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Size properties of wavelet packets generated using finite filters

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Abstract

We show that asymptotic estimates for the growth in $L^p(\mathbb{R})$-norm of a certain subsequence of the basic wavelet packets associated with a finite filter can be obtained in terms of the spectral radius of a subdivision operator associated with the filter. We obtain lower bounds for this growth for $p \gg 2$ using finite dimensional methods. We apply the method to get estimates for the wavelet packets associated with the Daubechies, least asymmetric Daubechies, and Coiflet filters. A consequence of the estimates is that such basis wavelet packets cannot constitute a Schauder basis for $L^p(\mathbb{R})$ for $p \gg 2$. Finally, we show that the same type of results are true for the associated periodic wavelet packets in $L^p[0,1)$.

1. Introduction and main results.

Let $\{V_j\}$ be a multiresolution analysis with associated scaling function ϕ, wavelet ψ, and associated low-pass filters (m_0, m_1). The basic wavelet packets $\{w_n\}_{n=0}^{\infty}$ are defined recursively by $w_0 = \phi$, $w_1 = \psi$, and for $n \in \mathbb{N}$ with binary expansion

$$n = \sum_k \varepsilon_k \, 2^{k-1} \, ,$$

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we let

$$\widehat{w}_n(\xi) = \prod_{j=1}^{\infty} m_{\varepsilon_j}\Big(\frac{\xi}{2^j}\Big)\,.$$

Such functions were introduced in [1], [2] to improve the frequency localization of wavelets at high frequency. It was proved in [1] that the collection $\{w_n\}_n$ of basic wavelet packets associated with the Lemarié-Meyer multiresolution analysis are not uniformly bounded in $L^p(\mathbb{R})$-norm for p large. The technique used was to show that the family $\{\widehat{w}_n\}_n$ is not bounded in L^1-norm. This works because the Lemarié-Meyer low-pass filter m_0 is a nonnegative functions so each $\widehat{w}_n$ is just a modulation of a nonnegative function. It is therefore possible to recover the L^∞-norm of w_n from the L^1-norm of $\widehat{w}_n$. However, this technique fails in general since all finite filters associated with a multiresolution analysis are *not* nonnegative functions (see [3]). The growth in L^1-norm of the Fourier transform of basic wavelet packets associated with finite filters was studied in detail by É. Séré in [6], where he proves that the subsequence of the basic wavelet packets with worst asymptotic growth is $\{w_{2^n-1}\}_{n=0}^{\infty}$.

In the present paper we introduce a technique to estimate the $L^p(\mathbb{R})$-norm of the subsequence $\{w_{2^n-1}\}_{n=0}^{\infty}$ associated with finite filters (m_0, m_1). The key is to study the subdivision operator S, associated with the finite high-pass filter $m_1(\xi) = \sum_{k\in\mathbb{Z}} g_k\, e^{ik\xi}$, defined by

$$(Sc)_i = \sum_{j\in\mathbb{Z}} g_{i-2j}\, c_j\;, \qquad i \in \mathbb{Z}\,. \tag{1}$$

for $c \in \ell^p(\mathbb{Z})$, $1 \le p \le \infty$. We let $\sigma_p[S]$ denote the spectral radius of S on $\ell^p(\mathbb{Z})$. The main observation of Section 2 is

Theorem 1.1. *Let $\{w_n\}_{n=0}^{\infty}$ be the wavelet packets generated by the finite filters (m_0, m_1) associated with a multiresolution analysis. Define $\widetilde{\sigma}_p$, $1 \le p \le \infty$, by*

$$\widetilde{\sigma}_p = \lim_{n\to\infty} \|w_{2^n-1}\|_p^{1/n}\;.$$

Then $\widetilde{\sigma}_p$ exists and $\widetilde{\sigma}_p = 2^{1-1/p}\sigma_p[S]$.

In Section 3 we derive numerical estimates using Theorem 1.1 for the growth in $L^p(\mathbb{R})$-norm, $p \gg 2$, for a number of Daubechies, least asymmetric Daubechies, and Coiflet filters. We find that such families of wavelet packets all have a subsequence with growth in $L^p(\mathbb{R})$-norm of order n^α, with n denoting the frequency, for $p \gg 2$ and for some $\alpha > 0$

(depending on p). Moreover, our technique provides a lower bound for the value of α and a surprising consequence of this is derived in Section 4, where we prove that such wavelet packets cannot constitute a Schauder basis for $L^p(\mathbb{R})$ for $p \gg 2$. This is in sharp contrast to the simplest wavelet packet system, the Walsh system, that do constitute a Schauder basis for $L^p(\mathbb{R})$ for $1 < p < \infty$. In Section 5 we consider the same but more difficult question about growth in $L^p[0,1)$-norm for the periodized wavelet packets $\{\widetilde{w_n}\}_{n=0}^{\infty}$ defined by

$$\widetilde{w_n}(x) = \sum_{k\in\mathbb{Z}} w_n(x-k)\,.$$

The following theorem will be proved in Section 5.

