RESEARCH

Real Analysis Exchange
Vol. 40(2), 2014/2015, pp. 371–382

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ON BAIRE CLASSIFICATION OF STRONGLY SEPARATELY CONTINUOUS FUNCTIONS

Abstract

We investigate strongly separately continuous functions on a product of topological spaces and prove that if X is a countable product of real lines, then there exists a strongly separately continuous function $f : X \to \mathbb{R}$ which is not Baire measurable. We show that if X is a product of normed spaces X_n, $a \in X$ and $\sigma(a) = \{x \in X : |\{n \in \mathbb{N} : x_n \neq a_n\}| < \aleph_0\}$ is a subspace of X equipped with the Tychonoff topology, then for any open set $G \subseteq \sigma(a)$, there is a strongly separately continuous function $f : \sigma(a) \to \mathbb{R}$ such that the discontinuity point set of f is equal to G.

1 Introduction

In 1998 Omar Dzagnidze [2] introduced a notion of a strongly separately continuous function $f : \mathbb{R}^n \to \mathbb{R}$. Namely, he calls a function f *strongly separately continuous* at a point $x^0 = (x_1^0, \dots, x_n^0) \in \mathbb{R}^n$ if the equality

$$\lim_{x \to x^0} |f(x_1, \dots, x_k, \dots, x_n) - f(x_1, \dots, x_k^0, \dots, x_n)| = 0$$

holds for every $k = 1, \dots, n$. Dzagnidze proved that a function $f : \mathbb{R}^n \to \mathbb{R}$ is strongly separately continuous at x^0 if and only if f is continuous at x^0.

Extending these investigations, J. Činčura, T. Šalát and T. Visnyai [1] consider strongly separately continuous functions defined on the space ℓ_2 of

Mathematical Reviews subject classification: Primary: 54C08, 54C30; Secondary: 26A21
Key words: strongly separately continuous function, Baire classification
Received by the editors October 11, 2014
Communicated by: Miroslav Zeleny

sequences $x = (x_n)_{n=1}^{\infty}$ of real numbers such that $\sum_{n=1}^{\infty} x_n^2 < +\infty$, endowed with the standard metric $d(x, y) = (\sum_{n=1}^{\infty}(x_n - y_n)^2)^{1/2}$. In particular, the authors gave an example of a strongly separately continuous everywhere discontinuous function $f : \ell_2 \to \mathbb{R}$.

Recently, T. Visnyai in [6] continued to study properties of strongly separately continuous functions on ℓ_2 and constructed a strongly separately continuous function $f : \ell_2 \to \mathbb{R}$ which belongs to the third Baire class and is not quasi-continuous at every point. Moreover, T. Visnyai gave a sufficient condition for a strongly separately continuous function to be continuous on ℓ_2.

In this paper, we study strongly separately continuous functions defined on a subspaces of a product $\prod_{t\in T} X_t$ of topological spaces X_t equipped with the Tychonoff topology of pointwise convergence. We show that if X is a product of a sequence $(X_n)_{n=1}^{\infty}$ of topological spaces X_n, $a \in X$ and $\sigma(a) = \{x \in X : |\{n \in \mathbb{N} : x_n \neq a_n\}| < \aleph_0\}$ is a subspace of X equipped with the Tychonoff topology, then every strongly separately continuous function $f : \sigma(a) \to \mathbb{R}$ belongs to the first stable Baire class. Moreover, we prove that if X is a countable product of real lines, then there exists a strongly separately continuous function $f : X \to \mathbb{R}$ which is not Baire measurable. In the last section we show that if X is a product of normed spaces, then for any open set $G \subseteq \sigma(a)$, there is a strongly separately continuous function $f : \sigma(a) \to \mathbb{R}$ such that the discontinuity point set of f is equal to G.

