Constant scalar curvature hypersurfaces with spherical boundary in Euclidean space

Luis J. Alías and J. Miguel Malacarne

Dedicated to Professor A. Gervasio Colares on the occasion of his 70th birthday

Abstract

It is still an open question whether a compact embedded hypersurface in the Euclidean space with constant mean curvature and spherical boundary is necessarily a hyperplanar ball or a spherical cap, even in the simplest case of a compact constant mean curvature surface in $\mathbb{R}^3$ bounded by a circle. In this paper we prove that this is true for the case of the scalar curvature. Specifically we prove that the only compact embedded hypersurfaces in the Euclidean space with constant scalar curvature and spherical boundary are the hyperplanar round balls (with zero scalar curvature) and the spherical caps (with positive constant scalar curvature). The same applies in general to the case of embedded hypersurfaces with constant r-mean curvature, with $r \geq 2$.

1. Introduction.

A classical result by Alexandrov [1] states that the spheres are the only *closed* embedded hypersurfaces with constant mean curvature in the Euclidean space $\mathbb{R}^{n+1}$. Here by closed hypersurfaces we mean *compact and without boundary* hypersurfaces. This result was extended to the case of

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the scalar curvature by Ros in [15], where it was shown that the spheres are the only closed embedded hypersurfaces with constant scalar curvature in $\mathbb{R}^{n+1}$, solving a problem proposed by Yau in [18]. More generally, Ros [16] also extended it to the case of the r-mean curvature H_r, showing that the spheres are the only closed embedded hypersurfaces with constant r-mean curvature in $\mathbb{R}^{n+1}$.

However, when one considers the corresponding problem for the case of compact hypersurfaces with non-empty boundary, it is still unknown whether a compact embedded hypersurface in $\mathbb{R}^{n+1}$ with non-zero constant mean curvature and spherical boundary is spherical, even in the simplest case of a compact constant mean curvature surface in $\mathbb{R}^3$ bounded by a circle. In [10] Kapouleas showed that there exist examples of higher genus compact, non-spherical immersed surfaces with constant mean curvature in $\mathbb{R}^3$ bounded by a circle. It was conjectured in [9] that a compact surface with non-zero constant mean curvature in $\mathbb{R}^3$ bounded by a circle is a spherical cap if either the surface has genus zero and it is immersed or the surface is embedded. In the last years, different authors have considered this problem obtaining several partial results [2], [4], [5], [6], [8], but the original conjecture remains open.

In [11], Koiso gave a new interpretation of this problem by studying under what conditions the symmetries of the boundary of a non-zero constant mean curvature hypersurface M embedded into $\mathbb{R}^{n+1}$ are inherited by the whole hypersurface. In particular, she showed that when the boundary Σ is a round $(n-1)$-sphere contained in a hyperplane Π of $\mathbb{R}^{n+1}$, and M does not intersect the outside of Σ in Π, then M is symmetric with respect to every hyperplane which contains the center of Σ and is perpendicular to Π, and hence M must be a spherical cap. On the other hand, Brito, Sá Earp, Meeks and Rosenberg [9] also showed that if Σ is a strictly convex $(n-1)$-dimensional submanifold contained in a hyperplane Π of $\mathbb{R}^{n+1}$, and M is a compact embedded hypersurface in $\mathbb{R}^{n+1}$ with constant mean curvature and $\partial M = \Sigma$ which is transverse to Π along the boundary ∂M, then M is contained in one of the half-spaces of $\mathbb{R}^{n+1}$ determined by Π and M has all the symmetries of Σ. More recently, Rosenberg [17] extended some of these results to the case of the r-mean curvatures.

In this paper we will prove that the conjecture is true for embedded hypersurfaces in the case of the scalar curvature. Specifically we will prove the following.

Theorem 1. *The only compact embedded hypersurfaces in the Euclidean space with constant scalar curvature and spherical boundary are the hyperplanar round balls (with zero scalar curvature) and the spherical caps (with positive constant scalar curvature).*

In particular, for the 2-dimensional case we conclude that a compact embedded surface in $\mathbb{R}^3$ with constant Gaussian curvature and circular boundary is necessarily a flat planar disc or a spherical cap.

