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ON FUNCTIONS OF BOUNDED SEMIVARIATION

Abstract

The concept of bounded variation has been generalized in many ways. In the frame of functions taking values in Banach space, the concept of bounded semivariation is an important generalization. The aim of this paper is to provide an accessible summary of this notion, to illustrate it with an appropriate body of examples, and to outline its connection with the integration theory due to Kurzweil.

1 Introduction

Several notions of variation appear when dealing with infinite dimensional problems. Among them, semivariation is frequent, being commonly found in studies involving convolution, Stieltjes type integration, and also in topics related to vector measures.

Initially called the w-property, the concept of bounded semivariation for operator-valued functions was introduced in 1936 by M. Gowurin in his paper on the Stieltjes integral in Banach spaces, [18]. Some decades later, the Gowurin w-property was show to be useful in the investigation of integral representations of continuous linear transformations. See [41] and [13].

Nowadays, although papers make use of the concept of bounded semivariation, the majority of results are stated without proofs or proper references.

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Besides that, the literature lacks of material where basic results are collected in a unified way.

While working with Professor Milan Tvrdý in Generalized Differential Equations, due to the relevance of semivariation to our research, we have carried out some studies concerning this notion. Hoping that the finds from our work could be a valuable material for other researchers dealing with semivariation, we prepared this notes which summarizes the present knowledge in such a topic and supplements it with some remarks and new results. The presentation does not reflect the chronological order of the discoveries, but rather attempts to organize results in a logical framework. Moreover, in order to make these notes self-contained, most results are presented with a detailed proof and some illustrative examples are given. We trust that our citations and bibliography sufficiently identify the appropriate antecedent.

Besides basic results and properties, this survey also includes a section dedicated to the investigation of the relation between semivariation and non-absolute integrals.

First, let us fix some notation. Throughout this survey X and Y denote Banach spaces and $L(X,Y)$ stands for the Banach space of bounded linear operators from X to Y. By $\|\cdot\|_X$ and $\|\cdot\|_{L(X,Y)}$ we denote the norm in X and the usual operator norm in $L(X,Y)$, respectively. In particular, we write $L(X)=L(X,X)$ and $X^*=L(X,\mathbb{R})$. For an arbitrary function $f:[a,b]\to X$ we set $\|f\|_\infty=\sup_{t\in[a,b]}\|f(t)\|_X$.

Consider a nondegenerate closed interval $[a,b]$ and denote by $\mathcal{D}[a,b]$ the set of all finite divisions of $[a,b]$ of the form

$$D=\{\alpha_0,\alpha_1,\dots,\alpha_{\nu(D)}\},\quad a=\alpha_0<\alpha_1<\ldots<\alpha_{\nu(D)}=b\,,$$

where $\nu(D)\in\mathbb{N}$ corresponds to the number of subintervals in which $[a,b]$ is divided.

With these concepts in hand we are ready to define the semivariation of an operator-valued function.

Definition 1. Given a function $F:[a,b]\to L(X,Y)$ and $D\in\mathcal{D}[a,b]$, let

$$V(F,D,[a,b])=\sup\left\{\left\|\sum_{j=1}^{\nu(D)}\big[F(\alpha_j)-F(\alpha_{j-1})\big]x_j\right\|_Y\,:\,x_j\in X,\ \|x_j\|_X\le 1\right\}.$$

The *semivariation* of F on $[a,b]$ is then defined by

$$\mathrm{SV}_a^b(F)=\sup\{V(F,D,[a,b])\,:\,D\in\mathcal{D}[a,b]\}.$$

If $\mathrm{SV}_a^b(F) < \infty$, we say that the function F is of bounded semivariation on $[a,b]$. The set of all functions $F : [a,b] \to L(X,Y)$ of bounded semivariation on $[a,b]$ we denote by $SV([a,b], L(X,Y))$.

If no misunderstanding can arise, we write simply $V(F,D)$ instead of $V(F,D,[a,b])$.

Remark 2. The concept presented in Definition 1, called w-property in [18], is also known as $\mathcal{B}$-variation, with respect to the bilinear triple $\mathcal{B} = (L(X,Y), X, Y)$. For details, see [34] and [15]. The terminology used in this paper is consistent with that found in the book by Hönig [22] and seem to be the most frequent in literature. However, we call the readers attention to the fact that the term 'semivariation' might also appear with a slightly different formulation - for example, when applied to measure theory or to functions with values in a general Banach space. See, for instance, [7], [9] or, in the context of functions with values in locally convex spaces, [11].

It is not hard to see that semivariation is more general than the notion of variation in the sense of Jordan. Indeed, for $F : [a,b] \to L(X,Y)$, we have

$$\mathrm{SV}_a^b(F) \le \mathrm{var}_a^b(F)$$

where $\mathrm{var}_a^b(F)$ stands for the variation of F on $[a,b]$ and is given by

$$\mathrm{var}_a^b(F) = \sup\left\{ \sum_{j=1}^{\nu(D)} \|F(\alpha_j) - F(\alpha_{j-1})\|_{L(X,Y)} \,:\, D \in \mathcal{D}[a,b] \right\}.$$

Let $BV([a,b], L(X,Y))$ denote the set of all functions $F : [a,b] \to L(X,Y)$ of bounded variation on $[a,b]$ (i.e $\mathrm{var}_a^b(F) < \infty$). Then clearly,

$$BV([a,b], L(X,Y)) \subseteq SV([a,b], L(X,Y)).$$

The relation between these two sets will be analysed in more detail in Section 4.

The following example, constructed based on some ideas from [40], illustrates the calculation of the semivariation of a function. Moreover, this example will be important later while investigating the relation between semivariation and the variation in the sense of Jordan.

Example 3. Let ℓ_2 be the Banach space of sequences $x = \{x_n\}_n$ in $\mathbb{R}$ such that the series $\sum_{n=1}^{\infty} |x_n|^2$ converges, equipped with the norm

$$\|x\|_2 = \Big(\sum_{n=1}^{\infty} |x_n|^2\Big)^{1/2}.$$

Let e_k, $k \in \mathbb{N}$, denote the canonical Schauder basis of ℓ_2, where e_k is the sequence whose k-th term is 1 and all other terms are zero. For each $k \in \mathbb{N}$, consider $y_k \in \ell_2$ given by $y_k = \frac{1}{k} e_k$, that is, $y_k = \{y_n^{(k)}\}_n$ where

$$y_n^{(k)} = \begin{cases} \dfrac{1}{k} & \text{if } \ n = k \\ 0 & \text{otherwise} \end{cases} \qquad \text{for } \ n \in \mathbb{N}$$

Note that the series $\sum_{k=1}^{\infty} y_k$ converges in ℓ_2 and denote its sum by S.