Theorem 1.2. *Let $\{w_n\}_n$ be a wavelet packet basis associated with the finite filters (m_0, m_1). Choose N such that* $\operatorname{diam}\operatorname{supp}(w_n) \leq 2^N$. *Fix $L \in 2\,\mathbb{Z}^+ + 1$. If*

$$(m_0^{-1}(0) \cup m_1^{-1}(0)) \cap \Big(\bigcup_{k=1}^{N} 2^{-k}(2\,\mathbb{Z}+1)\pi \Big) = \varnothing$$

then there exist finite constants $c_p, C_p > 0$ (depending on L) such that

$$c_p\,\|w_{2^n-1}\|_p \leq \|\widetilde{w_{2^{n+N}-L}}\|_{L^p[0,1)} \leq C_p\,\|w_{2^n-1}\|_p\;,$$

for $n \ni 2^{n+N} - L \geq 1$.

This theorem is then applied to the periodized versions of the wavelet packets mentioned above. The conclusion is that they all have a subsequence with growth in $L^p[0,1)$-norm of order n^α, $\alpha > 0$, for $p \gg 2$. Moreover, we prove that such periodic wavelet packets cannot constitute a Schauder basis for $L^p[0,1)$ for large p.

2. L^p-norms of wavelet packets.

In this section we some fundamental results about multiresolution analyses and scaling functions to calculate the $L^p(\mathbb{R})$-norm of wavelet packets associated with finite filters. We will assume that $\{V_j\}$ be a multiresolution analysis with associated scaling function ϕ satisfying $|\phi(x)| \leq C\,(1+|x|)^{-1-\varepsilon}$ for some $\varepsilon > 0$, and associated low-pass filters (m_0, m_1). In [5] one can find the following lemma,

Lemma 2.1. *There exist finite constants $c_p, C_p > 0$ such that for every finite sequence $\{c_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$ we have*

$$c_p \,\|\{c_k\}\|_{\ell^p(\mathbb{Z})} \le \Big\| \sum_{k\in\mathbb{Z}} c_k\, \phi(x-k)\Big\|_p \le C_p \,\|\{c_k\}\|_{\ell^p(\mathbb{Z})}\ ,$$

which gives us a sharp estimate of the $L^p(\mathbb{R})$ norm of a wavelet packet associated with a multiresolution analysis.

Lemma 2.2. *There exist finite positive constants c_p and C_p such that the $L^p(\mathbb{R})$-norm, $1 \le p \le \infty$, of the wavelet packet w_n, defined by*

$$\widehat{w}_n(\xi) = \Big(\prod_{j=1}^{N} m_{\varepsilon_j}\Big(\frac{\xi}{2^j}\Big)\Big)\widehat{\phi}\Big(\frac{\xi}{2^N}\Big)\,,$$

is bounded by

$$c_p\, 2^N\, 2^{-N/p}\, \|\{c_k\}\|_{\ell^p(\mathbb{Z})} \le \|w_n\|_p \le C_p\, 2^N\, 2^{-N/p}\, \|\{c_k\}\|_{\ell^p(\mathbb{Z})}\ ,$$

where

$$m_{\varepsilon_N}(\xi)\, m_{\varepsilon_{N-1}}(2\,\xi) \cdots m_{\varepsilon_1}(2^{N-1}\xi) = \sum_{k\in\mathbb{Z}} c_k\, e^{ik\xi}\,.$$

Proof. We have

$$\widehat{w}_n(\xi) = \Big(\prod_{j=1}^{N} m_{\varepsilon_j}\Big(\frac{\xi}{2^j}\Big)\Big)\widehat{\phi}\Big(\frac{\xi}{2^N}\Big)\,,$$

so

$$\widehat{w}_n(2^N\xi) = \Big(\prod_{j=0}^{N-1} m_{\varepsilon_{N-j}}(2^j\,\xi)\Big)\widehat{\phi}(\xi)\,.$$

Taking the inverse Fourier Transform of (2) shows that $2^{-N}w_n(2^{-N}x)$ is a linear combination of the functions $\{\phi(x-k)\}_k$ and that the expansion coefficients are given by the coefficients of the Fourier series

$$m_{\varepsilon_N}(\xi)\, m_{\varepsilon_{N-1}}(2\,\xi) \cdots m_{\varepsilon_1}(2^{N-1}\xi) = \sum_{k\in\mathbb{Z}} c_k\, e^{ik\xi}\,.$$

Note that $\|2^{-N}w_n(2^{-N}\cdot)\|_p = 2^{-N}2^{N/p}\|w_n\|_p$ for $1 \le p \le \infty$. It now follows from Lemma 2.2 that there exist constants c_p and C_p (independent of n) such that

$$c_p\, 2^N 2^{-N/p}\, \|\{c_k\}\|_{\ell^p(\mathbb{Z})} \le \|w_n\|_p \le C_p\, 2^N 2^{-N/p}\, \|\{c_k\}\|_{\ell^p(\mathbb{Z})}\ .$$

