrix* $P(R_i) = (p_{ij})_{i,j\geq 0}$ *given by* $p_{ii} = 1$*,*

$$p_{ij} = -[R_j/M(R_j) \,:\, R_i/M(R_i)]$$

if $R_j \in \mathcal{P}(R_i)$ *and* $p_{ij} = 0$ *for the rest.*

Note that $P(R_i)$ is an upper triangular matrix. When V is the valuation ring of a rank one valuation v of the quotient field of $R = R_0$, we will also write

$$P(R_i) = P(V) = P_v.$$

Remark 13 *Note that we have the following dictionary.*

1) $p_{i\,i+1} \neq 0$*, for* $i \geq 0$*.*

2) If $p_{ij} \neq 0$*, then* $p_{ih} \neq 0$ *for* $i < h \leq j$*.*

3) If V *is the valuation ring of a rank one valuation* v *of the quotient field of* $R = R_0$*, then for fixed* i*,* $p_{ij} = 0$ *for all but finitely many index* j*.*

5. Multiplicity sequence

As in the last section, let

$$(R_i) \equiv R = R_0 \subset R_1 \subset R_2 \subset \cdots \subset R_n \subset \cdots$$

be a fixed sequence of regular noetherian local rings of the same dimension, such that R_i is a quadratic transform of R_{i-1}, for $i \geq 1$.

We will assume that

$$V = \bigcup_{i\geq 0} R_i$$

is the ring of a rank one valuation v of the quotient field of $R = R_0$.

For $i \geq 0$, we will denote the set of proximate points of R_i by

$$\mathcal{P}(R_i) = \{R_j;\ R_j \text{ is proximate to } R_i\}.$$

Note that $\mathcal{P}(R_i)$ is a finite set by statement 3) of Lemma 11. We will denote by

$$n_i = \min\{v(z);\ z \in M(R_i)\}, \quad i \geq 0.$$

(Note that as R_i is a noetherian ring then n_i is well defined, $i \geq 0$).

Definition 14 *With the above notations, the multiplicity sequence is given by*

$$\{(n_0/n_i, e_i)\}_{i\geq 1},$$

where

$$e_i = [R_i/M(R_i) : R_0/M(R_0)],$$

for $i \geq 1$. Note that n_0/n_i is a real number with $1 \leq n_0/n_i$, $i \geq 1$.

Remark 15 *If $z_j \in M(R_j)$ with $v(z_j) = n_j$, then*

$$R_{j+1} = \left(R_j[z_j^{-1}M(R_j)]\right)_{\mathfrak{q}_j},$$

with $\mathfrak{q}_j$ a prime ideal of $R_j[z_j^{-1}M(R_j)]$ such that $M(R_j)R_j[z_j^{-1}M(R_j)] \subset \mathfrak{q}_j$.

In particular, $z_j \in M(R_{j+1}) - \{0\}$ and $n_{j+1} \leq n_j$, for all $j \geq 0$.

Lemma 16 *With the above notations, we have $n_{i+1} = n_i$ if and only if $R_{i+2} \notin \mathcal{P}(R_i)$, $i \geq 0$.*

Proof: Let us write $M(R_i)R_{i+1} = z_i R_{i+1}$, with $v(z_i) = n_i > 0$.

Let $E_i^{i+1} = (R_{i+1}, D_i^{i+1})$ be the exceptional divisor attached to R_i and $E_i^j = (R_j, D_i^j)$ its strict transform in R_j, $i < j$. We have $D_i^{i+1} = z_i R_{i+1}$.

Now, if $n_{i+1} = n_i$ then $D_i^{i+2} = R_{i+2}$ and $R_{i+2} \notin \mathcal{P}(R_i)$.

On the other hand, if $n_{i+1} < n_i$, then there exists $z_{i+1} \in M(R_{i+1})$ with $v(z_{i+1}) = n_{i+1}$. Thus,

$$D_i^{i+2} = \frac{z_i}{z_{i+1}} R_{i+2} \neq R_{i+2}$$

and $R_{i+2} \in \mathcal{P}(R_i)$. ■

Lemma 17 *With the above notations, we have*

$$n_i = \sum_{R_j \in \mathcal{P}(R_i)} n_j, \quad i \geq 0\,.$$

Proof: Let us consider

$$h(i) = \max\{k;\ R_{i+k} \in \mathcal{P}(R_i)\}, \quad i \geq 0.$$

We have $1 \leq h(i)$ and $R_{i+k} \in \mathcal{P}(R_i)$, $1 \leq k \leq h(i)$.