Our proof will be a consequence of more general symmetry result (Theorem 3), and it also applies in general to the case of hypersurfaces with constant r-mean curvature H_r, when $r \geq 2$ (Corollary 4). The results of this paper are part of a more general project where we derive some other new related results for compact hypersurfaces with constant higher order mean curvature in Euclidean space and, more generally, in hyperbolic and spherical spaces [3].

2. Preliminaries.

In this section we will review some standard facts about the geometry of Euclidean hypersurfaces. Let $\psi : M^n \longrightarrow \mathbb{R}^{n+1}$ be an orientable n-dimensional connected hypersurface immersed into the Euclidean space $\mathbb{R}^{n+1}$ (eventually with non-empty smooth boundary ∂M). Since M is orientable, we may choose along M a globally defined unit normal vector field N, and we may assume that M is oriented by N. Let ∇° be the flat Levi-Civita connection on $\mathbb{R}^{n+1}$, and let ∇ denote the Levi-Civita connection on M. Then the Gauss and Weingarten formulae for the hypersurface are given by

$$\nabla^\circ_X Y = \nabla_X Y + \langle AX, Y \rangle N \,,$$

and

$$A(X) = -\nabla^\circ_X N \,,$$

for all tangent vector fields $X, Y \in \mathcal{X}(M)$. Here $A : \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$ defines the shape operator (or the second fundamental form) of the hypersurface with respect to N. The curvature tensor R of the hypersurface M is described in terms of A by the Gauss equation, which is given by

$$(1) \qquad R(X,Y)Z = \langle AX, Z \rangle AY - \langle AY, Z \rangle AX \,,$$

for all tangent vector fields $X, Y, Z \in \mathcal{X}(M)$. Observe that our criterion here for the definition of the curvature tensor is the one in [13].

As is well known, A is a self-adjoint linear operator, and its eigenvalues $\kappa_1, \ldots, \kappa_n$ are the principal curvatures of the hypersurface. Associated to the shape operator there are n algebraic invariants, which are the elementary symmetric functions of its eigenvalues, given by

$$S_r = S_r(\kappa_1, \ldots, \kappa_n) = \sum_{i_1 < \cdots < i_r} \kappa_{i_1} \cdots \kappa_{i_r}, \qquad 1 \leq r \leq n \,.$$

Observe that the characteristic polynomial of A can be written in terms of the S_r's as

$$(2) \qquad \det(tI - A) = \sum_{r=0}^{n} (-1)^r S_r t^{n-r} .$$

The r-mean curvature H_r of the hypersurface is then defined by

$$\binom{n}{r} H_r = S_r .$$

In particular, when $r = 1$ then $H_1 = (1/n) \operatorname{trace}(A) = H$ is the mean curvature of M, which is the main extrinsic curvature of the hypersurface. On the other hand, when $r = 2$, H_2 defines a geometric quantity which is related to the (intrinsic) scalar curvature of the hypersurface. Indeed, from (1) we know that the Ricci curvature of M is given by

$$\operatorname{Ric}(X, Y) = nH\langle AX, Y \rangle - \langle AX, AY \rangle ,$$

for $X, Y \in \mathcal{X}(M)$, so that its scalar curvature S is

$$(3) \qquad S = \operatorname{trace}(\operatorname{Ric}) = n(n-1) H_2 .$$

In general, it follows from the Gauss equation (1) that when r is odd H_r is extrinsic (and its sign depends on the chosen orientation), while when r is even H_r is an intrinsic geometric quantity.

The classical Newton transformations $P_r : \mathcal{X}(M) \longrightarrow \mathcal{X}(M)$ are defined inductively from A by

$$(4) \qquad P_0 = I \qquad \text{and} \qquad P_r = S_r I - A P_{r-1} , \qquad 1 \leq r \leq n ,$$

where I denotes the identity in $\mathcal{X}(M)$, or equivalently by

$$P_r = S_r I - S_{r-1} A + \cdots + (-1)^{r-1} S_1 A^{r-1} + (-1)^r A^r .$$

Note that by the Cayley-Hamilton theorem, we have $P_n = 0$. Let us recall that each P_r is also a self-adjoint linear operator which commutes with A. Indeed, A and P_r can be simultaneously diagonalized; if $\{e_1, \ldots, e_n\}$ are the eigenvectors of A corresponding to the eigenvalues $\kappa_1, \ldots, \kappa_n$, respectively, then they are also the eigenvectors of P_r corresponding to the eigenvalues of P_r, and $P_r(e_i) = \mu_{i,r} e_i$ with

$$\mu_{i,r} = \frac{\partial S_{r+1}}{\partial \kappa_i} = \sum_{i_1 < \cdots < i_r, i_j \neq i} \kappa_{i_1} \cdots \kappa_{i_r} , \qquad 1 \leq i \leq n .$$