In what follows, we will restrict our attention to subsequences of the form $\{w_{2^n-1}\}_n$. The main reason is that the binary expansion of $2^n - 1$ consists of $n-1$ 1's and nothing else which simplifies the estimates given by Lemma 2.2. The key to getting good estimates is to consider the operator S defined by (1) on $\ell^p(\mathbb{Z})$, $1 \leq p \leq \infty$. S is called the (stationary) subdivision operator associated with the filter m_1. Note that S is just the bi-infinite matrix $(g_{i-2j})_{ij}$ considered as a bounded operator on $\ell^p(\mathbb{Z})$. It is also easy to check that S can be represented (formally) as the multiplication operator

$$Sf(\xi) = m_1(\xi)\, f(2\,\xi)\,,$$

for $f(\xi) = \sum_{k\in\mathbb{Z}} c_k\, e^{ik\xi}$.

We are interested in calculating the spectral radius $\sigma_p[S]$ of S on $\ell^p(\mathbb{Z})$. The multiplicative representation of S suggests that the product

$$m_1(\xi)\, m_1(2\,\xi) \cdots m_1(2^{n-1}\xi)$$

might be useful for that purpose. Indeed, the following result can be found in [4]:

Theorem 2.1. *Let m_1 be a finite high-pass filter, and let S be defined by* (1). *Define the sequence $\{g_k^n\}_k$ by*

$$\sum_{k\in\mathbb{Z}} g_k^n\, e^{in\xi} = m_1(\xi)\, m_1(2\,\xi) \cdots m_1(2^{n-1}\xi)\,.$$

Then

$$\sigma_p[S] = \lim_{n\to\infty} \|\{g_k^n\}_k\|_{\ell^p(\mathbb{Z})}^{1/n}\,.$$

We now combine Theorem 2.1 and Lemma 2.2 to get the proof of Theorem 1.1.

Proof of Theorem 1.1. We have, using the same notation as in Lemma 2.2,

$$c_p\, 2^n\, 2^{-n/p}\, \|\{c_k^n\}\|_{\ell^p(\mathbb{Z})} \leq \|w_{2^n-1}\|_p \leq C_p\, 2^n\, 2^{-n/p}\, \|\{c_k^n\}\|_{\ell^p(\mathbb{Z})}\,.$$

The result then follows from Theorem 2.1 by taking the n'th root of the above inequalities and letting $n \longrightarrow \infty$.

2.1. Estimates for $\sigma_p[S]$.

We want to find the asymptotic behavior of the subsequence $\{w_{2^n-1}\}_n$ in $L^p(\mathbb{R})$. By Theorem 1.1 this reduces to calculating the spectral radius $\sigma_p[S]$. Unfortunately, there is no general method available to calculate $\sigma_p[S]$. However, the following lemma shows that we only have worry about $\sigma_\infty[S]$ to estimate $\sigma_p[S]$ for p large. Note that the lemma is a Bernstein type inequality.

Lemma 2.3. *Let $\{w_n\}$ be a wavelet packet system associated with a multiresolution analysis $\{V_j\}$ with scaling function ϕ. Let $n > 0$, $2^{j-1} \leq n < 2^j$. Then there is a finite constant C_p, independent of j, such that for $p \in [1, \infty]$*

$$\|w_n\|_\infty \leq C_p \, 2^{j/p} \, \|w_n\|_p \, .$$

Proof. We have $w_n \in V_j$ so

$$w_n(x) = \sum_{k \in \mathbb{Z}} c_k \, \phi_{j,k} \ ,$$

for some finite sequence $\{c_k\}$. Then, using Lemma 2.1,

$$\begin{aligned} \|w_n\|_\infty &\leq C_\infty \, 2^{j/2} \, \|\{c_k\}\|_{\ell^\infty(\mathbb{Z})} \\ &\leq C_\infty \, 2^{j/2} \, \|\{c_k\}\|_{\ell^p(\mathbb{Z})} \\ &= C_\infty \, 2^{j/p} \, (2^{j/2 - j/p} \|\{c_k\}\|_{\ell^p(\mathbb{Z})}) \\ &\leq C_p \, 2^{j/p} \, \|w_n\|_p \, . \end{aligned}$$

And we have

Corollary 2.1. *Let $\{w_n\}$ be a wavelet packet system associated a multiresolution analysis. Then*

$$\widetilde{\sigma}_p \geq 2^{-1/p} \, \widetilde{\sigma}_\infty \, .$$

2.2. Lower bounds for σ_∞.

We are left with the following problem; how do we obtain a lower bound for $\sigma_\infty[S]$? It turns out that the calculation of $\sigma_\infty[S]$ can be reduced to a finite dimensional problem. We need the following definition and theorem

Definition 2.1. *Let A_0 and A_1 be two $n \times n$-matrices. The joint spectral radius of A_0 and A_1 is given by*

$$\rho(A_0, A_1) = \limsup_{r\to\infty} \max_{\varepsilon\in\{0,1\}^r} \|A_{\varepsilon_1} A_{\varepsilon_2} \cdots A_{\varepsilon_r}\|^{1/r},$$

where $\|\cdot\|$ is any (matrix) norm on $\mathbb{R}^{n\times n}$.