Let $F : [0,1] \to L(\mathbb{R}, \ell_2)$ be given by

$$\big(F(t)\big)\, x = \begin{cases} x \displaystyle\sum_{k=1}^{n} y_k & \text{if } \ t \in (\frac{1}{n+1}, \frac{1}{n}], \ n \in \mathbb{N}, \\ x\, S & \text{if } \ t = 0 \end{cases}$$

for $t \in [0,1]$ and $x \in \mathbb{R}$.

In order to prove that $F \in SV([0,1], L(\mathbb{R}, \ell_2))$, let us consider $D \in \mathcal{D}[0,1]$ with $D = \{\alpha_0, \alpha_1, \ldots, \alpha_{\nu(D)}\}$. Put

$$n_j = \max\{k \in \mathbb{N} : k\alpha_j \le 1\} \ \text{ for } \ j = 1, \ldots, \nu(D),$$

and $\Lambda = \{j : n_j < n_{j-1}\} \subset \{2, \ldots, \nu(D)\}$. For $x_j \in \mathbb{R}$, $j = 1, \ldots, \nu(D)$ with $|x_j| \le 1$ we have $F(\alpha_j)x_j = \sum_{k=1}^{n_j} y_k$, and consequently

$$\begin{aligned} \sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]x_j &= x_1 \Big(\sum_{k=1}^{n_1} y_k - S\Big) + \sum_{j=2}^{\nu(D)} x_j \Big(\sum_{k=1}^{n_j} y_k - \sum_{m=1}^{n_{j-1}} y_m\Big) \\ &= -x_1 \Big(\sum_{k=n_1+1}^{\infty} y_k\Big) - \sum_{j\in\Lambda} x_j \Big(\sum_{k=n_j+1}^{n_{j-1}} y_k\Big). \end{aligned}$$

For $k \in \mathbb{N}$, define

$$\lambda_k = \begin{cases} -x_j & \text{if } \ n_j < k \le n_{j-1}, \quad j \in \Lambda \\ -x_1 & \text{if } \ k > n_1 \\ 0 & \text{otherwise} \end{cases} . \tag{1}$$

Using this sequence and the definition of y_k, we can write

$$\Big\| \sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]\, x_j \Big\|_2^2 = \Big\| \sum_{k=1}^{\infty} \lambda_k\, y_k \Big\|_2^2 = \sum_{k=2}^{\infty} \Big| \frac{\lambda_k}{k} \Big|^2 \leq \sum_{k=2}^{\infty} \frac{1}{k^2}.$$

This shows that $\mathrm{SV}_0^1(F) \leq \Big(\sum_{k=2}^{\infty} \frac{1}{k^2} \Big)^{1/2} = \sqrt{\frac{\pi^2}{6} - 1}$

We claim that $\mathrm{SV}_0^1(F) = \sqrt{\frac{\pi^2}{6} - 1}$. Indeed, fixed an arbitrary $N \in \mathbb{N}$, consider $D_N \in \mathcal{D}[0,1]$ given by

$$D_N = \Big\{ 0, \frac{1}{N}, \frac{1}{N-1}, \dots, \frac{1}{2}, 1 \Big\},$$

Thus, for $x_j \in \mathbb{R}$, $j = 1, \dots, N$ with $|x_j| \leq 1$ we have

$$\begin{aligned} &\Big\| \sum_{\ell=1}^{N-1} [F(\tfrac{1}{\ell}) - F(\tfrac{1}{\ell+1})] x_\ell + [F(\tfrac{1}{N}) - F(0)] x_N \Big\|_2 \\ &= \Big\| \sum_{\ell=1}^{N-1} x_\ell y_{\ell+1} + \sum_{k=N+1}^{\infty} x_N y_k \Big\|_2 = \Big(\sum_{k=2}^{\infty} \Big| \frac{\tilde{x}_k}{k} \Big|^2 \Big)^{1/2} \end{aligned}$$

where $\tilde{x}_k = x_{k-1}$ if $k = 2, \dots, N$, and $\tilde{x}_k = x_N$ for $k \in \mathbb{N}$, $k > N$. Taking the supremum over all possible choices of $x_j \in \mathbb{R}$, $j = 1, \dots, N$ with $|x_j| \leq 1$ we obtain

$$V(F, D_N, [0,1]) = \Big(\sum_{k=2}^{\infty} \frac{1}{k^2} \Big)^{1/2},$$

which proves the claim.

Remark 4. In the particular case $X = \mathbb{R}$ the space $SV([a,b], L(\mathbb{R}, Y))$ can be regarded as the space of the functions of weak bounded variation, usually denoted by $BW([a,b], Y)$, cf. [21]. This is clear once we recall that the weak variation of a function $f : [a,b] \to Y$ is given by

$$\mathrm{W}_a^b(f) = \sup\{ W(f, D) \,:\, D \in \mathcal{D}[a,b] \}$$

where for each $D \in \mathcal{D}[a,b]$

$$W(f, D) = \sup \left\{ \Big\| \sum_{j=1}^{\nu(D)} \big[f(\alpha_j) - f(\alpha_{j-1}) \big]\, \lambda_j \Big\|_Y \,:\, \lambda_j \in \mathbb{R},\, |\lambda_j| \leq 1 \right\}.$$

Therefore, we can say that the semivariation of $F : [0,1] \to L(\mathbb{R}, \ell_2)$ in the Example 3 coincides with the weak variation of the function $f : [0,1] \to \ell_2$ given by $f(t) = (F(t))1$ for $t \in [0,1]$.

2 Semivariation: basic results

This section summarizes basic properties of the semivariation that are often mentioned without proof in papers which use this notion. In order to make the current work as complete as possible, all the proofs are included. Most of the results can be found, for instance, in [21], [22] and [37]. We start by noting that $SV([a,b],L(X,Y))$ is a vector space.