From here it can be easily seen that

$$\text{trace}(P_r) = (n - r)S_r = c_r H_r$$

and

$$\text{trace}(AP_r) = (r + 1)S_{r+1} = c_r H_{r+1} \, ,$$

where

$$c_r = (n - r)\binom{n}{r} = (r + 1)\binom{n}{r + 1} .$$

For the details, we refer the reader to the classical paper by Reilly [14] (see also [17] for a more accesible modern treatment by Rosenberg).

3. A geometric configuration.

Throughout this paper, we will consider the following geometric configuration in Euclidean space $\mathbb{R}^{n+1}$. Let $\Pi \subset \mathbb{R}^{n+1}$ be a hyperplane in $\mathbb{R}^{n+1}$, and let $\Sigma^{n-1} \subset \Pi$ be an orientable $(n - 1)$-dimensional compact submanifold contained in Π. Let $\psi : M^n \longrightarrow \mathbb{R}^{n+1}$ be an orientable compact hypersurface immersed into $\mathbb{R}^{n+1}$ with smooth boundary ∂M. As usual, M is said to be a hypersurface with boundary Σ if the immersion ψ restricted to the boundary ∂M is a diffeomorphism onto Σ. In order to orient this geometric configuration, let us consider the hypersurface M oriented by a globally defined unit normal vector field N. The orientation of M induces a natural orientation on its boundary as follows: given a point $p \in \partial M$, a basis $\{v_1, \ldots, v_{n-1}\}$ for $T_p(\partial M)$ is said to be positively oriented if $\{u, v_1, \ldots, v_{n-1}\}$ is a positively oriented basis for T_pM, whenever $u \in T_pM$ is outward pointing. We will denote by ν the outward pointing unit conormal vector field along ∂M. By means of the diffeomorphism $\psi|_{\partial M} : \partial M \longrightarrow \Sigma$, the orientation of ∂M is induced on each connected component of Σ. On the other hand, let η be the unitary vector field normal to Σ in Π which points outward with respect to the domain in Π bounded by Σ. Now, we choose a the unique unitary vector field normal to Π which is compatible with η and with the orientation of Σ. With this choice, given a point $p \in \Sigma$, a basis $\{v_1, \ldots, v_{n-1}\}$ for $T_p\Sigma$ is positively oriented if and only if $\{\eta(p), v_1, \ldots, v_{n-1}\}$ is a positively oriented basis for Π.

Let $\{e_1, \ldots, e_{n-1}\}$ be a (locally defined) positively oriented frame field along ∂M. Using this frame, we can write $\nu = e_1 \times \cdots \times e_{n-1} \times N$, and

similarly $\eta = e_1 \times \cdots \times e_{n-1} \times a$, since $\det(\nu, e_1, \ldots, e_{n-1}, N) = 1 = \det(\eta, e_1, \ldots, e_{n-1}, a)$. From these expressions we easily find that

$$(5) \qquad \langle \eta, N \rangle = -\langle a, \nu \rangle .$$

Let A_Σ denote the shape operator of $\Sigma^{n-1} \subset \Pi$ with respect to the unit normal vector field η. It then follows that

$$\nabla^{\mathrm{o}}_{e_i} e_j = \sum_{k=1}^{n-1} \langle \nabla^{\mathrm{o}}_{e_i} e_j, e_k \rangle e_k + \langle \nabla^{\mathrm{o}}_{e_i} e_j, \nu \rangle \nu + \langle A e_i, e_j \rangle N ,$$

for every $1 \le i, j \le n - 1$, and also

$$\nabla^{\mathrm{o}}_{e_i} e_j = \sum_{k=1}^{n-1} \langle \nabla^{\mathrm{o}}_{e_i} e_j, e_k \rangle e_k + \langle A_\Sigma e_i, e_j \rangle \eta ,$$

so that from (5) we have that

$$(6) \qquad \langle A e_i, e_j \rangle = -\langle A_\Sigma e_i, e_j \rangle \langle a, \nu \rangle , \qquad 1 \le i, j \le n - 1 .$$