The following general theorem about subdivision operators is proved in [4].

Theorem 2.2. *Let $m_1(\xi) = \sum_{n=-1}^{N} g_n\, e^{in\xi}$ be a high-pass filter associated with a multiresolution analysis. Form the two matrices*

$$A_0 = (g_{-i+2j})_{i,j=-1}^{N-1}, \qquad A_1 = (g_{1-i+2j})_{i,j=-1}^{N-1}.$$

Then

$$\sigma_\infty[S] = \rho(A_0, A_1).$$

It is, in general, difficult to calculate the joint spectral radius of the matrices A_0, A_1 introduced in Theorem 2.2. However, we just want a lower bound for σ_∞ so for our purpose it suffices to notice that $\rho(A_0, A_1) \geq \rho(A_0)$. Hence, the spectral radius of the matrix A_0 gives us a lower bound on σ_∞, *i.e.*, we have reduced the problem to a finite dimensional eigenvalue problem that can be solved (numerically, at least) for any finite filter.

3. Growth in L^p-norm of some familiar wavelet packets.

We now apply this method to some much used filters. We have calculated lower bounds for $\widetilde{\sigma}_\infty$ for some of the standard Daubechies filters, least asymmetric Daubechies filters, and Coiflet filters (see [3] for definitions). The estimates, which were calculated using Matlab and verified using the power method, appear in Tables 1, 2, and 3, respectively. The columns related to "$\widetilde{\sigma}_1$" and "p_0" will be explained in Section 4. It is interesting to note the difference in the estimates obtained for the Daubechies filter and the least asymmetric Daubechies filter of the same length since their transfer functions agree in absolute value. It suggests that the phase of the transfer function does influence the behavior of the associated wavelet packets in $L^p(\mathbb{R})$.

Daub$_N$	Lower bounds for			
	$\tilde{\sigma}_1$	$\tilde{\sigma}_\infty$	$\tilde{\sigma}_1\tilde{\sigma}_\infty$	p_0
2	0.918558	$\frac{\sqrt{11+\sqrt{3}}}{4}$	1.159376	4.687617
3	0.946828	1.182094	1.119240	6.153068
4	0.964076	1.128085	1.087560	8.257957
5	0.975229	1.178557	1.149363	4.979198
6	0.982686	1.120631	1.101229	7.188270
7	0.987780	1.088578	1.075275	9.550474
8	0.991312	1.120338	1.110605	6.607374
9	0.993788	1.081554	1.074836	9.604556
10	0.995538	1.050467	1.045780	15.48460
11	0.996783	1.077456	1.073990	9.710528
12	0.997673	1.053657	1.051206	13.87991
13	0.998313	1.023405	1.021679	32.31807
14	0.998774	1.047230	1.045946	15.42983
15	0.999107	1.034474	1.033551	21.00407
16	0.999349	1.007608	1.006952	100.0505
17	0.999524	1.027401	1.026913	26.10002
18	0.999652	1.021871	1.021515	32.56199
19	0.999745	1.001009	1.000754	919.3268
20	0.999813	1.015251	1.015061	46.36799

Lower bounds for $\widetilde{\sigma}_1$, $\widetilde{\sigma}_\infty$, and p_0 for the first 20 Daubechies filters (with filter length from 4 to 40).

Least asymmetric Daubechies$_N$	Lower bounds for			
	$\tilde{\sigma}_1$	$\tilde{\sigma}_\infty$	$\tilde{\sigma}_1\tilde{\sigma}_\infty$	p_0
4	0.964076	1.192708	1.149862	4.963745
5	0.975229	1.087374	1.060439	11.81179
6	0.982686	1.146192	1.126374	5.825744
7	0.987780	1.133295	1.119446	6.143067
8	0.991312	1.111158	1.101505	7.169679
9	0.993788	1.047619	1.041111	17.20426
10	0.995538	1.084002	1.079118	9.095479

Lower bounds for $\widetilde{\sigma}_1$, $\widetilde{\sigma}_\infty$, and p_0
for the least asymmetric Daubechies filters of length 8 to 20.