Let $z_{i+l} \in M(R_{i+l})$ be such that $v(z_{i+l}) = n_{i+l}$, $0 \leq l \leq h(i)$. Also let $E_i^{i+1} = (R_{i+1}, D_i^{i+1})$ be the exceptional divisor attached to R_i and $E_i^j = (R_j, D_i^j)$ its strict transform in R_j, $i < j$.

We have $D_i^{i+1} = z_i R_{i+1}$ and

$$D_i^{i+k} = \frac{z_i}{z_{i+1} \cdots z_{i+k-1}} R_{i+k}, \quad 1 < k \leq h(i).$$

As $R_{i+h(i)+1} \notin \mathcal{P}(R_i)$, then

$$n_{i+h(i)} = v\left(\frac{z_i}{z_{i+1} \cdots z_{i+h(i)-1}}\right).$$

Thus

$$n_i = \sum_{R_j \in \mathcal{P}(R_i)} n_j = n_{i+1} + n_{i+2} + \ldots + n_{i+h(i)}.$$

■

Let

$$\alpha = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cdots}}$$

be a finite or infinite continued fraction. Let $[q_0] := q_0$ and define recursively $[q_0, \ldots, q_j] = [q_0, q_1, \ldots, q_{j-1} + 1/q_j]$ for $j \geq 1$ (as long as $q_j \neq 0$). Then we have

$$\alpha = \lim_{n \to \infty} [q_0, q_1, \ldots, q_n].$$

As usually, we will write $\alpha = [q_0, q_1, q_2 \ldots]$

Theorem 18 *With the above notations, the refined proximity matrix determines the multiplicity sequence and vice-versa.*

Proof: First we will see that the multiplicity sequence determines the refined proximity matrix $P(R_i)$.

Let us consider

$$h(i) = \max\{k;\ n_{i+1} + n_{i+2} + \cdots + n_{i+k} \leq n_i\}, \qquad i \geq 0.$$

Note that

$$h(i) = \max\{k;\ R_{i+k} \in \mathcal{P}(R_i)\}.$$

We compute the row i of $P(R_i)$ as follows: $p_{ij} = 0$ if $j < i$ or $i + h(i) < j$, $p_{ii} = 1$, and

$$p_{ij} = -e_j/e_i,$$

for $i + 1 \leq j \leq i + h(i)$.

Conversely the refined proximity matrix determines the multiplicity sequence. Let us consider

$$h(i) = \max\{k;\ p_{i,i+k} \neq 0\}, \qquad i \geq 0.$$

Note that

$$h(i) = \max\{k;\ R_{i+k} \in \mathcal{P}(R_i)\} = \max\{k;\ n_{i+1} + n_{i+2} + \cdots + n_{i+k} \leq n_i\}$$

and

$$n_i = n_{i+1} + \cdots + n_{i+h(i)}, \qquad i \geq 0.$$

It is easily checked that

$$e_i = (-1)^i p_{01} p_{12} \ldots p_{i-1\,i}, \qquad i \geq 1.$$

On the other hand, let us write

$$\frac{n_0}{n_i} = [q_0^i, q_1^i, q_2^i, \ldots]$$

the continued fraction associated to n_0/n_i.

To finish the proof we point out that q_0^i, q_1^i, q_2^i, ... are determined by i and the sequence $(h(i))_{i\geq 0}$.

The above is a consequence of Corollary 23 given in the next section for a more general situation. ■

6. Technical results

In this section, we will study sequences of real numbers with properties as the multiplicity sequence.

Let $(n_i)_{i\geq 0}$ be a fixed sequence of positive real numbers and $(h(i))_{i\geq 0}$ be a sequence of positive integers such that $h(i) > 0$ and

$$n_i = n_{i+1} + \cdots + n_{i+h(i)}, \quad i \geq 0.$$

Lemma 19 *With the above notations, let us consider $A = \sum_{j=0}^{k} a_j n_j$ and $B = \sum_{j=0}^{l} b_j n_j$, such that $a_j \in \{0,1\}$ for $0 \leq j \leq k$, $a_k \neq 0$, $b_j \in \{0,1\}$ for $0 \leq j \leq l$, $b_0 = \cdots = b_k = 0$ and $b_l \neq 0$. Then we can write $A = \sum_{j=0}^{s} \alpha_j n_j$ and $B = \sum_{j=0}^{s} \beta_j n_j$ such that α_j, β_j are non-negative integers and either $\alpha_j \geq \beta_j$ for $0 \leq j \leq s$ or $\beta_j \geq \alpha_j$ for $0 \leq j \leq s$. Moreover, s, $\alpha_0, \ldots, \alpha_s$, $\beta_0, \ldots, \beta_s$ can be constructed by using only $a_0, \ldots, a_k$, $b_0, \ldots, b_l$, and the sequence $(h(i))_{i\geq 0}$.*

Proof: Let j_0 be the non-negative integer given by

$$j_0 = \min\{j;\ 0 \le j \le l \text{ and } \ b_j \ne 0\}.$$

Note that $k < j_0 \le l$.