Proposition 5. *Let F, $G \in SV([a,b],L(X,Y))$ and $\lambda \in \mathbb{R}$ be given. Then both functions $(F+G)$ and (λF) are of bounded semivariation on $[a,b]$, and*

$$\mathrm{SV}_a^b(F+G) \leq \mathrm{SV}_a^b(F) + \mathrm{SV}_a^b(G) \quad \textit{and} \quad \mathrm{SV}_a^b(\lambda F) = |\lambda| \, \mathrm{SV}_a^b(F). \tag{2}$$

Proof. The assertions follow from the fact that the relations

$$V(F+G,D) \leq V(F,D) + V(G,D) \quad \text{and} \quad V(\lambda F, D) = |\lambda| \, V(F,D)$$

hold for every division $D \in \mathcal{D}[a,b]$. □

According to (2), $\mathrm{SV}_a^b(\,\cdot\,)$ defines a seminorm on the space of functions of bounded semivariation. On the other hand, if we put

$$\|F\|_{SV} = \|F(a)\|_{L(X,Y)} + \mathrm{SV}_a^b(F) \quad \text{for} \quad F \in SV([a,b],L(X,Y), \tag{3}$$

then $SV([a,b],L(X,Y))$ becomes a normed space. This fact is a consequence of (2) together with the following assertion.

Proposition 6. *Let $F \in SV([a,b],L(X,Y))$. Then $\mathrm{SV}_a^b(F) = 0$ if and only if $F \equiv C$ for some fixed operator $C \in L(X,Y)$.*

Proof. Clearly, the semivariation of a constant function is zero. Conversely, assume that $\mathrm{SV}_a^b(F) = 0$. Given $t \in (a,b]$, if we consider the division $D = \{a,t,b\}$ of $[a,b]$, for any $x \in X$ with $\|x\|_X \leq 1$ we have

$$\|F(t)x - F(a)x\|_Y = \big\|[F(t)-F(a)]x + [F(b)-F(t)]0\big\|_Y \leq V(F,D,[a,b]).$$

Therefore $[F(t)-F(a)] = 0 \in L(X,Y)$; that is, F is a constant function. □

Remark 7. It is worth mentioning that in the definition of the norm $\|\cdot\|_{SV}$ we can use the fixed value of the function in any point of the interval, that is, taking $c \in [a,b]$, we can consider

$$\|F\|_{SV} = \|F(c)\|_{L(X,Y)} + \mathrm{SV}_a^b(F), \quad F \in SV([a,b],L(X,Y)).$$

The choice of the left-ending point of the interval seems to be the most common in the literature, though. Therefore, in this work, we assume the norm in $SV([a,b],L(X,Y))$ to be as introduced in (3).

Note that, for $F : [a,b] \to L(X,Y)$ and $t \in [a,b]$ we have

$$\|F(t)\|_{L(X,Y)} \le \|F(a)\|_{L(X,Y)} + \mathrm{SV}_a^b(F).$$

Hence, every function $F \in SV([a,b], L(X,Y))$ is bounded and

$$\|F\|_\infty \le \|F\|_{SV}.$$

In view of this, we can see that the topology induced in $SV([a,b], L(X,Y))$ by the supremum norm is weaker than the one induced by $\|\cdot\|_{SV}$.

In the sequel we prove that the space of functions of bounded semivariation is complete when equipped with the norm $\|\cdot\|_{SV}$, cf. [37, Proposition 4] or [21, I.3.3]. To this end, we will need the following convergence result.

Lemma 8. *Let $F : [a,b] \to L(X,Y)$. Assume that a sequence $\{F_n\}_n$ in $SV([a,b], L(X))$ and a constant $M > 0$ are such that*

$$\mathrm{SV}_a^b(F_n) \le M \quad \textit{for every} \ \ n \in \mathbb{N},$$

and

$$\lim_{n\to\infty} \|F_n(t)x - F(t)x\|_Y = 0 \quad \textit{for every} \ \ t \in [a,b] \ \ \textit{and} \ \ x \in X.$$

Then $\mathrm{SV}_a^b(F) \le M$.

Proof. Let $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\}$ be a division of $[a,b]$ and let $x_j \in X$, $j = 1, \dots, \nu(D)$ with $\|x_j\|_X \le 1$. Note that, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} \Big\| \sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]\, x_j \Big\|_Y &\le \Big\| \sum_{j=1}^{\nu(D)} [F_n(\alpha_j) - F_n(\alpha_{j-1})]\, x_j \Big\|_Y \\ &\quad + \Big\| \sum_{j=1}^{\nu(D)} [F(\alpha_j) - F_n(\alpha_j) - F(\alpha_{j-1}) + F_n(\alpha_{j-1})]\, x_j \Big\|_Y \\ \le M + \sum_{j=1}^{\nu(D)} \|[F(\alpha_j) - F_n(\alpha_j)]x_j\|_Y &+ \sum_{j=1}^{\nu(D)} \|[F(\alpha_{j-1}) - F_n(\alpha_{j-1})]x_j\|_Y. \end{aligned} \tag{4}$$

Given $\varepsilon > 0$, there is $N_D \in \mathbb{N}$ such that

$$\|[F(\alpha_j) - F_{N_D}(\alpha_j)]x_i\|_Y < \frac{\varepsilon}{2\,\nu(D)} \quad \text{for} \ \ j = 0, 1, \dots, \nu(D),$$

where $i = j$ or $i = j+1$; whenever it makes sense. Therefore, taking $n = N_D$ in (4) we obtain

$$\Big\| \sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]\, x_j \Big\|_Y < M + \varepsilon$$

which implies that

$$V(F, D) \le M + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $V(F, D) \le M$ for every $D \in \mathcal{D}[a, b]$, and consequently $\mathrm{SV}_a^b(F) \le M$. $\square$

The previous convergence result usually appears in connection with an integration theory. See, for instance, [22, Theorem I.5.8] and [10]. This type of result, often mentioned as a Helly-Bray Theorem, will be studied in Section 5 in the context of Kurzweil-Stieltjes integral. But for now, we are ready to prove the completeness of the space $SV([a, b], L(X, Y))$.