2 Strongly separately continuous functions and $\mathcal{S}$-open sets

Let $X = \prod_{t\in T} X_t$ be a product of a family of sets X_t with $|X_t| > 1$ for all $t \in T$. If $S \subseteq S_1 \subseteq T$, $a = (a_t)_{t\in T} \in X$, $x = (x_t)_{t\in S_1} \in \prod_{t\in S_1} X_t$, then we denote by a_S^x a point $(y_t)_{t\in T}$, where

$$y_t = \begin{cases} x_t, & t \in S, \\ a_t, & t \in T \setminus S. \end{cases}$$

In the case $S = \{s\}$, we shall write a_s^x instead of $a_{\{s\}}^x$.

If $n \in \mathbb{N}$, then we set

$$\sigma_n(x) = \{y = (y_t)_{t\in T} \in X : |\{t \in T : y_t \neq x_t\}| \leq n\}$$

and

$$\sigma(x) = \bigcup_{n=1}^{\infty} \sigma_n(x).$$

Each of the sets of the form $\sigma(x)$ for an $x \in X$ is called *a σ-product of the space X*.

We denote by τ the Tychonoff topology on a product $X = \prod_{t\in T} X_t$ of topological spaces X_t. If $X_0 \subseteq X$, then the symbol (X_0, τ) means the subspace X_0 equipped with the Tychonoff topology induced from (X, τ).

If $X_t = Y$ for all $t \in T$, then the product $\prod_{t\in T} X_t$ we also denote by $Y^{\mathfrak{m}}$, where $\mathfrak{m} = |T|$.

A set $E \subseteq \prod_{t\in T} X_t$ is called *$\mathcal{S}$-open* if

$$\sigma_1(x) \subseteq E$$

for all $x \in E$.

Let $\mathcal{S}(X)$ denote the collection of all $\mathcal{S}$-open subsets of X. We notice that $\mathcal{S}(X)$ is a topology on X. We will denote by $(X, \mathcal{S})$ the product $X = \prod_{t\in T} X_t$ equipped with the topology $\mathcal{S}(X)$.

The next properties follow easily from the definitions.

Proposition 2.1. *Let $X = \prod_{t\in T} X_t$, $|X_t| > 1$ for all $t \in T$ and $E \subseteq X$. Then*

1. *$E \in \mathcal{S}(X)$ if and only if $X \setminus E \in \mathcal{S}(X)$;*
2. *$E \in \mathcal{S}(X)$ if and only if $E = \bigcup_{x\in E} \sigma(x)$;*
3. *if $x \in X$, then $\sigma(x)$ is the smallest $\mathcal{S}$-open set which contains x;*
4. *if $E \in \mathcal{S}(X)$, then E is dense in (X, τ);*
5. *there exists a non-trivial $\mathcal{S}$-open subset of X if and only if $|T| \geq \aleph_0$.*

It follows from Proposition 2.1 that σ-products of two distinct points of $\prod_{t\in T} X_t$ either coincide, or do not intersect. Consequently, the family of all σ-products of an arbitrary $\mathcal{S}$-open set $E \subseteq \prod_{t\in T} X_t$ generates a partition of E on mutually disjoint $\mathcal{S}$-open sets, which we will call *$\mathcal{S}$-components of E*.

Definition 2.2. Let $(X_t : t \in T)$ be a family of topological spaces, let Y be a topological space, and let $E \subseteq \prod_{t\in T} X_t$ be an $\mathcal{S}$-open set. A mapping $f : E \to Y$ is said to be *separately continuous at a point $a = (a_t)_{t\in T} \in E$ with respect to the t-th variable* provided that the mapping $g : X_t \to Y$ defined by the rule $g(x) = f(a_t^x)$ for all $x \in X_t$ is continuous at the point $a_t \in X_t$.