Coiflet$_N$	Lower bounds for			
	$\widetilde{\sigma}_1$	$\widetilde{\sigma}_\infty$	$\widetilde{\sigma}_1\widetilde{\sigma}_\infty$	p_0
3	0.939727	1.075437	1.010617	65.63136
6	0.967122	1.197928	1.158542	4.710071
9	0.984923	1.151143	1.133787	5.520289
12	0.992775	1.114805	1.106750	6.833865
15	0.996445	1.086199	1.082338	8.760274

Lower bounds for $\widetilde{\sigma}_1$, $\widetilde{\sigma}_\infty$, and p_0 for the "Coiflet" filters with filter length 6 to 30.

The following result generalizes the results obtained in [1] for the Meyer wavelets.

Corollary 3.1. *For each wavelet packet system associated with one of the filters listed in Tables* 1, 2, *and* 3 *there is a* $p_0 > 2$ *such that for* $p \geq p_0$ *we have a constant* $r_p > 1$ *such that* $\|w_{2^n-1}\|_p \geq C_p\, r_p^n$.

We would like to know if the previous theorem is sharp in the sense that there is a p, $2 < p < p_0$, such that $\sup_n \|w_{2^n-1}\|_p < \infty$. The answer is, in general, negative as the following result shows.

Theorem 3.1. *Let* m_0 *be the Daubechies filter of length* 4 *and let* $\{w_n\}$ *be the associated wavelet packets. Then*

$$\|w_{2^n-1}\|_p \stackrel{n\to\infty}{\longrightarrow} \infty\,,$$

for every $p > 2$.

Proof. If we can prove that $\|w_{2^n-1}\|_1 \stackrel{n\to\infty}{\longrightarrow} 0$ then the result will follow by Hölder's inequality since $\|w_{2^n-1}\|_2 = 1$. It suffices to show that $\sigma_1[S] < 1$. Note that if we can find an N such that $\sum_k |c_k^N| = \alpha < 1$, where

$$m_1(\xi)\cdots m_1(2^{N-1}\xi) = \sum_{k\in\mathbb{Z}} c_k^N\, e^{ik\xi}\,,$$

then $\sigma_1[S] \leq \alpha^{1/N} < 1$. But one can check that

$$\sum_{k\in\mathbb{Z}} |c_k^7| = \frac{9517 + 13043\sqrt{3}}{32768} < 0.98\,.$$

4. Failure of some wavelet packet systems to be a basis for $L^p(\mathbb{R})$.

It is well known that the simplest example of a wavelet packet system, the Walsh system, do form a Schauder basis for $L^p(\mathbb{R})$, $1 < p < \infty$, so one might conjecture that such a result holds for any reasonable wavelet packet system. However, it turns out that the assertion is not true for many nice finite filters such as the Daubechies, least asymmetric Daubechies, and Coiflet filters. They all fail because of the following result:

Lemma 4.1. *If $\{w_n(x-k)\}_{k,n}$ is a Schauder basis for $L^p(\mathbb{R})$, $1 < p < \infty$, then there exists a finite constant C_p such that*

$$\|w_n\|_p \, \|w_n\|_{p'} \le C_p \,, \qquad n = 0, 1, \ldots$$

Proof. It is a well known result (see [7]) that a Schauder basis $\{e_n\}$ in a Banach space $\mathcal{B}$ with associated coefficient functionals $\{f_n\}$ satisfies

$$\sup_n \|e_n\|_{\mathcal{B}} \, \|f_n\|_{\mathcal{B}^*} < +\infty \,.$$

So it suffices to show that $w_n \in L^{p'}(\mathbb{R})$ is the coefficient functional of $w_n \in L^p(\mathbb{R})$. However, this follows easily using that $\{w_n(x-k)\}_{n,k}$ is an orthonormal system in $L^2(\mathbb{R})$ and the fact that bi-orthogonal sequences for Schauder bases are unique [7].

The idea is to find a subsequence of a given wavelet packet system for which (4.1) fails. We have the following useful result.

Lemma 4.2. *If*

$$\widetilde{\sigma}_1[S] \, \widetilde{\sigma}_\infty[S] = \alpha > 1 \,,$$

then the associated wavelet packet system $\{w_n(\cdot - k)\}_{n,k}$ (in any ordering) fails to be a Schauder basis for $L^p(\mathbb{R})$ for $p > p_0$, where $p_0 = 1/\log_2(\alpha)$.

Proof. Since the functions $\{w_n\}$ all have support contained in some fixed finite interval, we have $\|w_n\|_1 \le C_p \|w_n\|_p$. Thus, for $p > 2$,

$$\begin{aligned} \|w_{2^n-1}\|_{p'} \, \|w_{2^n-1}\|_p &\ge C_p \, \|w_{2^n-1}\|_1 \, \|w_{2^n-1}\|_p \\ &\ge \widetilde{C}_p \, 2^{-n/p} \, \|w_{2^n-1}\|_1 \, \|w_{2^n-1}\|_\infty \,, \end{aligned}$$

where we have used Lemma 2.3. Note that

$$2^{-n/p}\,\|w_{2^n-1}\|_1\,\|w_{2^n-1}\|_\infty \overset{n\to\infty}{\longrightarrow} \infty$$

for $p > p_0$, and from Lemma 4.1 it follows that $\{w_n(x-k)\}_{n,k}$ fails to be a Schauder basis for such $L^p(\mathbb{R})$.