Theorem 9. *$SV([a, b], L(X, Y))$ is a Banach space with respect to the norm $\|\cdot\|_{SV}$.*

Proof. Let $\{F_n\}_n$ be a Cauchy sequence in $SV([a, b], L(X, Y))$ with respect to the norm $\|\cdot\|_{SV}$. This means that given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\|F_n(t) - F_m(t)\|_{L(X,Y)} \le \|F_n - F_m\|_{SV} < \varepsilon, \quad n, m \ge n_0 \text{ and } t \in [a, b]. \tag{5}$$

Hence, for each $t \in [a, b]$, $\{F_n(t)\}_n$ is a Cauchy sequence in $L(X, Y)$ which implies that there exists $F(t) \in L(X, Y)$ such that

$$\lim_{n\to\infty} \|F_n(t) - F(t)\|_{L(X,Y)} = 0.$$

Moreover, due to (5), this convergence is uniform on $[a, b]$. By the fact that $\{F_n\}_n$ is a Cauchy sequence there exists $M > 0$ such that $\mathrm{SV}_a^b(F_n) \le M$ for every $n \in \mathbb{N}$. Therefore by Lemma 8, $F \in SV([a, b], L(X, Y))$.

It remains to show that the convergence is also true in the topology induced by the norm $\|\cdot\|_{SV}$. To this aim, consider a division $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\}$ of $[a, b]$ and arbitrary $x_j \in X$, $j = 1, \dots, \nu(D)$ with $\|x_j\|_X \le 1$. By (5), for $n, m \ge n_0$, we have

$$\Big\| \sum_{j=1}^{\nu(D)} [F_n(\alpha_j) - F_m(\alpha_j) - F_n(\alpha_{j-1}) + F_m(\alpha_{j-1})]\, x_j \Big\|_Y < \varepsilon.$$

Thus, taking the limit $m \to \infty$ we obtain

$$\Big\| \sum_{j=1}^{\nu(D)} [F_n(\alpha_j) - F(\alpha_j) - F_n(\alpha_{j-1}) + F(\alpha_{j-1})]\, x_j \Big\|_Y \leq \varepsilon.$$

That is, $V((F_n - F), D) \leq \varepsilon$, for $n \geq n_0$. Since the division $D \in \mathcal{D}[a, b]$ is arbitrary, it follows that $\lim_{n\to\infty} \mathrm{SV}(F_n - F) = 0$, concluding the proof. □

The following theorem, which was borrowed from [22, Lemma I.1.11], proves that the functions of bounded variation are multipliers for the space $SV([a, b], L(X, Y))$.

Theorem 10. *Let $F \in SV([a, b], L(X, Y))$ and $G \in BV([a, b], L(X))$. Consider the function $FG : [a, b] \to L(X, Y)$ given by $(FG)(t) = F(t)\, G(t)$ for $t \in [a, b]$. Then $F\,G \in SV([a, b], L(X, Y))$ and*

$$\mathrm{SV}_a^b(F\,G) \leq \|F\|_\infty \operatorname{var}_a^b(G) + \|G\|_\infty \mathrm{SV}_a^b(F).$$

PROOF. Consider a division $D = \{\alpha_0, \alpha_1, \ldots, \alpha_{\nu(D)}\}$ of $[a, b]$ and let $x_j \in X$, $j = 1 \ldots, \nu(D)$ with $\|x_j\|_X \leq 1$. Note that

$$\begin{aligned} \sum_{j=1}^{\nu(D)} [(F\,G)(\alpha_j) - (F\,G)(\alpha_{j-1})]\, x_j &= \sum_{j=1}^{\nu(D)} F(\alpha_j)[G(\alpha_j) - G(\alpha_{j-1})]\, x_j \\ &\quad + \sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]\, G(\alpha_{j-1})\, x_j. \end{aligned} \tag{6}$$

Moreover, we have

$$\begin{aligned} &\Big\| \sum_{j=1}^{\nu(D)} F(\alpha_j)[G(\alpha_j) - G(\alpha_{j-1})]\, x_j \Big\|_Y \\ &\leq \|F\|_\infty \sum_{j=1}^{\nu(D)} \|G(\alpha_j) - G(\alpha_{j-1})\|_{L(X)} \leq \|F\|_\infty \operatorname{var}_a^b(G) \end{aligned}$$

and

$$\begin{aligned} &\Big\| \sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]\, G(\alpha_{j-1})\, x_j \Big\|_Y \\ &\leq \|G\|_\infty \left\| \sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})] \frac{G(\alpha_{j-1})\, x_j}{\|G\|_\infty} \right\|_Y \leq \|G\|_\infty \mathrm{SV}_a^b(F) \end{aligned}$$

These inequalities, together with (6), imply that

$$\Big\| \sum_{j=1}^{\nu(D)} [(F\,G)(\alpha_j) - (F\,G)(\alpha_{j-1})]\, x_j \Big\|_Y \leq \|F\|_\infty \operatorname{var}_a^b(G) + \|G\|_\infty \operatorname{SV}_a^b(F).$$

Consequently,

$$V((FG), D) \leq \|F\|_\infty \operatorname{var}_a^b(G) + \|G\|_\infty \operatorname{SV}_a^b(F)$$

for every $D \in \mathcal{D}[a,b]$, wherefrom the result follows. □

The next theorem presents some algebraic properties of the semivariation.

Theorem 11. *If $F : [a,b] \to L(X,Y)$ and $[c,d] \subset [a,b]$, then*

$$\operatorname{SV}_c^d(F) \leq \operatorname{SV}_a^b(F).$$

Moreover,

$$\operatorname{SV}_a^b(F) \leq \operatorname{SV}_a^c(F) + \operatorname{SV}_c^b(F) \quad \textit{for} \;\; c \in [a,b]. \tag{7}$$

Proof. It is easy to see that, for every division D of $[c,d]$, taking $\tilde{D} = D \cup \{a,b\}$, we have $\tilde{D} \in \mathcal{D}[a,b]$ and

$$V(F, D, [c,d]) \leq V(F, \tilde{D}, [a,b]) \leq \operatorname{SV}_a^b(F).$$

Therefore $\operatorname{SV}_c^d(F) \leq \operatorname{SV}_a^b(F)$.

