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WEAKLY SYMMETRIC FUNCTIONS AND WEAKLY SYMMETRICALLY CONTINUOUS FUNCTIONS

Abstract

We prove that there exists a nowhere weakly symmetric function $f : \mathbb{R} \to \mathbb{R}$ that is everywhere weakly symmetrically continuous and everywhere weakly continuous. Existence of a nowhere weakly symmetrically continuous function $f : \mathbb{R} \to \mathbb{R}$ that is everywhere weakly symmetric remains open.

1 Introduction

Throughout this paper, $f : \mathbb{R} \to \mathbb{R}$ is a function. It is known that there does not exist an everywhere symmetrically continuous function that is nowhere symmetric. However, there exists an everywhere symmetric function that is nowhere symmetrically continuous. It is easy to see from a theorem of K. Ciesielski and L. Larson that there exists a nowhere weakly symmetric and nowhere weakly symmetrically continuous function. In this paper, we prove that there exists a nowhere weakly symmetric function $f : \mathbb{R} \to \mathbb{R}$ that is everywhere weakly symmetrically continuous and everywhere weakly continuous.

Definitions. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then f is said to be symmetrically continuous (respectively symmetric) at $x \in \mathbb{R}$ if $\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$ (respectively $\lim_{h\to 0}[f(x+h) + f(x-h) - 2f(x)] = 0$); f is weakly symmetrically continuous (respectively weakly symmetric) at x if there exists a sequence $h_n \searrow 0$ such that $\lim_{n\to\infty}[f(x+h_n) - f(x-h_n)] = 0$ (respectively $\lim_{n\to\infty}[\ f(x+h_n) + f(x-h_n) - 2f(x)] = 0$); f is weakly

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continuous at x if there exist sequences $h_n \searrow 0$ and $k_n \searrow 0$ such that $\lim_{n\to\infty} f(x-h_n) = f(x) = \lim_{n\to\infty} f(x+k_n)$.

Notations. The symbols $SC_w(f)$ and $S_w(f)$ denote, respectively, the set of all points where f is weakly symmetrically continuous and the set of all points where f is weakly symmetric. The set of all points where f is weakly continuous is denoted by $C_w(f)$.

2 Theorems and examples

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Theorem 1. *[5] There does not exist an everywhere symmetrically continuous function $f : \mathbb{R} \to \mathbb{R}$ that is nowhere symmetric.*

Corollary 7.3.3 in [2] implies the following.

Example 1. There exists an everywhere symmetric function$f : \mathbb{R} \to \mathbb{Q}$ that is nowhere symmetrically continuous.

We analyze similar statements for weakly symmetrically continuous function and weakly symmetric functions.

Existence of a function $f : \mathbb{R} \to \mathbb{N}$ with $S_w(f) = \emptyset = SC_w(f)$ follows from a theorem in [1].

Theorem 2. *There exists a function $f : \mathbb{R} \to \mathbb{N}$ that is nowhere weakly symmetric and nowhere weakly symmetrically continuous.*

Proof. Theorem 1.1 in [1] shows that there exists a partition $\{P_n : n \in \mathbb{N}\}$ of $\mathbb{R}$ such that for each $x \in \mathbb{R}$, the set $\cup_{n\in\mathbb{N}}\{h > 0 : x-h, x+h \in P_n\}$ is finite. The function $f : \mathbb{R} \to \mathbb{N}$ defined by $f(x) = 2^n$ for $x \in P_n$ is nowhere weakly symmetric and nowhere weakly symmetrically continuous. □

We prove that there exists a function $f : \mathbb{R} \to \{\frac{1}{n} : n \in \mathbb{N}\}$ such that $S_w(f) = \emptyset$ and $SC_w(f) = \mathbb{R} = C_w(f)$.