Definition 2.3. Let $E \subseteq \prod_{t\in T} X_t$ be an $\mathcal{S}$-open set, let $\mathcal{T}$ be a topology on E, and let (Y, d) be a metric space. A mapping $f : (E, \mathcal{T}) \to Y$ is called *strongly separately continuous at a point $a \in E$ with respect to the t-th variable* if

$$\lim_{x\to a} d(f(x), f(x_t^a)) = 0.$$

Definition 2.4. A mapping $f : E \to Y$ is

- *(strongly) separately continuous at a point* $a \in E$ if f is (strongly) separately continuous at a with respect to each variable $t \in T$;
- *(strongly) separately continuous on the set* E if f is (strongly) separately continuous at every point $a \in E$ with respect to each variable $t \in T$.

Theorem 2.5. *Let $E \subseteq \prod_{t\in T} X_t$ be an $\mathcal{S}$-open set, and let (Y, d) be a metric space. A mapping $f : (E, \mathcal{S}) \to Y$ is continuous if and only if $f : (E, \mathcal{T}) \to Y$ is strongly separately continuous for an arbitrary topology $\mathcal{T}$ on E.*

Proof. *Necessity.* Fix a topology $\mathcal{T}$ on E and consider the partition $(\sigma(x_i) : i \in I)$ of the set E on $\mathcal{S}$-components $\sigma(x_i)$. We notice that $f|_{\sigma(x_i)} = y_i$, where $y_i \in Y$ for all $i \in I$, since f is continuous on $(E, \mathcal{S})$. Let $a = (a_t)_{t\in T} \in E$ and $t \in T$. If $x \in E$, then $x \in \sigma(x_i)$ for some $i \in I$. Moreover, $x_t^a \in \sigma(x_i)$. Then $f(x) = f(x_t^a) = y_i$. Hence, $d(f(x), f(x_t^a)) = 0$ for all $x \in E$. Hence, f is strongly separately continuous on $(E, \mathcal{T})$.

Sufficiency. Put $\mathcal{T} = \mathcal{S}$. Fix $x_0 \in E$ and show that f is continuous at x_0 on $(E, \mathcal{S})$. Let $x_0 \in \sigma(x_i)$ for some $i \in I$. Let us observe that $x \to x_0$ in $(E, \mathcal{S})$ if and only if $x \in \sigma(x_0)$. Since f is strongly separately continuous at x_0 and $\sigma(x_0) = \sigma(x_i)$, we have $d(f(x), f(x_t^{x_0})) = 0$ for all $x \in \sigma(x_i)$ and $t \in T$. Consequently, $f(x) = f(x_0)$ for all $x \in \sigma(x_i)$. Since the set $\sigma(x_i)$ is open in $(E, \mathcal{S})$, f is continuous at x_0. □

Let $(\sigma_i : i \in I)$ be a partition of $X = \prod_{t\in T} X_t$ on $\mathcal{S}$-components, and let $f : \prod_{t\in T} X_t \to \mathbb{R}$ be a function such that $f|_{\sigma_i} \equiv \text{const}$ for all $i \in I$. Theorem 2.5 implies that f is strongly separately continuous on (X, τ), since for every $i \in I$, the set σ_i is clopen in $(X, \mathcal{S})$. The next example shows that it is not so in the case $f|_{\sigma_i}$ is a continuous function on (σ_i, τ) for every $i \in I$.

Example 2.6. *Let $X = \mathbb{R}^{\aleph_0}$, let $(\sigma_i : i \in I)$ be a partition of X on $\mathcal{S}$-components, and let $\sigma(m) = \{x = (x_n) \in X : |\{n \in \mathbb{N} : x_n \neq m\}| < \aleph_0\}$ for all $m \in \mathbb{N}$. Consider a function $f : X \to \mathbb{R}$,*

$$f(x) = \begin{cases} m \cdot (x_1 + \cdots + x_m), & \text{if } x \in \sigma(m), \\ 0, & \text{otherwise.} \end{cases}$$

Then $f|_{\sigma_i} : (\sigma_i, \tau) \to \mathbb{R}$ is continuous for every $i \in I$, but f is not strongly separately continuous at $x = 0$.