To prove the superadditivity, given $c \in [a,b]$ and $D \in \mathcal{D}[a,b]$, consider $D_1 = (D \cap [a,c]) \cup \{c\}$ and $D_2 = (D \cap [c,b]) \cup \{c\}$. Clearly, D_1 and D_2 are divisions of $[a,c]$ and $[c,b]$, respectively. In addition,

$$V(F, D, [a,b]) \leq V_a^b(F, D \cup \{c\}, [a,b]) \leq V(F, D_1, [a,c]) + V(F, D_2, [c,b]).$$

Hence,

$$V(F, D, [a,b]) \leq \operatorname{SV}_a^c(F) + \operatorname{SV}_c^b(F) \quad \text{for every} \;\; D \in \mathcal{D}[a,b],$$

which leads to the inequality (7). □

According to the previous theorem: *If $F \in SV([a,b], L(X,Y))$, then F is of bounded semivariation on each closed subinterval of* $[a,b]$. As a consequence, we have the following:

Corollary 12. *Let $F \in SV([a,b], L(X,Y))$ be given. Then*

1. *the mapping* $t \in [a,b] \longmapsto \mathrm{SV}_a^t(F)$ *is nondecreasing;*

2. *the mapping* $t \in [a,b] \longmapsto \mathrm{SV}_t^b(F)$ *is nonincreasing.*

Theorem 11 indicates that, unlike variation, semivariation need not be additive with respect to intervals. The next example shows that the inequality in (7) may be strict.

Example 13. Let $F : [0,1] \to L(\mathbb{R}, \ell_2)$ be the function given on Example 3; that is, for $t \in [0,1]$ and $x \in \mathbb{R}$,

$$\big(F(t)\big)\,x = \begin{cases} x \displaystyle\sum_{k=1}^{n} y_k & \text{if } \ t \in (\frac{1}{n+1}, \frac{1}{n}], \ n \in \mathbb{N}, \\ \quad x\,S & \text{if } \ t = 0 \end{cases}$$

where $S = \sum_{k=1}^{\infty} y_k$ and $y_k = \frac{1}{k} e_k \in \ell_2$, $k \in \mathbb{N}$; with e_k being an element of the canonical Schauder basis of ℓ_2.

To prove that

$$\mathrm{SV}_0^1(F) < \mathrm{SV}_0^{\frac{1}{2}}(F) + \mathrm{SV}_{\frac{1}{2}}^1(F), \tag{8}$$

we first calculate $\mathrm{SV}_0^{\frac{1}{2}}(F)$.

Let $D = \{\alpha_0, \alpha_1, \ldots, \alpha_{\nu(D)}\}$ be a division of $[0, \frac{1}{2}]$, and as in Example 3, put

$$n_j = \max\{k \in \mathbb{N} \, : \, k\alpha_j \leq 1\} \ \text{ for } \ j = 1, \ldots, \nu(D),$$

and $\Lambda = \{j \, : \, n_j < n_{j-1}\} \subset \{2, \ldots, \nu(D)\}$. For $x_j \in \mathbb{R}$, $j = 1, \ldots, \nu(D)$ with $|x_j| \leq 1$ we have

$$\Big\| \sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]\, x_j \Big\|_2 = \Big\| \sum_{k=1}^{\infty} \lambda_k\, y_k \Big\|_2 = \Big(\sum_{k=3}^{\infty} \Big| \frac{\lambda_k}{k} \Big|^2 \Big)^{1/2} \leq \Big(\sum_{k=3}^{\infty} \frac{1}{k^2} \Big)^{1/2},$$

where λ_k for $k \in \mathbb{N}$, $k \geq 3$, is given as in (1) (note that the corresponding n_j satisfies $n_j \geq 2$, $j = 1, \ldots, \nu(D)$). In view of this, it is clear that

$$\mathrm{SV}_0^{\frac{1}{2}}(F) \leq \Big(\sum_{k=3}^{\infty} \frac{1}{k^2} \Big)^{1/2} = \sqrt{\frac{\pi^2}{6} - \frac{5}{4}}.$$

The equality $\mathrm{SV}_0^{\frac{1}{2}}(F) = \sqrt{\frac{\pi^2}{6} - \frac{5}{4}}$ is a consequence of the fact that

$$V(F, D_N, [0, \frac{1}{2}]) = \Big(\sum_{k=3}^{\infty} \frac{1}{k^2} \Big)^{1/2}$$

for any division $D_N = \left\{0, \frac{1}{N}, \frac{1}{N-1}, \dots, \frac{1}{2}\right\}$ with $N \in \mathbb{N}$.

On the other hand, it is not hard to see that $\mathrm{SV}_{\frac{1}{2}}^{1}(F) = \frac{1}{2}$. Indeed, for any division $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\}$ of $[\frac{1}{2}, 1]$ and for any choice of $x_j \in \mathbb{R}$, $j = 1, \dots, \nu(D)$ with $|x_j| \le 1$ we have

$$\sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]\, x_j = [F(\alpha_1) - F(\frac{1}{2})]\, x_1 = -x_1\, y_2 = -\frac{x_1}{2}\, e_2.$$

Hence, $V(F, D, [\frac{1}{2}, 1]) = \frac{1}{2}$. Recalling that $\mathrm{SV}_0^1(F) = \sqrt{\frac{\pi^2}{6} - 1}$, we conclude that (8) holds.

In the sequel we provide further characterizations of the semivariation of a function. The first one, Theorem 14, can be found, for instance, in [35, Proposition 1.1] or [22, Theorem I.4.4]. Basically, it connects the notions of semivariation and $\mathcal{B}^*$-variation, with respect to the bilinear triple $\mathcal{B}^* = (L(X,Y), L(X), L(X,Y))$. For definitions see [22].

Theorem 14. *For $F : [a,b] \to L(X,Y)$ and $D \in \mathcal{D}[a,b]$ put*

$$V^*(F, D) = \sup\left\{ \Big\| \sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, G_j \Big\|_{L(X,Y)} \,:\, \|G_j\|_{L(X)} \le 1 \right\}.$$

Then

$$\mathrm{SV}_a^b(F) = \sup\{V^*(F, D) \,:\, D \in \mathcal{D}[a,b]\}.$$

Proof. It is enough to show that

$$V^*(F, D) = V(F, D) \quad \text{for every} \quad D \in \mathcal{D}[a,b].$$

Let $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\}$ be a division of $[a,b]$. For $G_j \in L(X)$, $j{=}1, \dots, \nu(D)$ with $\|G_j\|_{L(X)} \le 1$ we have

$$\begin{aligned}
&\Big\| \sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, G_j \Big\|_{L(X,Y)} \\
&= \sup_{\|z\|_X \le 1} \Big\| \Big(\sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, G_j \Big) z \Big\|_Y \\
&= \sup_{\|z\|_X \le 1} \Big\| \sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, (G_j\, z) \Big\|_Y \le V(F, D)
\end{aligned}$$

where the last inequality is due to the fact that $\|G_j\, z\|_X \le 1$, j=$1,\dots,\nu(D)$, provided $\|z\|_X \le 1$. Hence, $V^*(F,D) \le V(F,D)$.