Theorem 3. *There exists a nowhere weakly symmetric function $f : \mathbb{R} \to \{\frac{1}{n} : n \in \mathbb{N}\}$ that is everywhere weakly symmetrically continuous and everywhere weakly continuous.*

Proof. Let $B = \{b_\xi : \xi < c\}$ be a linear basis of $\mathbb{R}$ over $\mathbb{Q}$ and let $0 \neq x \in \mathbb{R}$. Then x can be written uniquely as $x = \sum_{i=1}^{k} q_i b_{\alpha_i}$, where $q_i \in \mathbb{Q}\setminus\{0\}$, $b_{\alpha_i} \in B$ and $(\alpha_i)_{1\leq i\leq k}$ is an increasing sequence. The length $L(x)$ of x is defined to be k. Define a function $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = \sum_{i=1}^{k} q_i b_i$ and $g(0) = 0$. An equivalence relation $\sim$ on $\mathbb{R}$ is defined as follows. For $x, y \in \mathbb{R}$, $x \sim y$ if and only if $g(x) = g(y)$. Since the set $\{g(x) : x \in \mathbb{R}\}$ is countably infinite and the equivalence classes are either disjoint or the same, the set of distinct equivalence classes is countably infinite. Let $E = \{[e_{u,\,v}] : u,\, v \in \mathbb{N}\} \cup \{[0]\}$ be the set of all distinct equivalence classes, where $L(e_{u,\,v}) = u\ \forall v \in \mathbb{N}$. For $x, h \in \mathbb{R} \setminus \{0\}$, let

$$x = \sum_{i=1}^{k} q_i b_{\alpha_i},\ x + h = \sum_{i=1}^{m} x_i b_{\beta_i} \text{ and } x - h = \sum_{i=1}^{n} y_i b_{\gamma_i},$$

where $q_i,\, x_i,\, y_i \in \mathbb{Q} \setminus \{0\}$ and $(\alpha_i)_{1\le i\le k}$, $(\beta_i)_{1\le i\le m}$, $(\gamma_i)_{1\le i\le n}$ are increasing sequences. Define a function $f : \mathbb{R} \to \mathbb{Q}$ by $f(x) = \frac{1}{5^u 11^v}$ for $x \in [e_{u,\,v}]$ and $f(0) = 1$.

Claim 1. $f(x + h) + f(x - h) \neq 2f(x)\ \forall h \in \mathbb{R} \setminus \{0\}$.

Proof of the claim. Assume, to the contrary, that $f(x+h)+f(x-h) = 2f(x)$. It can be shown that, for $s, t, u, v, a, b \in \mathbb{N}$,

$$\frac{1}{5^s 11^t} + \frac{1}{5^u 11^v} = \frac{2}{5^a 11^b}$$

if and only if $s = u = a$ and $t = v = b$. Consequently, $[x + h] = [x] = [x - h]$, which implies that $g(x+h) = g(x) = g(x-h)$ and $k = m = n$. If $\beta_j \notin \{\alpha_i : 1 \le i \le k\}$ for some $1 \le j \le k$, then, since $2x = (x+h)+(x-h)$ and $x_j \neq 0$, we have $0 = x_j + y_j$, which contradicts that $g(x+h) = g(x-h)$. Therefore $(\alpha_i)_{1\le i\le k} = (\beta_i)_{1\le i\le k}$. Similarly, $(\alpha_i)_{1\le i\le k} = (\gamma_i)_{1\le i\le k}$. So $x + h = \sum_{i=1}^{k} x_i b_{\alpha_i}$, $x - h = \sum_{i=1}^{k} y_i b_{\alpha_i}$ and $\sum_{i=1}^{k} x_i b_i = g(x + h) = g(x - h) = \sum_{i=1}^{k} y_i b_i$. This implies that $x_i = y_i\ \forall 1 \le i \le k$ and $x + h = x - h$, a contradiction.