Proof. For every $m \in \mathbb{N}$ we put

$$u^m = \Big(\underbrace{\frac{1}{m}, \dots, \frac{1}{m}}_{m}, m, m, \dots\Big).$$

Then $u^m \in \sigma(m)$ and $u^m \to 0$ in (X,τ). Note that $f(u^m) = m$ and $f((u^m)_1^x) = m-1$. Therefore, $|f(u^m)-f((u^m)_1^x)| = 1$ for all $m \in \mathbb{N}$. Consequently, f is not strongly separately continuous at $x = 0$ with respect to the first variable. □

Theorem 2.7. *Let $E \subseteq \prod_{t\in T} X_t$ be an $\mathcal{S}$-open subset of a product of topological spaces X_t, let (Y,d) be a metric space, and let $f : (E,\tau) \to Y$ be a strongly separately continuous mapping at the point $a = (a_t)_{t\in T} \in E$. Then f is continuous at the point a if and only if*

$$\begin{gathered}\forall\varepsilon > 0\ \exists T_0 \subseteq T,\ |T_0| < \aleph_0 \\ \exists U\text{– a neighborhood of } a \text{ in } (E,\mathcal{T})\,| \\ d(f(a), f(x^a_{T_0})) < \varepsilon \quad \forall\, x \in U.\end{gathered} \tag{1}$$

PROOF. *Necessity.* Suppose f is continuous at the point a and $\varepsilon > 0$. Take a basic neighborhood U of a such that $d(f(x), f(a)) < \varepsilon$ for all $x \in U$, and put $T_0 = \emptyset$. Then $x^a_{T_0} = x$, which implies condition (1).

Sufficiency. Fix $\varepsilon > 0$. Using the condition of the theorem, we take a finite set $T_0 \subseteq T$ and a neighborhood U of a in (E,τ) such that

$$d(f(a), f(x^a_{T_0})) < \frac{\varepsilon}{2}$$

for every $x \in U$. If $T_0 = \emptyset$, then $d(f(x), f(a)) < \varepsilon$ for all $x \in U$, which implies the continuity of f at a. Now assume $T_0 = \{t_1,\dots,t_n\}$. Since f is strongly separately continuous at a, for every $k = 1,\dots,n$, we choose a neighborhood V_k of the point a such that

$$d(f(x), f(x^a_{t_k})) < \frac{\varepsilon}{2n}$$

for all $x \in V_k$. We take a basic neighborhood W of a such that

$$W \subseteq U \cap \big(\bigcap_{k=1}^{n} V_k\big).$$

Observe that $x^a_{\{t_1,\dots,t_k\}} \in W$ for every $k = 1,\dots,n$ and for every $x \in W$. Then for all $x \in W$, we have

$$\begin{aligned} d(f(x), f(a)) &\le d(f(x), f(x^a_{T_0})) + d(f(x^a_{T_0}), f(a)) \\ &< d(f(x), f(x^a_{\{t_1\}})) + d(f(x^a_{\{t_1\}}), f(x^a_{\{t_1,t_2\}})) \\ &\quad + \cdots + d(f(x^a_{\{t_1,\dots,t_{n-1}\}}), f(x^a_{\{t_1,\dots,t_n\}})) + \frac{\varepsilon}{2} \\ &< \frac{\varepsilon}{2n}\cdot n + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, f is continuous at the point a. □

The following corollary generalizes the result of Dzagnidze [2, Theorem 2.1].

Corollary 2.8. *Let E be an $\mathcal{S}$-open subset of a product $\prod_{t\in T} X_t$ of topological spaces X_t, $|T| < \aleph_0$, and let (Y,d) be a metric space. Then any strongly separately continuous mapping $f:(E,\tau)\to Y$ is continuous.*

Proof. Fix an arbitrary point $a\in E$ and a strongly separately continuous mapping $f:(E,\tau)\to Y$. For $\varepsilon>0$, we put $T_0=T$ and $U=E$. Then for all $x\in U$, we have $x^a_{T_0}=a$, and consequently

$$d(f(a),f(x^a_{T_0}))=0<\varepsilon.$$

Hence, f is continuous at the point a by Theorem 2.7. □

The proposition below shows that Corollary 2.8 is not valid for a product of infinitely many topological spaces.