We now suppose that the basis $\{e_1, \ldots e_{n-1}\}$ on the boundary is chosen such that, at the point $p \in \partial M$, it is formed by the eigenvectors of A_Σ, and let us denote its corresponding eigenvalues by $\tau_1, \ldots, \tau_{n-1}$. In other words,

$$A_\Sigma e_i = \tau_i e_i , \qquad 1 \le i \le n - 1 .$$

Hence by (6), $\langle A e_i, e_j \rangle = 0$ when $i \ne j$, and for each $p \in \partial M$, the matrix of A in the orthonormal basis $\{e_1, \ldots e_{n-1}, \nu\}$ of $T_p M$ is given by

$$(7) \qquad A = \begin{pmatrix} -\tau_1 \langle a, \nu \rangle & 0 & \cdots & 0 & \langle A\nu, e_1 \rangle \\ 0 & -\tau_2 \langle a, \nu \rangle & \cdots & 0 & \langle A\nu, e_2 \rangle \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\tau_{n-1} \langle a, \nu \rangle & \langle A\nu, e_{n-1} \rangle \\ \langle A\nu, e_1 \rangle & \langle A\nu, e_2 \rangle & \cdots & \langle A\nu, e_{n-1} \rangle & \langle A\nu, \nu \rangle \end{pmatrix} .$$

This expression invites us to compute the characteristic polynomial of A. To do that, we begin by observing that for each k, $1 \le k \le n - 1$,

$$(8) \qquad \begin{aligned} &\det(t I_{k+1} - \Lambda_k) \\ &= (t + \tau_k \langle a, \nu \rangle) \det(t I_k - \Lambda_{k-1}) - \langle A\nu, e_k \rangle^2 \prod_{j=1}^{k-1} (t + \tau_j \langle a, \nu \rangle) , \end{aligned}$$

where $\Lambda_k = \Lambda(\tau_1, \dots, \tau_k)$ is the $(k+1)$-dimensional matrix given by

$$
\Lambda_k = \begin{pmatrix}
-\tau_1 \langle a, \nu \rangle & 0 & \cdots & 0 & \langle A\nu, e_1 \rangle \\
0 & -\tau_2 \langle a, \nu \rangle & \cdots & 0 & \langle A\nu, e_2 \rangle \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -\tau_k \langle a, \nu \rangle & \langle A\nu, e_k \rangle \\
\langle A\nu, e_1 \rangle & \langle A\nu, e_2 \rangle & \cdots & \langle A\nu, e_k \rangle & \langle A\nu, \nu \rangle
\end{pmatrix}.
$$

In particular, $\Lambda_{n-1} = A$ and therefore, applying a simple induction argument in (8), we obtain that the characteristic polynomial of A is given by

$$
\det(tI_n - A) = (t - \langle A\nu, \nu \rangle) \prod_{i=1}^{n-1} (t + \tau_i \langle a, \nu \rangle)
$$

$$
- \sum_{i=1}^{n-1} \langle A\nu, e_i \rangle^2 \prod_{j=1, j \neq i}^{n-1} (t + \tau_j \langle a, \nu \rangle)
$$

$$
= (t - \langle A\nu, \nu \rangle) \left(t^{n-1} + \sum_{i=1}^{n-1} s_i \langle a, \nu \rangle^i t^{n-1-i} \right)
$$

$$
- \sum_{i=1}^{n-1} \langle A\nu, e_i \rangle^2 \left(t^{n-2} + \sum_{j=1}^{n-2} s_j(\widehat{\tau}_i) \langle a, \nu \rangle^j t^{n-2-j} \right),
$$