Remark. Notice that the negative result of Lemma 4.2 is independent of the ordering of the system $\{w_n\}$. Thus, whenever a wavelet packet system fails to be a Schauder basis due to this result we can be sure that the reason is not that we have chosen the "wrong" ordering of the system. Lemma 4.2 is coarse in the sense that it does not take into account the interaction between different wavelet packets, and all we can say in the case where $\alpha = 1$ is such a wavelet packet system *might* be a Schauder basis for $L^p(\mathbb{R})$. One such example is the Walsh system.

We already have estimates of $\sigma_\infty[S]$. The following result takes care of $\sigma_1[S]$,

Lemma 4.3. *Let $m_1(\xi)$ be a finite high-pass filter with real coefficients associated with a multiresolution analysis. Then*

$$\sigma_1[S] \geq \Big|m_1\Big(\frac{2\pi}{3}\Big)\Big| .$$

Proof. Note that the set $\{-2\pi/3, 2\pi/3\}$ is invariant under the transformation $\xi \longrightarrow 2\,\xi \pmod{2\pi}$. Also,

$$\Big|m_1\Big(\frac{2\pi}{3}\Big)\Big| = \Big|m_1\Big(-\frac{2\pi}{3}\Big)\Big|$$

since m_1 has real coefficients. Thus,

$$\Big|m_1\Big(\frac{2\pi}{3}\Big)\cdots m_1\Big(2^{n-1}\,\frac{2\pi}{3}\Big)\Big| = \Big|m_1\Big(\frac{2\pi}{3}\Big)\Big|^n .$$

Let

$$\sum_{k\in\mathbb{Z}} c_k^n\, e^{ik\xi} = m_1(\xi)\, m_1(2\,\xi)\cdots m_1(2^{n-1}\,\xi) .$$

Then

$$\Big|m_1\Big(\frac{2\pi}{3}\Big)\Big|^n \leq \|m_1(\xi)\cdots m_1(2^{n-1}\,\xi)\|_{L^\infty[0,2\pi)} \leq \sum_{k\in\mathbb{Z}} |c_k^n| ,$$

and the results follows from Lemma 2.1.

We have the following unfortunate result about the basic wavelet packets associated with one of the filters listed in Tables 1, 2, and 3,

Corollary 4.1. *For each wavelet packet system $\{w_n\}$ associated with one of the filters listed in Tables 1, 2, and 3 there exists a (finite) $p_0 > 2$ such that for $p > p_0$, the system $\{w_n(\cdot - k)\}_{n,k}$ (in any ordering) fails to be a Schauder basis of $L^p(\mathbb{R})$.*

Lower bounds for p_0 can be found in Tables 1, 2, and 3.

5. Periodic wavelet packets.

We want to calculate the growth in $L^p[0,1)$-norm of the periodic wavelet packets associated with wavelet packet systems generated using finite filters. The main result is Theorem 1.2 below, which we will prove using the next lemma.

Lemma 5.1. *Let $\{m_k\}_{k\in\mathbb{Z}}$ be a 2^N-periodic sequence with*

$$\alpha = \inf_k |m_k| > 0\,.$$

Then the operator T, defined on $L^2[0,1)$ by

$$T\Big(\sum_{k\in\mathbb{Z}} a_k e^{2\pi ikx}\Big) = \sum_{k\in\mathbb{Z}} m_k\, a_k\, e^{2\pi ikx}\,,$$

extends to an isomorphism on $L^p[0,1)$, $1 < p < \infty$.

Proof. Define the operator $A_N : L^p[0,1) \longrightarrow L^p[0,1)$ by

$$A_N(g)(x) := \frac{1}{2^N}\sum_{\ell=0}^{2^N-1} g\Big(x - \frac{\ell}{2^N}\Big)\,,$$

where g is considered a 1-periodic function. It is clear that A_N is bounded on $L^p[0,1)$. We claim that T has the representation

$$Tf(x) = \sum_{k=0}^{2^N-1} m_k\, A_N(fe^{-2\pi ik\cdot})\, e^{2\pi ikx}\,,$$

for every trigonometric polynomial $f = \sum_{n\in\mathbb{Z}} a_n\, e^{2\pi i n x}$. To see this we notice that

$$A_N(fe^{-2\pi k\,\cdot})(x) = \frac{1}{2^N}\sum_{\ell=0}^{2^N-1}\sum_{n\in\mathbb{Z}} a_n\, e^{2\pi i(n-k)x}\, e^{-2\pi i(n-k)\ell/2^N}$$

$$= \sum_{n\in\mathbb{Z}} a_n\, e^{2\pi i(n-k)x}\Big(\frac{1}{2^N}\sum_{\ell=0}^{2^N-1} e^{-2\pi i(n-k)\ell/2^N}\Big)$$

$$= \sum_{n\in k+2^N\mathbb{Z}} a_n\, e^{2\pi i(n-k)x}\,,$$

from which the claim follows at once. Hence, T is bounded on $L^p[0,1)$, $1<p<\infty$, and applying the same argument to the multiplier sequence $\lambda_k = 1/m_k$ we get that T extends to an isomorphism on $L^p[0,1)$.