Proposition 2.9. *Let $X=\prod_{t\in T} X_t$ be a product of topological spaces X_t, where $|X_t|>1$ for every $t\in T$, let $|T|>\aleph_0$, and let (Y,d) be a metric space with $|Y|>1$. Then there exists a strongly separately continuous everywhere discontinuous mapping $f:(X,\tau)\to Y$.*

Proof. Fix $x_0\in X$ and $y_1,y_2\in Y$, $y_1\neq y_2$. According to Proposition 2.1(5), $\sigma(x_0)\neq\emptyset\neq X\setminus\sigma(x_0)$. Set $f(x)=y_1$ if $x\in\sigma(x_0)$ and $f(x)=y_2$ if $x\in X\setminus\sigma(x_0)$. We prove that f is everywhere discontinuous on X. Indeed, let $a\in X$ and $f(a)=y_1$. Take an open neighborhood V of y_1 such that $y_2\notin V$. If U is an arbitrary neighborhood of a in (X,τ), then there is $x\in U\setminus\sigma(x_0)$ by Proposition 2.1 (4). Then $f(x)=y_2\notin V$. Therefore, f is discontinuous at a. Similarly one can show that f is discontinuous at a in the case $f(a)=y_2$.

Since the set $\sigma(x_0)$ is clopen in $(X,\mathcal{S})$, the mapping $f:(X,\mathcal{S})\to Y$ is continuous. It remains to apply Theorem 2.5. □

3 Baire measurable strongly separately continuous functions

Let $B_0(X,Y)$ be the collection of all continuous mappings $f:X\to Y$. Assume that the classes $B_\xi(X,Y)$ are already defined for all $0\le\xi<\alpha$, where $\alpha<\omega_1$. Then $f:X\to Y$ is said to be *of the α-th Baire class*, $f\in B_\alpha(X,Y)$, if f is a pointwise limit of a sequence of mappings $f_n\in B_{\xi_n}(X,Y)$, where $\xi_n<\alpha$. Denote

$$\mathcal{B}(X,Y)=\bigcup_{0\le\alpha<\omega_1} B_\alpha(X,Y).$$

We say that $f : X \to Y$ *is a Baire measurable mapping* if $f \in \mathcal{B}(X, Y)$.

Let $0 \le \alpha < \omega_1$, let X be a metrizable space, let Y be a topological space and let Z be a locally convex space. W. Rudin [5] proved that every mapping $f : X \times Y \to Z$ which is continuous with respect to the first variable and is of the α-th Baire class with respect to the second one belongs to the $(\alpha + 1)$-th Baire class on $X \times Y$. The following proposition is an easy corollary of the Rudin Theorem.

Proposition 3.1. *Let $n \in \mathbb{N}$, let $X_1, \dots, X_n$ be metrizable spaces, and let Z be a locally convex space. Then every separately continuous mapping $f : \prod_{i=1}^{n} X_i \to Z$ belongs to the $(n-1)$-th Baire class.*

Proof. The assertion of the proposition is evident if $n = 1$ and is exactly the Rudin Theorem if $n = 2$. Now assume that the proposition is true for all $2 \le k < n$ and prove it for $k = n$. Denote $X = \prod_{i=1}^{n-1} X_i$. Then $f : X \times X_n \to Z$ belongs to the $(n-2)$-th Baire class with respect to the first variable by the inductive assumption, and f is continuous with respect to the second variable. Applying the Rudin Theorem we have $f \in B_{n-1}(X \times X_n, Z)$. □

The next result shows that the corollary of Rudin's Theorem is not valid for infinite products.