where s_r (respectively $s_r(\widehat{\tau}_i)$) stands for the elementary symmetric functions of $\tau_1, \dots, \tau_{n-1}$, (respectively $\tau_1, \dots, \widehat{\tau}_i, \dots, \tau_{n-1}$), and, as usual, $s_0 = s_0(\widehat{\tau}_i) = 1$ by definition. Comparing the terms of above polynomials, we conclude from (2) that the r-mean curvature H_r of the hypersurface M, at a boundary point $p \in \partial M$, is given by

$$
nH_1 = S_1 = -s_1 \langle a, \nu \rangle + \langle A\nu, \nu \rangle,
$$

$$
(9) \qquad \binom{n}{2} H_2 = S_2 = s_2 \langle a, \nu \rangle^2 - s_1 \langle a, \nu \rangle \langle A\nu, \nu \rangle - \sum_{i=1}^{n-1} \langle A\nu, e_i \rangle^2,
$$

$$
\binom{n}{r} H_r = S_r = (-1)^r s_r \langle a, \nu \rangle^r + (-1)^{r-1} s_{r-1} \langle a, \nu \rangle^{r-1} \langle A\nu, \nu \rangle
$$

$$
+ (-1)^{r-1} \langle a, \nu \rangle^{r-2} \sum_{i=1}^{n-1} s_{r-2}(\widehat{\tau}_i) \langle A\nu, e_i \rangle^2,
$$

for $3 \leq r \leq n$, where $s_n = 0$ by definition.

Now we are ready to prove the following essential auxiliary result.

Lemma 2. *Let Σ be an orientable $(n-1)$-dimensional compact submanifold contained in a hyperplane $\Pi = a^\perp$ of $\mathbb{R}^{n+1}$. Let $\psi : M^n \longrightarrow \mathbb{R}^{n+1}$ be an orientable hypersurface with boundary $\Sigma = \psi(\partial M)$, and let ν stands for the outward pointing unit conormal vector field along $\partial M \subset M$. Then, along the boundary ∂M and for every $1 \leq r \leq n-1$, it holds*

$$(10) \qquad \langle P_r \nu, \nu \rangle = (-1)^r s_r \langle a, \nu \rangle^r \,,$$

where s_r is the r-th symmetric function of the principal curvatures of $\Sigma \subset \Pi$ with respect to the outward pointing unitary normal.

Proof. We will use induction on r. Observe that for $r = 1$, we have that

$$\langle P_1 \nu, \nu \rangle = S_1 - \langle A\nu, \nu \rangle \,,$$

which jointly with (9) implies

$$\langle P_1 \nu, \nu \rangle = -s_1 \langle a, \nu \rangle \,.$$

For a given $2 \leq r \leq n-1$, suppose that

$$(11) \qquad \langle P_j \nu, \nu \rangle = (-1)^j s_j \langle a, \nu \rangle^j$$

holds for all $1 \leq j \leq r-1$. Observe that

$$A\nu = \sum_{i=1}^{n-1} \langle A\nu, e_i \rangle e_i + \langle A\nu, \nu \rangle \nu \,,$$

so that from the inductive definition of P_r and (11) we conclude that

$$\langle P_r \nu, \nu \rangle = S_r - \langle P_{r-1}\nu, A\nu \rangle$$

$$(12) \qquad = S_r - \langle P_{r-1}\nu, \nu \rangle \langle A\nu, \nu \rangle - \sum_{i=1}^{n-1} \langle P_{r-1}\nu, e_i \rangle \langle A\nu, e_i \rangle$$

$$= S_r + (-1)^r s_{r-1} \langle a, \nu \rangle^{r-1} \langle A\nu, \nu \rangle - \sum_{i=1}^{n-1} \langle P_{r-1}\nu, e_i \rangle \langle A\nu, e_i \rangle \,.$$

On the other hand, we also know from (7) that

$$Ae_i = -\tau_i\langle a, \nu\rangle\, e_i + \langle A\nu, e_i\rangle\, \nu\,,$$

so that from our induction hypothesis (11) we have for every $1 \leq j \leq r-1$,

$$\langle P_j\nu, e_i\rangle = -\langle P_{j-1}\nu, Ae_i\rangle$$
$$= \tau_i\langle a, \nu\rangle\, \langle P_{j-1}\nu, e_i\rangle - (-1)^{j-1}s_{j-1}\langle a, \nu\rangle^{j-1}\langle A\nu, e_i\rangle\,.$$