Let us consider $(r_i)_{i\ge 0}$, the sequence of non-negative integers given by $r_0 = k$, and $r_{i+1} = r_i + h(r_i)$, for $i \ge 0$. We have

$$r_0 < r_1 < \cdots < r_i < \cdots .$$

Let r_m be such that $r_{m-1} < j_0 \le r_m$.

Let us write

$$A = \sum_{j=0}^{r_m} a'_j n_j,$$

where $a'_j = a_j$ for $0 \le j < k = r_0$, $a'_j = 1$ if $j \ne r_i$, $0 \le i < m$, $r_0 = k \le j \le r_m$ and $a'_{r_i} = 0$ for $0 \le i < m$. (Note that $a'_j \in \{0,1\}$ for $0 \le j \le r_m$).

Let $\delta \ge 1$ be the positive integer given by

$$\delta = \sum_{j=j_0}^{l} b_j.$$

If $\delta = 1$, then $j_0 = l$. Thus the result follows by taking $s = r_m$, $\alpha_j = a'_j$, $0 \le j \le s = r_m$, $\beta_{j_0} = 1$ and $\beta_j = 0$ for $j \ne j_0$, $0 \le j \le s = r_m$.

If $\delta > 1$, we consider $C = \sum_{j=j_0}^{r_m} b_j n_j$, $A'' = \sum_{j=0}^{r_m} a''_j n_j$ and $B'' = \sum_{j=r_m+1}^{l} b_j n_j$, where $a''_j = a'_j$ for $0 \le j < j_0$, $a''_j = 0$ if $b_j \ne 0$ for $j_0 \le j \le r_m$ and $a''_j = a'_j$ if $b_j = 0$ for $j_0 \le j \le r_m$. (Note that if $l < r_m$, then $B'' = 0$, $B = C$, $A = A'' + C$ and there is nothing to do).

In any case, we have $A = A'' + C$ and $B = B'' + C$.

At this point, applying induction on δ we get $A'' = \sum_{j=0}^{s} \alpha''_j n_j$ and $B'' = \sum_{j=0}^{s} \beta''_j n_j$ such that α''_j, β''_j are non-negative integers and either $\alpha''_j \ge \beta''_j$ for $0 \le j \le s$ or $\beta''_j \ge \alpha''_j$ for $0 \le j \le s$. (Note that we can assume $r_m \le s$).

Now, the result follows by taking $\alpha_j = \alpha''_j$ (resp. $\beta_j = \beta''_j$), for $0 \le j < j_0$ or $r_m < j \le s$, $\alpha_j = \alpha''_j + b_j$ (resp. $\beta_j = \beta''_j + b_j$), for $j_0 \le j \le r_m$. ■

Lemma 20 *With the above notations, let us consider* $A = n_i$ *and* $B = \sum_{j=0}^{l} b_j n_j$, *with* $b_j \in \{0,1\}$ *for* $0 \le j \le l$ *and* $b_l \ne 0$. *Then we can write* $A = \sum_{j=0}^{s} \alpha_j n_j$ *and* $B = \sum_{j=0}^{s} \beta_j n_j$ *such that* α_j, β_j *are non-negative integers and either* $\alpha_j \ge \beta_j$ *for* $0 \le j \le s$ *or* $\beta_j \ge \alpha_j$ *for* $0 \le j \le s$. *Moreover,* s, $\alpha_0, \ldots,$ α_s, $\beta_0, \ldots, \beta_s$ *are determined by* i, $b_0, \ldots, b_l$, *and the sequence* $(h(i))_{i\ge 0}$.

Proof: Let us consider

$$j_0 = \min\{j;\ 0 \leq j \leq l \text{ and } \ b_j \neq 0\}.$$

We have two possibilities:

(1) $j_0 \leq i$. In this case, let l_1 be the non-negative integer given by

$$l_1 = \max\{j;\ 0 \leq j \leq i \text{ and } \ b_j \neq 0\}.$$

We have $j_0 \leq l_1 \leq i$. If $l_1 = i$ then there is nothing to do ($b_i = 1$).

Now, let us assume $l_1 < i$ and consider the sequence of non-negative integers $(r_k)_{k\geq 0}$, given by $r_0 = l_1$, and $r_{k+1} = r_k + h(r_k)$, for $k \geq 0$. We have

$$r_0 < r_1 < \cdots < r_k < \cdots .$$

Let r_m be such that $r_{m-1} < i \leq r_m$.