Proposition 3.2. *There exists a strongly separately continuous function $f : (\mathbb{R}^{\aleph_0}, \tau) \to \mathbb{R}$ which is not Baire measurable.*

Proof. Consider a partition $(\sigma_i : i \in I)$ of $\mathbb{R}^{\aleph_0}$ on $\mathcal{S}$-components σ_i. It is not hard to verify that $|I| = \mathfrak{c}$. Denote by $\mathcal{F}$ the collection of all functions $f : \mathbb{R}^{\aleph_0} \to \mathbb{R}$ such that $f|_{\sigma_i} = \text{const}$ for all $i \in I$. Then $|\mathcal{F}| = 2^{|I|} = 2^{\mathfrak{c}}$. Moreover, since $(\mathbb{R}^{\aleph_0}, \tau)$ is separable, $|B_0(\mathbb{R}^{\aleph_0}, \mathbb{R})| = \mathfrak{c}$, and consequently, $|\mathcal{B}(X, Y)| = \mathfrak{c}$. Hence, there exists $f \in \mathcal{F} \setminus \mathcal{B}(\mathbb{R}^{\aleph_0}, \mathbb{R})$. Since for every $i \in I$ the set σ_i is clopen in $(\mathbb{R}^{\aleph_0}, \mathcal{S})$, f is continuous on $(\mathbb{R}^{\aleph_0}, \mathcal{S})$. Then f is strongly separately continuous on $(\mathbb{R}^{\aleph_0}, \tau)$ according to Proposition 2.5. □

Let $1 \le \alpha < \omega_1$. A mapping $f : X \to Y$ belongs to the *α-th stable Baire class*, $f \in B_\alpha^d(X, Y)$, if there exists a sequence of mappings $f_n \in B_{\alpha_n}(X, Y)$, where $\alpha_n < \alpha$, such that for every $x \in X$, there exists $N \in \mathbb{N}$ such that $f_n(x) = f(x)$ for all $n \ge N$.

Theorem 3.3. *Let $(X_n)_{n=1}^{\infty}$ be a sequence of topological spaces, $a \in \prod_{n=1}^{\infty} X_n$, $E = \sigma(a)$, and let $f : (E, \tau) \to \mathbb{R}$ be a function.*

1. *If f is strongly separately continuous, then $f \in B_1^d(E, \mathbb{R})$.*
2. *If f is separately continuous and X_n is metrizable for every $n \in \mathbb{N}$, then $f \in B_{\omega_0}^d(E, \mathbb{R})$.*

Proof. For every $n \in \mathbb{N}$, we put

$$E_n = \prod_{i=1}^{n} X_i \times \prod_{i=n+1}^{\infty} \{a_i\},$$

$g_n = f|_{E_n}$, and

$$f_n(x) = g_n(x_1, \dots, x_n, a_{n+1}, \dots)$$

for all $x \in E$. Clearly, $E = \bigcup_{n=1}^{\infty} E_n$, $E_n \subseteq E_{n+1}$, and every space (E_n, τ) is homeomorphic to $(\prod_{i=1}^{n} X_i, \tau)$.

If f is strongly separately continuous, then by Theorem 2.8, every g_n is continuous on E_n. Then $f_n : (E, \tau) \to \mathbb{R}$ is a continuous extension of g_n.

In the second case, $g_n \in B_{n-1}(E_n, Z)$ by Proposition 3.1 for every n. It is not hard to verify that $f_n \in B_{n-1}(E, Z)$.