To obtain the reverse inequality we use the Hahn-Banach Theorem, cf. [20, Theorem 2.7.4], to choose $w \in X$ and $\varphi \in X^*$ such that $\|w\|_X = 1$, $\|\varphi\|_{X^*} = 1$ and $\varphi(w) = 1$. Given $x_j \in X$, $j = 1,\dots,\nu(D)$ with $\|x_j\|_X \le 1$ consider $G_j \in L(X)$ defined by

$$G_j\, x = \varphi(x)\, x_j \quad \text{for} \quad x \in X.$$

Note that, $\|G_j\|_{L(X)} \le 1$ and $G_j\, w = x_j$ for $j = 1,\dots,\nu(D)$. Thus,

$$\begin{aligned}
\Big\| \sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, x_j \Big\|_Y \\
&= \Big\| \sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, (G_j\, w) \Big\|_Y \\
&= \Big\| \Big(\sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, G_j \Big)\, w \Big\|_Y \\
&\le \Big\| \sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, G_j \Big\|_{L(X,Y)} \|w\|_X
\end{aligned}$$

which yields $V(F,D) \le V^*(F,D)$ concluding the proof. □

Remark 15. In a more general formulation, $V^*(F,D)$ in Theorem 14 can be defined so that the supremum is taken over all possible choices of $G_j \in L(Z,X)$ with $\|G_j\|_{L(Z,X)} \le 1$, $j = 1,\dots,\nu(D)$; where Z is an arbitrary Banach space.

The following theorem, stated in [22, 3.6, Chapter I], will be useful for the investigation of continuity type results for semivariation. The characterization presented involves functions $(y^* \circ F) : [a,b] \to X^*$, obtained by the composition of $F : [a,b] \to L(X,Y)$ and a functional $y^* \in Y^*$ which reads as follows

$$(y^* \circ F)(t)(x) = y^*\big(F(t)\, x\big) \quad \text{for} \quad t \in [a,b],\ x \in X. \tag{9}$$

Theorem 16. *The semivariation of a function $F : [a,b] \to L(X,Y)$ is given by*

$$\mathrm{SV}_a^b(F) = \sup\big\{ \operatorname{var}_a^b (y^* \circ F) \,:\, y^* \in Y^*,\ \|y^*\|_{Y^*} \le 1 \big\}. \tag{10}$$

Moreover, $F \in SV([a,b], L(X,Y))$ if and only if $(y^* \circ F) \in BV([a,b], X^*)$ for all $y^* \in Y^*$.

Proof. Let $D \in \mathcal{D}[a,b]$, $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\}$, be given. For $x_j \in X$, $j = 1, \dots, \nu(D)$ with $\|x_j\|_X \le 1$ we have

$$\begin{aligned}
&\Big\| \sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]\, x_j \Big\|_Y \\
&\quad = \sup \left\{ \Big| y^*\Big(\sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]\, x_j \Big) \Big| : \|y^*\|_{Y^*} \le 1 \right\} \\
&\quad \le \sup \left\{ \sum_{j=1}^{\nu(D)} \big| [y^* \circ F(\alpha_j) - y^* \circ F(\alpha_{j-1})]\, x_j \big| : \|y^*\|_{Y^*} \le 1 \right\}.
\end{aligned}$$

Therefore,

$$V(F,D) \le \sup \left\{ \sum_{j=1}^{\nu(D)} \|y^* \circ F(\alpha_j) - y^* \circ F(\alpha_{j-1})\|_{X^*} : \|y^*\|_{Y^*} \le 1 \right\}$$

for every $D \in \mathcal{D}[a,b]$, and consequently

$$\mathrm{SV}_a^b(F) \le \sup \big\{ \operatorname{var}_a^b (y^* \circ F) : y^* \in Y^*, \|y^*\|_{Y^*} \le 1 \big\}. \tag{11}$$

On the other hand, given $y^* \in Y^*$ with $\|y^*\|_{Y^*} \le 1$ and $\varepsilon > 0$, for each $j = 1, \dots, \nu(D)$, there exists $x_j \in X$ with $\|x_j\|_X \le 1$ such that

$$\big\| y^* \circ F(\alpha_j) - y^* \circ F(\alpha_{j-1}) \big\|_{X^*} - \frac{\varepsilon}{\nu(D)} < \big| [y^* \circ F(\alpha_j) - y^* \circ F(\alpha_{j-1})]\, x_j \big|.$$

If we put $\lambda_j := \operatorname{sgn}\big([y^* \circ F(\alpha_j) - y^* \circ F(\alpha_{j-1})]\, x_j \big)$ and $\tilde{x}_j = \lambda_j\, x_j$ for each $j = 1, \dots, \nu(D)$, then we obtain

$$\begin{aligned}
&\sum_{j=1}^{\nu(D)} \|y^* \circ F(\alpha_j) - y^* \circ F(\alpha_{j-1})\|_{X^*} - \varepsilon \le \sum_{j=1}^{\nu(D)} [y^* \circ F(\alpha_j) - y^* \circ F(\alpha_{j-1})]\, \tilde{x}_j \\
&\quad \le \Big| \sum_{j=1}^{\nu(D)} [y^* \circ F(\alpha_j) - y^* \circ F(\alpha_{j-1})]\, \tilde{x}_j \Big| = \Big| y^*\Big(\sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]\, \tilde{x}_j \Big) \Big| \\
&\quad \le \|y^*\|_{Y^*} \Big\| \sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]\, \tilde{x}_j \Big\|_Y \le \mathrm{SV}_a^b(F).
\end{aligned}$$

Taking the surpremum over all $D \in \mathcal{D}[a,b]$, we get $\mathrm{var}_a^b(y^* \circ F) \leq \varepsilon + \mathrm{SV}_a^b(F)$. Since $\varepsilon > 0$ is arbitrary, it follows that

$$\mathrm{var}_a^b(y^* \circ F) \leq \mathrm{SV}_a^b(F) \quad \text{for } y^* \in Y^* \text{ with } \|y^*\|_{Y^*} \leq 1,$$

which, together with (11), proves the result. □

The equality in (10) is used in a more general way to define the notion of semivariation in the context of functions with values in an arbitrary Banach space (cf. [4]). More precisely, if Z is a Banach space, the semivariation of $f : [a,b] \to Z$ is given by

$$\mathrm{semivar}_a^b(f) = \sup \left\{ \ \mathrm{var}_a^b \,(z^* \circ f) \ : \ z^* \in Z^*, \ \|z^*\|_{Z^*} \leq 1 \right\}$$

where the functions $(z^* \circ f) : [a,b] \to \mathbb{R}$ are defined as in (9) with an obvious adaptation.