We can write $\beta_j = b_j$, $0 \leq j < l_1 = r_0$, $r_m < j \leq l$, $\beta_{r_k} = 0$, $0 \leq k < m$, $\beta_j = b_j + 1$, $i \leq j \leq r_m$ and $\beta_j = 1$, $l_1 \leq j < i$ and $j \notin \{r_0, r_1, \ldots, r_{m-1}\}$. Note that $\beta_i = 1$ and the result follows by taking $\alpha_j = 0$ if $j \neq i$ and $\alpha_i = 1$.

(2) $j_0 > i$. In this case, we can apply Lemma 19 to obtain the result. ■

Lemma 21 *With the above notations, let us consider* $A = \sum_{j=0}^{k} a_j n_j$ *and* $B = \sum_{j=0}^{l} b_j n_j$*, with* a_j *a non-negative integer for* $0 \leq j \leq k$, $a_k \neq 0$, $b_j \in \{0,1\}$ *for* $0 \leq j \leq l$ *and* $b_l \neq 0$*. Then we can write* $A = \sum_{j=0}^{s} \alpha_j n_j$ *and* $B = \sum_{j=0}^{s} \beta_j n_j$ *such that* α_j, β_j *are non-negative integers and either* $\alpha_j \geq \beta_j$ *for* $0 \leq j \leq s$ *or* $\beta_j \geq \alpha_j$ *for* $0 \leq j \leq s$*. Moreover,* s, $\alpha_0, \ldots, \alpha_s$, $\beta_0, \ldots, \beta_s$ *can be constructed by using only* $a_0, \ldots, a_k$, $b_0, \ldots, b_l$*, and the sequence* $(h(i))_{i\geq 0}$.

Proof: Let $\delta \geq 1$ be the positive integer given by

$$\delta = \sum_{j=0}^{k} a_j.$$

If $\delta = 1$ the result follows from Lemma 20. Let us assume $\delta > 1$. We have three possibilities:

(1) $l = k$.

Let us consider $A' = \sum_{j=0}^{k} a'_j n_j$ and $B' = \sum_{j=0}^{k} b'_j n_j$ where $a'_j = a_j$, $b'_j = b_j$ for $0 \leq j < k = l$, and $a'_k = a_k - 1$, $b'_k = b_k - 1 = 0$.

Now, applying induction on δ we get $A' = \sum_{j=0}^{s} \alpha'_j n_j$ and $B' = \sum_{j=0}^{s} \beta'_j n_j$ such that α'_j, β'_j are non-negative integers and either $\alpha'_j \geq \beta'_j$ for $0 \leq j \leq s$ or $\beta'_j \geq \alpha'_j$ for $0 \leq j \leq s$. Note that we can assume $l = k \leq s$.

To have the result, we must only write $\alpha_j = \alpha'_j$ (resp. $\beta_j = \beta'_j$), $j \neq k$, $0 \leq j \leq s$ and $\alpha_k = \alpha'_k + 1$ (resp. $\beta_k = \beta'_k + 1$).

(2) $l < k$.

In this case, we consider the sequence of non-negative integers $(r_i)_{i\geq 0}$, given by $r_0 = l$, and $r_{i+1} = r_i + h(r_i)$, for $i \geq 0$. We have

$$r_0 < r_1 < \ldots < r_i < \ldots.$$

Let r_m be such that $r_{m-1} < k \leq r_m$.

We write $B = \sum_{j=0}^{r_m} b'_j n_j$, where $b'_j = b_j$ for $0 \leq j < l = r_0$, $b'_j = 1$ if $j \neq r_i$, $0 \leq i < m$, $r_0 = l \leq j \leq r_m$ and $b'_{r_i} = 0$ for $0 \leq i < m$. Note that $b'_j \in \{0, 1\}$, $0 \leq j \leq r_m$ and $b'_k = 1$.

Now, consider $C = n_k$, $A'' = \sum_{j=0}^{k} a''_j n_j$ and $B'' = \sum_{j=0}^{r_m} b''_j n_j$, where $a''_j = a_j$, $0 \leq j < k$, $a''_k = a_k - 1$, $b''_j = b'_j$, $j \neq k$, $0 \leq j \leq r_m$ and $b''_k = b'_k - 1 = 0$.

We have $A = A'' + C$ and $B = B'' + C$.