This implies by a recursive argument that

$$(13) \qquad \langle P_{r-1}\nu, e_i\rangle = (-1)^{r-1}\langle A\nu, e_i\rangle\, \langle a, \nu\rangle^{r-2}\sum_{j=0}^{r-2}(-1)^j s_{r-2-j}\tau_i^j$$

$$= (-1)^{r-1}\langle A\nu, e_i\rangle\, \langle a, \nu\rangle^{r-2}\, s_{r-2}(\widehat{\tau_i})\,,$$

since it is not difficult to see that

$$s_m(\widehat{\tau_i}) = \sum_{j=0}^{m}(-1)^j s_{m-j}\, \tau_i^j\,,$$

for every $1 \leq m \leq n-1$. Using now (13) in (12), along with (9), we conclude that

$$\langle P_r\nu, \nu\rangle = S_r + (-1)^r s_{r-1}\, \langle a, \nu\rangle^{r-1}\, \langle A\nu, \nu\rangle$$

$$+ (-1)^r\, \langle a, \nu\rangle^{r-2}\sum_{i=1}^{n-1} s_{r-2}(\widehat{\tau_i})\, \langle A\nu, e_i\rangle^2$$

$$= (-1)^r s_r\langle a, \nu\rangle^r\,.$$

This finishes the proof of Lemma 2.

4. Proof of the main results.

Our Theorem 1 will be a consequence of the following symmetry result, which extends a previous result given by Rosenberg [17, Theorem 7.3] (observe that the case $r = n$ was first proved by Rosenberg).

Theorem 3. *Let Σ be an strictly convex compact $(n-1)$-dimensional submanifold contained in a hyperplane $\Pi \subset \mathbb{R}^{n+1}$, and let $\psi : M^n \longrightarrow \mathbb{R}^{n+1}$*

be an embedded compact hypersurface with boundary Σ. Let us assume that for a given $2 \leq r \leq n$, the r-mean curvature H_r of M is a non-zero constant. Then M is contained in one of the half-spaces of $\mathbb{R}^{n+1}$ determined by Π and M has all the symmetries of Σ.

Proof. By choosing an appropriate orientation of M, we may assume without loss of generality that H_r is a positive constant. Actually, it is not difficult to see that under the hypothesis above there exists at least one interior elliptic point of M, that is, an interior point of M where all the principal curvatures are positive. In fact, since M is not part of a hyperplane (because of $H_r \neq 0$), then one easily finds a radius $R > 0$ and a point $a \in \mathbb{R}^{n+1}$ such that the closed round ball $\overline{B}(a, R)$ contains M and such that there is a point $p_0 \in \text{int}(M) \cap \partial B(a, R)$ (englobe M with spheres of large radius until such a sphere touches M on one side at an interior point). This fact, jointly with the fact that H_r is a positive constant, allows us to conclude that the Newton transformation P_{r-1} is positive definite on M (see [7, Proposition 3.2] and [17, p. 232]). In particular, by (10) it follows that $\langle a, \nu \rangle(p) \neq 0$ at every boundary point $p \in \partial M$, what means that M is transverse to Π along the boundary. Our result then is a consequence of [17, Theorem 7.3].

Now Theorem 1 easily follows from Theorem 3. Indeed, if M is not a hyperplanar round ball, then the constant scalar curvature, or equivalently H_2 (see equation (3)), must be necessarily positive because there exists at least one interior elliptic point of M. The same applies in general for the case of hypersurfaces with constant r-mean curvature H_r, when $r \geq 2$.

Corollary 4. *The only embedded compact hypersurfaces in the Euclidean space with constant r-mean curvature H_r (with $2 \leq r \leq n$) and spherical boundary are the hyperplanar round balls (with $H_r = 0$) and the spherical caps (with H_r a non-zero constant).*

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Luis J. Alías[*]
Departamento de Matemáticas
Universidad de Murcia
E-30100 Espinardo, Murcia, SPAIN
ljalias@um.es

and

J. Miguel Malacarne[†]
Departamento de Matemática
Universidade Federal do Espírito Santo
29060-900 Vitória-ES, BRAZIL
jmiguel@cce.ufes.br