Thereafter, for operator-valued functions two notions of semivariation can be derived. However, no direct connection between them is established since such a connection would rely on a characterization of the dual of $L(X,Y)$. On the other hand, as observed in [4], given a function $F : [a,b] \to L(X,Y)$, those two notions are related as follows:

$$\text{for each } x \in X \text{ the function } F_x : t \in [a,b] \mapsto F(t)x \in Y \text{ satisfies}$$
$$\mathrm{semivar}_a^b(F_x) \leq \mathrm{SV}_a^b(F).$$

3 Semivariation and variation

We have mentioned in Section 1 that every function of bounded variation is also of bounded semivariation, that is,

$$BV([a,b], L(X,Y)) \subseteq SV([a,b], L(X,Y)). \tag{12}$$

This section is devoted to the study of conditions ensuring the equality of these two sets. To start, we investigate the case when Y is the real line.

Theorem 17. *Let $F : [a,b] \to X^*$ be given. Then, $F \in SV([a,b], X^*)$ if and only if $F \in BV([a,b], X^*)$. In this case,* $\mathrm{SV}_a^b(F) = \mathrm{var}_a^b(F)$.

Proof. This proof is analogous to the proof of Theorem 16. In summary, it is a consequence of the fact that we can write

$$\begin{aligned} V(F,D) &= \sup\left\{ \Big| \sum_{j=1}^{\nu(D)} \big[F(\alpha_j) - F(\alpha_{j-1})\big]\, x_j \Big| \,:\, x_j \in X,\ \|x_j\|_X \le 1 \right\} \\ &= \sum_{j=1}^{\nu(D)} \big\|F(\alpha_j) - F(\alpha_{j-1})\big\|_{X*} \end{aligned}$$

for every division $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\}$ of $[a,b]$. □

Remark 18. Given $n \in \mathbb{N}$ consider a function $F : [a,b] \to L(X, \mathbb{R}^n)$. Writing $F = (F_1, \dots, F_n)$ with $F_j : [a,b] \to X^*$, $j = 1, \dots, n$, it is clear that

$$F \in BV([a,b], L(X,\mathbb{R}^n)) \text{ if and only if } F_j \in BV([a,b], X^*),\ j = 1, \dots, n$$

and similarly

$$F \in SV([a,b], L(X,\mathbb{R}^n)) \text{ if and only if } F_j \in SV([a,b], X^*),\ j = 1, \dots, n.$$

With this in mind, the assertion in Theorem 17 can be extended to the case when Y is an Euclidean space. More generally, since every finite dimensional space is isomorphic to some $\mathbb{R}^n$, $n \in \mathbb{N}$, we have the following:

Theorem 19. *If Y is a finite dimensional Banach space, we have*

$$F \in SV([a,b], L(X,Y)) \textit{ if and only if } F \in BV([a,b], L(X,Y)).$$

The following example presents a function of bounded semivariation whose variation is not finite.

Example 20. Let $F : [0,1] \to L(\mathbb{R}, \ell_2)$ be the function given on Example 3; that is, for $t \in [0,1]$ and $x \in \mathbb{R}$,

$$\big(F(t)\big)\, x = \begin{cases} x \displaystyle\sum_{k=1}^{n} y_k & \text{if } t \in (\frac{1}{n+1}, \frac{1}{n}],\ n \in \mathbb{N}, \\ x\, S & \text{if } t = 0 \end{cases} \tag{13}$$

where $S = \sum_{k=1}^{\infty} y_k$ and $y_k = \frac{1}{k} e_k \in \ell_2$, $k \in \mathbb{N}$, with e_k being an element of the canonical Schauder basis of ℓ_2.

We know that $F \in SV([0,1], L(\mathbb{R}, \ell_2))$. On the other hand, since

$$\|F(\tfrac{1}{k}) - F(\tfrac{1}{k+1})\|_{L(\mathbb{R},\ell_2)} = \|y_{k+1}\|_2 = \frac{1}{k+1} \quad \text{for every} \quad k \in \mathbb{N},$$

we have

$$\sum_{k=1}^{N+1} \frac{1}{k} \leq \sum_{k=1}^{N} \|F(\tfrac{1}{k}) - F(\tfrac{1}{k+1})\|_{L(\mathbb{R},\ell_2)} + \|F(\tfrac{1}{N+1}) - F(0)\|_{L(\mathbb{R},\ell_2)} \leq \operatorname{var}_0^1(F),$$

for any choice of $N \in \mathbb{N}$. Therefore, $\operatorname{var}_0^1(F) = \infty$.

The main tool for the construction of the function in the example above was the sequence $\{y_n\}_n$ in ℓ_2 whose series converges but not absolutely. It was shown by Dvoretzky and Rogers, [12], that every infinite dimensional Banach space contains such a sequence, and hence *finite dimension* is a necessary and sufficient condition for the equivalence between bounded variation and bounded semivariation. We provide the full details below, and remark that in [40, Theorem 2] this equivalence was actually proved for functions defined on a ring of sets.

Theorem 21. *If the dimension of the space Y is infinite, then there exists $F \in SV([a,b], L(X,Y))$ such thar $\operatorname{var}_a^b(F) = \infty$.*

Proof. By the Dvoretzky-Rogers Theorem A5 and its Corollary A6 in the Appendix, there exists a sequence $\{y_n\}_n$ in Y such that the series $\sum_{n=1}^{\infty} y_n$ is unconditionally convergent but not absolutely convergent. Considering an increasing sequence $\{t_n\}_n$ in (a,b) converging to b and fixing an arbitrary $\varphi \in X^*$, with $\|\varphi\|_{X^*} = 1$, let

$$F(t)\,x = \sum_{t_k < t} \varphi(x)\, y_k \quad \text{for} \quad x \in X \quad \text{and} \quad t \in [a,b].$$

Note that, by Theorem A3 from the Appendix, $F : [a,b] \to L(X,Y)$ is well-defined.