Thus, $f(x + h) + f(x - h) \neq 2f(x)\ \forall h \in \mathbb{R} \setminus \{0\}$. □

We show that for any fixed $x \in \mathbb{R}$, there exists a positive number K such that $|f(x + h) + f(x - h) - 2f(x)| \ge K\ \forall h \in \mathbb{R} \setminus \{0\}$, and hence, $S_w(f) = \emptyset$. Recall that $f(0) = 1$. Clearly, the statement is true for $x = 0$. Let a and b be fixed positive integers,

$$f(x) = \frac{1}{5^a 11^b},\ f(x + h) = \frac{1}{5^s 11^t} \text{ and } f(x - h) = \frac{1}{5^u 11^v},$$

where $h \neq 0$. If s or $t > a+2b$, then $\frac{1}{5^s 11^t} < \frac{1}{5^a 11^b}$ and $\left|\frac{1}{5^s 11^t} - \frac{2}{5^a 11^b}\right| > \frac{1}{5^a 11^b}$. Let

$$L = \inf\left\{\left|\frac{1}{5^s 11^t} - \frac{2}{5^a 11^b}\right| : \forall s,\, t \in \mathbb{N}\right\}.$$

Since $\left\{\frac{1}{5^s 11^t} - \frac{2}{5^a 11^b} : s, t \le a + 2b\right\}$ is a finite set of nonzero numbers, $L > 0$. It is easy to see that there exists a positive integer $M > a$ such that

$\left|\frac{1}{5^s 11^t}+\frac{1}{5^u 11^v}-\frac{2}{5^a 11^b}\right|>\frac{L}{2}$ $\forall s,t,u,v\in\mathbb{N}$ and s,t,u or $v>M$. Notice that, by Claim 1, it is not possible that $s=u=a$ and $t=v=b$. Let K be the minimum of $\left\{\frac{L}{2},\left|\frac{1}{5^s 11^t}+\frac{1}{5^u 11^v}-\frac{2}{5^a 11^b}\right| : s,t,u \text{ and } v\le M\right\}$. Then $K>0$ and $|f(x+h)+f(x-h)-2f(x)|\ge K$ $\forall h\in\mathbb{R}\setminus\{0\}$. Consequently, $f(x+h_n)+f(x-h_n)-2f(x)\nrightarrow 0$ as $h_n\to 0$. Thus, $S_w(f)=\emptyset$.

To prove that $SC_w(f)=\mathbb{R}$ and $C_w(f)=\mathbb{R}$, choose a Hamel basis B such that $|B\cap I|=|\mathbb{R}|$ for every nonempty open interval I. Recall that $x=\sum_{i=1}^{k} q_i b_{\alpha_i}$. Let $(b_{\mu_n})_{1\le n<\omega}$ be a sequence in $B\cap(0,1)$ such that $(\mu_n)_{1\le n<\omega}$ is an increasing sequence and $\alpha_k<\mu_1<c$. Let $h_n=\frac{1}{2^n}\sum_{i=1}^{n} b_{\mu_i}$. Clearly, $L(x\pm h_n)\ge n$ and $0\le f(x\pm h_n)\le\frac{1}{5^n}$ and $h_n\to 0$. So $f(x+h_n)-f(x-h_n)\to 0$ and $SC_w(f)=\mathbb{R}$. There exists an increasing sequence $(q_k b_{\zeta_n})_{1\le n<\omega}$ and a decreasing sequence $(q_k b_{\lambda_n})_{1\le n<\omega}$ such that $\zeta_n>\alpha_k$ $\forall n$, $\lambda_n>\alpha_k$ $\forall n$ and both $(q_k b_{\zeta_n})_{1\le n<\omega}$ and $(q_k b_{\lambda_n})_{1\le n<\omega}$ converge to $q_k b_{\alpha_k}$. Let $p_n=q_k b_{\alpha_k}-q_k b_{\zeta_n}$ and $k_n=-q_k b_{\alpha_k}+q_k b_{\lambda_n}$. Then $p_n\searrow 0$ and $k_n\searrow 0$ and $f(x-p_n)=f(x)=f(x+k_n)$ $\forall n$. Thus, $C_w(f)=\mathbb{R}$. □

Problem 1. Does there exist a nowhere weakly symmetrically continuous function $f:\mathbb{R}\to\mathbb{R}$ that is everywhere weakly symmetric?

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