Now if $x \in E$, then there is $N \in \mathbb{N}$ such that $x \in E_n$ for all $n \geq N$. Therefore, $f_n(x) = f(x)$ for all $n \geq N$. Hence, $f \in B_1^d(E, \mathbb{R})$ in the first case and $f \in B_{\omega_0}^d(E, \mathbb{R})$ in the second one. □

Proposition 3.4. *Let $a = (0, 0, \dots) \in \mathbb{R}^{\aleph_0}$, $E = \sigma(a) \subseteq \mathbb{R}^{\aleph_0}$ and $Y = [0, 1]$. Then there exists a separately continuous function $f : E \to Y$ such that $f \notin \bigcup_{n=1}^{\infty} B_n((E, \tau), Y)$.*

Proof. For every $n \in \mathbb{N}$, we take a function $h_n \in B_{n+1}(\mathbb{R}, Y) \setminus B_n(\mathbb{R}, Y)$. According to the Lebesgue Theorem [4], for every $n \in \mathbb{N}$, there exists a separately continuous function $g_n : \mathbb{R}^{n+2} \to Y$ such that

$$g_n(\underbrace{x, x, \dots, x}_{n+2}) = h_n(x)$$

for each $x \in \mathbb{R}$. Evidently, g_n is not of the n-th Baire class on $\mathbb{R}^{n+2}$.

Let $\varphi : \mathbb{R} \to Y$ be any continuous function such that $\{0\} = \varphi^{-1}(0)$. For $n \in \mathbb{N}$, we consider a function $f_n : E \to Y$,

$$f_n(x_1, \dots, x_n, \dots) = \varphi(x_{n+2}) \cdot g_n(x_1, \dots, x_{n+2}).$$

Then the function $f_n : E \to Y$ is separately continuous as the product of two separately continuous functions. Moreover,

$$f_n|_{E_{n+2}} \notin B_n(E_{n+2}, Y)$$

for every $n \in \mathbb{N}$, where

$$E_n = \mathbb{R}^n \times \{0\} \times \{0\} \times \dots .$$

For every $x \in E$, we put

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} f_n(x).$$

Observe that $f : E \to \mathbb{R}$ is separately continuous as the sum of the uniformly convergent series of separately continuous functions.

It remains to show that $f \notin \bigcup_{n=1}^{\infty} B_n(E, Y)$. Assume to the contrary that $f \in B_n(E, Y)$ for some $n \in \mathbb{N}$. Then $f|_{E_{n+2}} \in B_n(E_{n+2}, Y)$. Notice that

$$f|_{E_{n+2}} = \sum_{k=1}^{n} \frac{1}{2^k} f_k|_{E_{n+2}},$$

since $f_k|_{E_{n+2}} = 0$ for all $k \geq n+1$. Denote

$$g = \sum_{k=1}^{n-1} \frac{1}{2^k} f_k|_{E_{n+2}}.$$

Then we have $g \in B_n(E_{n+2}, Y)$, since

$$f_k|_{E_{n+2}} \in B_{k+1}(E_{n+2}, Y) \subseteq B_n(E_{n+2}, Y)$$

for every $k = 1, \dots, n-1$. Therefore,

$$f_n|_{E_{n+2}} = (f|_{E_{n+2}} - g) \in B_n(E_{n+2}, Y),$$

which implies a contradiction. □

4 Discontinuities of strongly separately continuous mappings

For a mapping f between spaces X and Y, we denote the set of all points of continuity of f by $C(f)$. Let $D(f) = X \setminus C(f)$.

Theorem 4.1. *Let $X = \prod_{n=1}^{\infty} X_n$ be a product of normed spaces $(X_n, \|\cdot\|_n)$, and let $a \in X$. Then for any open set $G \subseteq (\sigma(a), \tau)$, there exists a strongly separately continuous function $f : (\sigma(a), \tau) \to \mathbb{R}$ such that $D(f) = G$.*

Proof. Without loss of generality we may assume that $a = (0, 0, \dots)$. For every $n \in \mathbb{N}$, we consider a norm $\|\cdot\|_n$ on the space X_n which generates its topological structure. Let d be a bounded metric on X which generates the