At this point, applying induction on δ we get $A'' = \sum_{j=0}^{s} \alpha''_j n_j$ and $B'' = \sum_{j=0}^{s} \beta''_j n_j$ such that α''_j, β''_j are non-negative integers and either $\alpha''_j \geq \beta''_j$ for $0 \leq j \leq s$ or $\beta''_j \geq \alpha''_j$ for $0 \leq j \leq s$. Note that we can assume $r_m \leq s$.

To have the result, we must only write $\alpha_j = \alpha''_j$ (resp. $\beta_j = \beta''_j$), $j \neq k$, $0 \leq j \leq s$ and $\alpha_k = \alpha''_k + 1$ (resp. $\beta_k = \beta''_k + 1$).

(3) $l > k$.

In this case, let j_1 be the non-negative integer given by

$$j_1 = \min\{j;\ b_j \neq 0 \text{ and } j \geq k\}.$$

Note that $k \leq j_1 \leq l$.

If $k = j_1$ let us consider $A' = \sum_{j=0}^{k} a'_j n_j$ and $B' = \sum_{j=0}^{l} b'_j n_j$ where $a'_j = a_j$, $b'_j = b_j$ if $j \neq j_1$, $a'_{j_1} = a_{j_1} - 1$ and $b'_{j_1} = b_{j_1} - 1 = 0$.

At this point, applying induction on δ we get $A' = \sum_{j=0}^{s} \alpha'_j n_j$ and $B' = \sum_{j=0}^{s} \beta'_j n_j$ such that α'_j, β'_j are non-negative integers and either $\alpha'_j \geq \beta'_j$ for $0 \leq j \leq s$ or $\beta'_j \geq \alpha'_j$ for $0 \leq j \leq s$. Note that we can assume $l \leq s$.

To have the result, we must only write $\alpha_j = \alpha'_j$ (resp. $\beta_j = \beta'_j$), $j \neq j_1$, $0 \leq j \leq s$ and $\alpha_{j_1} = \alpha'_{j_1} + 1$ (resp. $\beta_{j_1} = \beta'_{j_1} + 1$).

Now, let us assume $k < j_1 \leq l$ and consider the sequence of non-negative integers $(r_i)_{i\geq 0}$, given by $r_0 = k$, and $r_{i+1} = r_i + h(r_i)$, for $i \geq 0$. We have

$$r_0 < r_1 < \cdots < r_i < \cdots.$$

Let r_m be such that $r_{m-1} < j_1 \leq r_m$.

We can write $A = \sum_{j=0}^{r_m} a'_j n_j$, where $a'_j = a_j$, $0 \leq j < k = r_0$, $a'_k = a'_{r_0} = a_k - 1$, $a'_j = 1$ if $j \neq r_i$, $0 < i < m$, $r_0 = k < j \leq r_m$ and $a'_{r_i} = 0$ for $0 \leq i \leq m - 1$. (Note that $a'_j \in \{0, 1\}$ for $k < j \leq r_m$).

Let us write $C^1 = \sum_{j=j_1}^{r_m} b_j n_j$, $A^1 = \sum_{j=0}^{r_m} a_j^1 n_j$ and $B^1 = \sum_{j=0}^{l} b_j^1 n_j$, where $a_j^1 = a'_j$, $0 \leq j < j_1$, $a_j^1 = 0$ if $b_j \neq 0$, $j_1 \leq j \leq r_m$, $a_j^1 = a'_j$ if $b_j = 0$, $j_1 \leq j \leq r_m$, $b_j^1 = b_j$, $0 \leq j < j_1$, $b_j^1 = 0$, $j_1 \leq j \leq r_m$ and $b_j^1 = b_j$, $r_m < j \leq l$. (Here $b_j = 0$ if $l < j \leq r_m$).

We have $A = A^1 + C^1$ and $B = B^1 + C^1$. If $B^1 = 0$, then there is nothing to do.

If $a_j^1 = 0$ for $j > k$, then $\sum_{j=0}^{r_m} a_j^1 = \delta - 1$ and applying induction on δ we get $A^1 = \sum_{j=0}^{s} \alpha_j^1 n_j$ and $B^1 = \sum_{j=0}^{s} \beta_j^1 n_j$ such that α_j^1, β_j^1 are non-negative integers and either $\alpha_j^1 \geq \beta_j^1$ for $0 \leq j \leq s$ or $\beta_j^1 \geq \alpha_j^1$ for $0 \leq j \leq s$. We can assume $s \geq r_m$.

To have the result, we must only write $\alpha_j = \alpha_j^1$ (resp. $\beta_j = \beta_j^1$) for $0 \leq j < j_1$ or $r_m < j \leq s$ and $\alpha_j = \alpha_j^1 + b_j$ (resp. $\beta_j = \beta_j^1 + b_j$) for $j_1 \leq j \leq r_m$.