We can now prove Theorem 1.2.

Proof of Theorem 1.2. We have, using that $m_1(k\,\pi) = -(k \mod 2)$,

$$\widetilde{w_{2^{n+N}-L}}(x) = \sum_{k\in\mathbb{Z}} \widehat{w}_{2^{N+n}-L}(2\pi k)\, e^{2\pi i k x}$$

$$= \sum_{k\in\mathbb{Z}} m_1(\pi k)\, m_{\varepsilon_2}\Big(\frac{\pi k}{2}\Big)\cdots m_{\varepsilon_J}\Big(\frac{\pi k}{2^N}\Big)\widehat{w}_{2^n-1}\Big(\frac{\pi k}{2^N}\Big) e^{2\pi i k x}$$

$$= -\sum_{\ell\in\mathbb{Z}} m_{\varepsilon_2}\Big(\frac{(2\,\ell+1)\,\pi}{2}\Big)\cdots m_{\varepsilon_J}\Big(\frac{(2\,\ell+1)\,\pi}{2^N}\Big)$$

$$\cdot\, \widehat{w}_{2^n-1}\Big(\frac{(2\ell+1)\,\pi}{2^N}\Big) e^{2\pi i 2\ell x}\, e^{2\pi i x}\,,$$

where $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_J$ are the first J bits of the binary expansion of $2^{n+N} - L$. Note that $\varepsilon_1 = 1$ since L is odd and $\varepsilon_2, \ldots, \varepsilon_J$ do not depend on n, only on L. Thus,

$$\|\widetilde{w_{2^{n+N}-L}}\|_{L^p[0,1)}$$

$$= \Big\|\sum_{\ell\in\mathbb{Z}} m_{\varepsilon_2}\Big(\frac{(2\,\ell+1)\,\pi}{2}\Big)\cdots m_{\varepsilon_J}\Big(\frac{(2\,\ell+1)\,\pi}{2^N}\Big)$$

$$\cdot\, \widehat{w}_{2^n-1}\Big(\frac{(2\,\ell+1)\,\pi}{2^N}\Big) e^{2\pi i 2\ell x} e^{2\pi i x}\Big\|_{L^p[0,1)}$$

$$= \Big\| \sum_{\ell\in\mathbb{Z}} m_{\varepsilon_2}\Big(\frac{(2\,\ell+1)\,\pi}{2}\Big) \cdots m_{\varepsilon_J}\Big(\frac{(2\,\ell+1)\,\pi}{2^N}\Big)$$

$$\cdot\, \widehat{w}_{2^n-1}\Big(\frac{(2\,\ell+1)\,\pi}{2^N}\Big) e^{2\pi i 2\ell x}\Big\|_{L^p[0,1)}$$

$$= \Big\| \sum_{\ell\in\mathbb{Z}} m_{\varepsilon_2}\Big(\frac{(2\,\ell+1)\,\pi}{2}\Big) \cdots m_{\varepsilon_J}\Big(\frac{(2\,\ell+1)\,\pi}{2^N}\Big)$$

$$\cdot\, \widehat{w}_{2^n-1}\Big(\frac{(2\,\ell+1)\,\pi}{2^N}\Big) e^{2\pi i \ell x}\Big\|_{L^p[0,1)} .$$

Note that

$$\Big\{ m_{\varepsilon_2}\Big(\frac{(2\,\ell+1)\,\pi}{2}\Big) \cdots m_{\varepsilon_J}\Big(\frac{(2\,\ell+1)\,\pi}{2^N}\Big)\Big)\Big\}_{\ell\in\mathbb{Z}}$$

is a 2^N-periodic sequence. Moreover, the sequence is non-vanishing (by assumption). Hence, by Lemma 5.1 for $1 < p < \infty$,

$$\|\widetilde{w_{2^{n+N}-L}}\|^p_{L^p[0,1)} \simeq \Big\| \sum_{\ell\in\mathbb{Z}} \widehat{w}_{2^n-1}\Big(\frac{(2\,\ell+1)\,\pi}{2^N}\Big) e^{2\pi i \ell x}\Big\|^p_{L^p[0,1)}$$

$$= 2^{-N} \Big\| \sum_{\ell\in\mathbb{Z}} \widehat{w}_{2^n-1}\Big(\frac{2\,\ell\,\pi}{2^N} + \frac{\pi}{2^N}\Big) e^{2\pi i 2^{-N}\ell x}\Big\|^p_{L^p[0,2^N)} .$$