Tychonoff topology τ. Denote $X_0 = (\sigma(a), \tau)$ and $F = X_0 \setminus G$. For every $x = (x_n)_{n\in\mathbb{N}} \in X_0$, put

$$\varphi(x) = \begin{cases} d(x,F), & \text{if } F \neq \emptyset \\ 1, & \text{if } F = \emptyset \end{cases},$$

$$g(x) = \exp(-\sum_{n=1}^{\infty} \|x_n\|_n),$$

$$f(x) = \varphi(x) \cdot g(x).$$

We prove that $F \subseteq C(f)$. Indeed, if $x^0 \in F$ and $(x^m)_{m=1}^{\infty}$ is a convergent to x^0 sequence in X_0, then $\lim_{m\to\infty} \varphi(x^m)\cdot g(x^m) = 0$, since $\lim_{m\to\infty} \varphi(x^m) = \varphi(x^0) = 0$ and $|g(x^m)| \leq 1$ for every m. Hence, $\lim_{m\to\infty} f(x^m) = 0 = f(x^0)$.

Fix an arbitrary $x^0 \in G$ and show that $x^0 \in D(f)$. For every $m \in \mathbb{N}$, we choose $x_m \in X_m$ with $\|x_m\|_m = \ln 2 + \|x_m^0\|_m$ and set

$$x^m = (x_1^0, x_2^0, \ldots, x_{m-1}^0, x_m, x_{m+1}^0, \ldots).$$

Clearly, $x^m \to x^0$ in X_0. For every $m \in \mathbb{N}$, we have

$$\begin{aligned} g(x^m) - g(x^0) &= \exp(-\sum_{n=1}^{\infty} \|x_n^0\|_n)(\exp(\sum_{n=1}^{\infty} \|x_n^0\|_n - \sum_{n=1}^{\infty} \|x_n^m\|_n) - 1) \\ &= g(x^0)(\exp(-\ln 2) - 1) \\ &= -\frac{1}{2} g(x^0). \end{aligned}$$

Therefore, $g(x^m) = \frac{1}{2} g(x^0)$ and

$$f(x^m) - f(x^0) = \varphi(x^m)g(x^m) - \varphi(x^0)g(x^0) = g(x^0)(\frac{1}{2}\varphi(x^m) - \varphi(x^0))$$

for all $m \in \mathbb{N}$. Then

$$\lim_{m\to\infty} (f(x^m) - f(x^0)) = -\frac{1}{2}\varphi(x^0) \cdot g(x^0) < 0.$$

Hence, f is discontinuous at x^0. Consequently, $D(f) = G$.

It remains to check that f is strongly separately continuous on X_0. Evidently, f is strongly separately continuous on the set $C(f) = F$. Fix $x^0 \in G$, $k \in \mathbb{N}$ and an arbitrary convergent to x^0 sequence $(x^m)_{m=1}^{\infty}$ in X_0. For every m, we put $y^m = (x^m)_{\{k\}}^{x^0}$. Since G is open and $y^m \to x^0$, we may suppose that $x^m, y^m \in G$ for every m. We note that

$$\begin{aligned} f(x^m) - f(y^m) &= g(x^m)(\varphi(x^m) - \varphi(y^m)) + \varphi(y^m)(g(x^m) - g(y^m)) \\ &= g(x^m)(\varphi(x^m) - \varphi(y^m)) + \varphi(y^m)g(y^m)(\exp(\|x_k^0\|_k - \|x_k^m\|_k) - 1). \end{aligned}$$

It follows from the inequality

$$\exp(-\|x_k^0 - x_k^m\|) \leq \exp(\|x_k^0\|_k - \|x_k^m\|_k) \leq \exp(\|x_k^0 - x_k^m\|)$$

that

$$\lim_{m\to\infty}(\exp(\|x_k^0\|_k - \|x_k^m\|_k) - 1) = 0.$$

Taking into account that φ and g are bounded and that

$$\lim_{m\to\infty} \varphi(x^m) = \lim_{m\to\infty} \varphi(y^m) = \varphi(x^0),$$

we obtain that

$$\lim_{m\to\infty} (f(x^m) - f(y^m)) = 0.$$

Hence, f is strongly separately continuous on X_0. □

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