Let us assume $a_j^1 \neq 0$ for some $j > k$ and $B^1 \neq 0$. We have two possibilities:

a) $l \leq r_m$.

In this case, $b_j^1 = 0$ for $j \geq k$. Let us write

$$j_2 = \min\{j;\ a_j^1 \neq 0 \text{ and } j > k\}$$

and

$$l_2 = \max\{j;\ b_j^1 \neq 0\}.$$

We have $l_2 < k < j_2$.

Now, we consider the sequence of non-negative integers $(r_i^1)_{i \geq 0}$, given by $r_0^1 = l_2$, and $r_{i+1}^1 = r_i^1 + h(r_i^1)$, for $i \geq 0$. We have

$$r_0^1 < r_1^1 < \cdots < r_i^1 < \cdots .$$

Let $r_{m_1}^1$ be such that $r_{m_1-1}^1 < j_2 \leq r_{m_1}^1$.

We can write $B^1 = \sum_{j=0}^{r_{m_1}^1} b'_j n_j$, where $b'_j = b_j^1$, $0 \leq j < l_2 = r_0^1$, $b'_{r_i^1} = 0$, $0 \leq i \leq m_1 - 1$ and $b'_j = 1$, $l_2 \leq j \leq r_{m_1}^1$, $j \notin \{r_0^1, \ldots, r_{m_1-1}^1\}$. Note that $b'_j \in \{0, 1\}$ for $0 \leq j \leq r_{m_1}^1$.

Let us write $C^2 = \sum_{j=j_2}^{r_{m_1}^1} a_j^1 n_j$, $A^2 = \sum_{j=0}^{r_m} a_j^2 n_j$ and $B^2 = \sum_{j=0}^{r_{m_1}^1} b_j^2 n_j$, where $b_j^2 = b'_j$, $0 \leq j < j_2$, $b_j^2 = 0$ if $a_j^1 \neq 0$, $j_2 \leq j \leq r_{m_1}^1$, $b_j^2 = b'_j$ if $a_j^1 = 0$, $j_2 \leq j \leq r_{m_1}^1$, $a_j^2 = a_j^1$, $0 \leq j < j_2$, $a_j^2 = 0$, $j_2 \leq j \leq r_{m_1}^1$ and $a_j^2 = a_j^1$, $r_{m_1}^1 < j \leq r_m$. (Here $a_j^1 = 0$ if $r_m < j \leq r_{m_1}^1$).

We have $A^1 = A^2 + C^2$ and $B^1 = B^2 + C^2$. At this point, we have also two possibilities:

a.i) $r_m \leq r^1_{m_1}$. In this case, $a^2_j = 0$ for $j > k$. Then $\sum_{j=0}^{r_m} a^2_j = \delta - 1$ and applying induction on δ as above we get the Lemma.

a.ii) $r_m > r^1_{m_1}$. In this case, we can repeat in a similar way the reasoning of a). After a finite number, say d, of steps we get to one of the following statements:

α) $r_m \leq r^d_{m_d}$. In this situation, the Lemma follows by induction hypothesis on δ as in the case a.i).

β) $A = A^d + C^1 + \cdots + C^d$ and $B = B^d + C^1 + \cdots + C^d$, with $B^d = 0$. In this case, we also get the Lemma by taking expressions of A^d, C^1, …, C^d.

b) $l > r_m$.

In this case, let us write

$$j_2 = \min\{j;\ b^1_j \neq 0 \text{ and } j > r_m\} \quad \text{and} \quad k_2 = \max\{j;\ a^1_j \neq 0\}.$$

We have $k < k_2 \leq r_m < j_2$.

Now, we consider the sequence of non-negative integers $(r^1_i)_{i\geq 0}$, given by $r^1_0 = k_2$, and $r^1_{i+1} = r^1_i + h(r^1_i)$, for $i \geq 0$. We have

$$r^1_0 < r^1_1 < \cdots < r^1_i < \cdots .$$

Let $r^1_{m_1}$ be such that $r^1_{m_1-1} < j_2 \leq r^1_{m_1}$.

We can write $A = \sum_{j=0}^{r^1_{m_1}} a''_j n_j$, where $a''_j = a^1_j$, $0 \leq j < k_2 = r^1_0$, $a''_{r^1_i} = 0$, $0 \leq i \leq m_1 - 1$ and $a''_j = 1$, $k_2 \leq j \leq r^1_{m_1}$, $j \notin \{r^1_0, \ldots, r^1_{m_1-1}\}$. Note that $a''_j \in \{0, 1\}$ for $k < j \leq r^1_{m_1}$.