However,

$$2^{-N} \sum_{\ell\in\mathbb{Z}} \widehat{w}_{2^n-1}\Big(\frac{2\,\ell\,\pi}{2^N} + \frac{\pi}{2^N}\Big) e^{2\pi i 2^{-N}\ell x}$$

is just the Fourier series on $[0, 2^N)$ of the function

$$g(x) = \sum_{k\in\mathbb{Z}} f(x - 2^N k)\,,$$

where $f(x) = w_{2^n-1}(x)\, e^{-i\,2^{-N}\pi x}$. Also, $\|g\|_{L^p[0,2^N)} = \|w_{2^n-1}\|_{L^p(\mathbb{R})}$ since $\operatorname{diam}\operatorname{supp}(w_{2^n-1}) \le 2^N$. So we conclude that for $1 < p < \infty$

$$\|\widetilde{w_{2^{n+N}-L}}\|_{L^p[0,1)} \simeq \|w_{2^n-1}\|_{L^p(\mathbb{R})}\ ,$$

for n sufficiently large.

We now apply Theorem 1.2 to the wavelet packets of Section 3 to get the following result.

Corollary 5.1. *Let $\{w_n\}_n$ be a wavelet packet system generated using one of the filters listed in Tables* 1, 2, *and* 3. *Fix $L \in 2\,\mathbb{Z}^+ + 1$. Then there is a $p_0 > 2$ such that for $p \geq p_0$ there is a constant $r_p > 1$ (depending on L) such that*

$$\|\widetilde{w_{2^n-L}}\|_{L^p[0,1)} \geq C_p\, r_p^n\ ,$$

for n large.

Proof. Follows at once from Corollary 3.1 and Theorem 1.2, since the combined zero-set of the filters m_0 and m_1 is $\pi\,\mathbb{Z}$, and $(2\,\ell+1)/2^j \notin \mathbb{Z}$ for $j \geq 1$.

Corollary 5.1 can also be used to extend Corollary 3.1 to a larger index set. The following result emphasizes that it is the high-pass filter (m_1) that causes the growth in L^p-norm of the wavelet packets.

Corollary 5.2. *Let $\{w_n\}_n$ be a wavelet packet system generated using one of the filters listed in Tables* 1, 2, *and 3. Fix $L \in 2\,\mathbb{Z}^+ + 1$. Then there is a $p_0 > 2$ such that for $p \geq p_0$ there is a constant $r_p > 1$ (depending on L) such that*

$$\|w_{2^n-L}\|_{L^p(\mathbb{R})} \geq C_p\, r_p^n\ ,$$

for n large.

Proof. Follows at once from Corollary 5.1, Minkowski's inequality, and the fact that the wavelet packets all have support contained in some fixed interval.

We proved in the previous section that compactly supported wavelet packets may fail to be Schauder bases for the $L^p(\mathbb{R})$-spaces. We show in this section that a similar (unfortunate) result holds true for periodic wavelet packets. The failure is due to the following analog of Lemma 4.1.

Lemma 5.2. *If $\{\widetilde{w_n}\}_{n=0}^{\infty}$ is a Schauder basis for $L^p[0,1)$, $1 < p < \infty$, then there exists a finite constant C_p such that*

$$\|\widetilde{w_n}\|_{L^p[0,1)}\, \|\widetilde{w_n}\|_{L^{p'}[0,1)} \leq C_p\ , \qquad n = 0, 1, \ldots \tag{4}$$

Proof. Same as for Lemma 4.1.

We now use Theorem 1.2 and Lemma 5.2 to obtain the following result.

Corollary 5.3. *Let $\{w_n\}_n$ be a wavelet packet system generated using one of the filters listed in Tables* 1, 2, *and* 3. *Then there is a $p_0 > 2$ such that for $p \geq p_0$ the periodic wavelet packet system $\{\widetilde{w_n}\}_n$ (in any ordering) fails to be a Schauder basis for $L^p[0,1)$.*

Proof. Choose p_0 such that

$$\sup_n \|w_{2^n-1}\|_{p'} \, \|w_{2^n-1}\|_p = \infty \,,$$

for each $p \geq p_0$. Fix $p \geq p_0$. By Theorem 1.2, there is a constant $c_p \in (0,\infty)$ and an integer N such that

$$\|\widetilde{w_{2^{n+N}-1}}\|_{L^{p'}[0,1)} \, \|\widetilde{w_{2^{n+N}-1}}\|_{L^p[0,1)} \geq c_p \|w_{2^n-1}\|_{p'} \, \|w_{2^n-1}\|_p \,.$$

Hence,

$$\sup_j \|\widetilde{w_{2^j-1}}\|_{L^{p'}[0,1)} \, \|\widetilde{w_{2^j-1}}\|_{L^p[0,1)} = \infty \,.$$

The result then follows from Lemma 5.2.

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