Let us write $C^2 = \sum_{j=j_2}^{r^1_{m_1}} b^1_j n_j$, $A^2 = \sum_{j=0}^{r^1_{m_1}} a^2_j n_j$ and $B^2 = \sum_{j=0}^{l} b^2_j n_j$, where $a^2_j = a''_j$, $0 \leq j < j_2$, $a^2_j = 0$ if $b^1_j \neq 0$, $j_2 \leq j \leq r^1_{m_1}$, $a^2_j = a''_j$ if $b^1_j = 0$, $j_2 \leq j \leq r^1_{m_1}$, $b^2_j = b^1_j$, $0 \leq j < j_2$, $b^2_j = 0$, $j_2 \leq j \leq r^1_{m_1}$ and $b^2_j = b^1_j$, $r^1_{m_1} < j \leq l$. (Here $b^1_j = 0$ if $l < j \leq r^1_{m_1}$).

We have $A^1 = A^2 + C^2$ and $B^1 = B^2 + C^2$. At this point, we have also two possibilities:

b.i) $l \leq r^1_{m_1}$. In this case, $b^2_j = 0$ for $j > k$. If $a^2_j = 0$ for $j > k$, the result follows by induction on δ as above.

If there exists $a^2_j \neq 0$ for some $j > k$, we are in a similar situation as in the case a). So we also get the Lemma with a similar reasoning as case a).

b.ii) $l > r^1_{m_1}$. In this case, we can repeat in a similar way the reasoning in b). After a finite number, say d, of steps we get $l \leq r^d_{m_d}$. Therefore, we get the Lemma in a similar way as case b.i). ■

Theorem 22 *With the above notations, let us consider* $A = \sum_{j=0}^k a_j n_j$ *and* $B = \sum_{j=0}^k b_j n_j$*, with* a_j *and* b_j *non-negative integers for* $0 \leq j \leq k$*. If* $A \geq B > 0$ *then we can write* $A = \sum_{j=0}^s \alpha_j n_j$ *and* $B = \sum_{j=0}^s \beta_j n_j$ *such that* α_j, β_j *are non-negative integers and* $\alpha_j \geq \beta_j \geq 0$ *for* $0 \leq j \leq s$*. Moreover,* s*,* $\alpha_0, \ldots, \alpha_s$*,* $\beta_0, \ldots, \beta_s$ *can be constructed by using only* $a_0, \ldots, a_k$*,* $b_0, \ldots, b_k$*, and the sequence* $(h(i))_{i\geq 0}$*.*

Proof: Let $\delta \geq 1$ be the positive integer given by

$$\delta = \sum_{j=0}^k b_j.$$

If $\delta = 1$, the result follows from Lemma 21. If $\delta > 1$, let us assume $b_i \neq 0$.

By Lemma 21 applied to A and n_i, we can write $n_i = \sum_{j=0}^k b^*_j n_j$, such that a_j, b^*_j are non-negative integers and $a_j \geq b^*_j$ for $0 \leq j \leq k$. Note that $A \geq B \geq n_i > 0$.

We write $A' = \sum_{j=0}^k a'_j n_j$ and $B' = \sum_{j=0}^k b'_j n_j$, with $a'_j = a_j - b^*_j$, $0 \leq j \leq k$, $b'_j = b_j$ if $j \neq i$ and $b'_i = b_i - 1$.

As $\sum_{j=0}^k b'_j = \delta - 1$, applying induction on δ we get $A' = \sum_{j=0}^s \alpha'_j n_j$ and $B' = \sum_{j=0}^s \beta'_j n_j$ such that α'_j, β'_j are non-negative integers and $\alpha'_j \geq \beta'_j$ for $0 \leq j \leq s$. Note that $A' \geq B' > 0$ and we can assume $s \geq k$.

To obtain the result we must only write $\alpha_j = \alpha'_j + b^*_j$ (resp. $\beta_j = \beta'_j + b^*_j$) for $0 \leq j \leq s$. (Here $b^*_j = 0$ for $k < j \leq s$). ■

Corollary 23 *With the above notations, let us consider* $A = \sum_{j=0}^k a_j n_j$ *and* $B = \sum_{j=0}^k b_j n_j$*, with* a_j*,* b_j *non-negative integers for* $0 \leq j \leq k$ *and* $B \neq 0$*.*

Let $A/B = [q_0, q_1, q_2 \ldots]$ *be the continued fraction associated to* A/B*. Then* q_0*,* q_1*,* $\ldots$ *can be constructed by using only* $a_0, \ldots, a_k$ $b_0, \ldots, b_k$*, and the sequence* $(h(i))_{i\geq 0}$*.*

Proof: If $A \geq B$ then by Theorem 22 we can assume that a_j, b_j are non-negative integers and $a_j \geq b_j$ for $0 \leq j \leq k$, and $b_j > 0$ for some j. In this case

$$\frac{A}{B} = 1 + \frac{A^1}{B},$$

with $A^1 = \sum_{j=0}^k a^1_j n_j$ and $a^1_j = a_j - b_j$, $0 \leq j \leq k$.

If $A^1 \geq B$, by Theorem 22, we can assume that $a^1_j \geq b_j$ for $0 \leq j \leq k$.

Now, we can repeat the process with A^1/B and after q_0 steps we get

$$\frac{A}{B} = q_0 + \frac{A^{q_0}}{B} = q_0 + \frac{1}{\frac{B}{A^{q_0}}},$$

with $A^{q_0} = \sum_{j=0}^{k} a_j^{q_0} n_j$, $a_j^{q_0}$ a non-negative integer for $0 \leq j \leq k$ and $B > A^{q_0}$.

If $A < B$ then $q_0 = 0$ and we shall take $A^{q_0} = A$ to have

$$\frac{A}{B} = 0 + \frac{1}{\frac{B}{A^{q_0}}},$$

and $B > A^{q_0}$.

At this point, we can use induction to obtain the result in any case. ■

References

[1] Abhyankar, S. S.: On the valuations centered in a local domain. *Amer. J. Math.* **78** (1956), 321–348.

[2] Abhyankar, S. S.: Desingularization of plane curves. In *Singularities, Part 1 (Arcata, Calif., 1981)*, 1–45. Proc. Sympos. Pure Math. **40**. Amer. Math. Soc., Providence, RI, 1983.

[3] Abhyankar, S. S.: Good points of a hypersurface. *Adv. in Math.* **68** (1988), 87–256.

[4] Abhyankar, S. S.: *Algebraic Geometry for Scientists and Engineers*. Mathematical Surveys and Monographs **35**. American Mathematical Society, Providence, RI, 1990.

[5] Abhyankar, S. S.: *Ramification Theoretic Methods in Algebraic Geometry*. Annals of Mathematics Studies **43**. Princeton University Press, Princeton, N.J. 1959.

[6] Abhyankar, S. S.: *Resolution of Singularities of Embedded Algebraic Surfaces*. Springer-Verlag. New York Inc., 1998.

[7] Abhyankar, S. S.: Resolution of singularities and modular Galois theory. *Bull. Amer. Math. Soc. (N.S.)* **38** (2001), no. 2, 131–169.

[8] Aparicio, J., Granja, A. and Sánchez-Giralda, T.: On proximity relations for valuations dominating a two-dimensional regular local ring. *Rev. Mat. Iberoamericana* **15** (1999), no. 3, 621–634.

[9] Delgado, D., Galindo, C. and Núñez, A.: Saturation for valuations on two-dimensional regular rings. *Math. Z.* **234** (2000), 519–550.

[10] Granja, A. and Sánchez-Giralda, T.: Enriques graphs of plane curves. *Comm. Algebra* **20** (1992), no. 2, 527–562.

[11] Lipman, J.: Proximity inequalities for complete ideals in two-dimensional regular local rings. In *Commutative algebra: syzygies, multiplicities, and birational algebra (South Hadley, MA, 1992)*, 293–306. Contemp. Math. **159**. Amer. Math. Soc., Providence, RI, 1994.

[12] Nagata, M.: *Local Rings*. Interscience Tracts in Pure and Applied Mathematics **13**, Interscience Publishers, a division of John Wiley & Sons, New York-London, 1962.

[13] Shannon, D. L.: Monoidal transforms of regular rings. *Amer. J. Math.* **95** (1973), 294–320.

[14] Vaquié, M.: Valuations. In *Resolution of singularities (Obergurgl, 1997)*, 539–590. Progr. Math. **181**, Birkhäuser, Basel, 2000.

[15] Zariski, O.: Local uniformization on algebraic varieties. *Ann. of Math.* **41** (1940), 852–896.

[16] Zariski, O. and Samuel, P.: *Commutative Algebra, Vols. I and II.* D. Van Nostrand Company, Inc., Princeton, New Jersey, 1958, 1960.

Recibido: 3 de abril de 2002
Revisado: 8 de agosto de 2002

Ángel Granja
Department of Mathematics
University of León
Campus de Vegazana, s/n. 24071-León, Spain
demagb@unileon.es

Cristina Rodríguez
Department of Mathematics
University of León
Campus de Vegazana, s/n. 24071-León, Spain
demmrs@unileon.es

Both authors are partially supported by DGICYT, PB98-0753-C02-01.