We claim that the variation of F is not finite. Indeed, given $N \in \mathbb{N}$ consider the division $D_N = \{t_0, t_1, \ldots, t_{N+1}, b\}$ formed by elements of the sequence $\{t_n\}_n$ and $t_0 = a$. Noting that

$$\|y_k\|_Y = \|F(t_{k+1}) - F(t_k)\|_{L(X,Y)} \quad \text{for every} \quad k \in \mathbb{N},$$

we have

$$\sum_{k=1}^{N} \|y_k\|_Y \leq \sum_{j=1}^{N+1} \|F(t_j) - F(t_{j-1})\|_{L(X,Y)} + \|F(b) - F(t_{N+1})\|_{L(X,Y)} \leq \operatorname{var}_a^b(F).$$

Since $\sum_{n=1}^{\infty} y_n$ is not absolutely convergent, it follows that $\mathrm{var}_a^b(F) = \infty$.

Let us show that $F \in SV([a,b], L(X,Y))$. Consider a division $D = \{\alpha_0, \ldots, \alpha_{\nu(D)}\}$ of $[a,b]$ and let $x_j \in X$, $j = 1, \ldots, \nu(D)$ with $\|x_j\|_X \leq 1$. Thus,

$$\begin{aligned} &\Big\| \sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]\, x_j \Big\|_Y \\ &= \Big\| \sum_{j=1}^{\nu(D)} \big(\varphi(x_j) \sum_{t_k \in [\alpha_{j-1}, \alpha_j)} y_k\big) \Big\|_Y = \Big\| \sum_{k=1}^{\infty} \beta_k y_k \Big\|_Y \end{aligned}$$

where $\beta_k = \varphi(x_j)$ if $t_k \in [\alpha_{j-1}, \alpha_j)$, for some $j = 1, \ldots, \nu(D)$, otherwise $\beta_k = 0$. By Lemma A4 from the Appendix we know that the set

$$\left\{ \sum_{n=1}^{\infty} \lambda_n\, y_n \,:\, \lambda_n \in \mathbb{R} \ \text{ with } \ |\lambda_n| \leq 1,\ n \in \mathbb{N} \right\}$$

is bounded in Y. Thus, $\left\| \sum_{k=1}^{\infty} \beta_k y_k \right\|_Y$ is bounded (uniformly with respect to the choice of $x_j \in X$, $j = 1, \ldots, \nu(D)$) and, consequently, $\mathrm{SV}_a^b(F) < \infty$ which proves the result. □

According to Remark 18 and Theorem 21 we conclude that the notion of semivariation is relevant only in spaces with infinite dimension.

Corollary 22. *The following assertions are equivalent:*

- (i) *the dimension of the space Y is finite;*
- (ii) *every function $F \in SV([a,b], L(X,Y))$ is of bounded variation on $[a,b]$.*

It was shown on Example 13 that the semivariation of the function F in (13) is not additive with respect to intervals. It turns out that such an additivity type property can be used to identify whether a function of bounded semivariation has a bounded variation as well. This is the content of the following theorem.

Theorem 23. *Let $F \in SV([a,b], L(X,Y))$. Then $F \in BV([a,b], L(X,Y))$ if and only if*

$$M := \sup \left\{ \sum_{j=1}^{\nu(D)} \mathrm{SV}_{\alpha_{j-1}}^{\alpha_j}(F) \,:\, D \in \mathcal{D}[a,b] \right\} < \infty. \tag{14}$$

Moreover, in this case, $\mathrm{var}_a^b(F) = M$.

PROOF. Assume (14) holds. Given $\varepsilon > 0$, and $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\}$, we can choose $x_j \in X$, $j = 1, \dots, \nu(D)$ with $\|x_j\|_X \le 1$ such that

$$\|F(\alpha_j) - F(\alpha_{j-1})\|_{L(X,Y)} - \frac{\varepsilon}{\nu(D)} < \|\,[F(\alpha_j) - F(\alpha_{j-1})]\, x_j\|_Y.$$

Noting that, for $j = 1, \dots, \nu(D)$,

$$\|\,[F(\alpha_j) - F(\alpha_{j-1})]\, x_j\|_Y \le \mathrm{SV}_{\alpha_{j-1}}^{\alpha_j}(F),$$

it follows that

$$\sum_{j=1}^{\nu(D)} \|F(\alpha_j) - F(\alpha_{j-1})\|_{L(X,Y)} - \varepsilon < \sum_{j=1}^{\nu(D)} \mathrm{SV}_{\alpha_{j-1}}^{\alpha_j}(F) \le M.$$

Therefore, taking the supremum over all divisions $D \in \mathcal{D}[a,b]$ we obtain

$$\mathrm{var}_a^b(F) < M + \varepsilon.$$

Consequently $F \in BV([a,b], L(X,Y))$ and, since $\varepsilon > 0$ is arbitrary, we have $\mathrm{var}_a^b(F) \le M$.

On the other hand, for any division $D = \{\alpha_0, \alpha_1, \dots, \alpha_{\nu(D)}\}$ of $[a,b]$ we have

$$\sum_{j=1}^{\nu(D)} \mathrm{SV}_{\alpha_{j-1}}^{\alpha_j}(F) \le \sum_{j=1}^{\nu(D)} \mathrm{var}_{\alpha_{j-1}}^{\alpha_j}(F) = \mathrm{var}_a^b(F),$$

wherefrom we conclude that $\mathrm{var}_a^b(F) = M$. □

4 Semivariation, limits and continuity

It is well-known that a function of bounded variation is regulated, that is, the one-sided limits exist at every point of the domain. See, for instance, [21, Theorem I.2.7] or [15, Lemma 2.1]. In this section we investigate the connection between functions of bounded semivariation and regulated functions.

Following the notation in [22], if $f : [a,b] \to X$ is a regulated function $[a,b]$, we write $f \in G([a,b], X)$, and the one-sided limits are denoted by

$$f(t-) = \lim_{s \to t-} f(s) \quad \text{and} \quad f(t+) = \lim_{s \to t+} f(s)$$

for $t \in [a,b]$ with the convention $f(a-) = f(a)$ and $f(b+) = f(b)$.

Another useful notion throughout this section is the semivariation on half-closed intervals.

Definition 24. Given $F : [a,b] \to L(X,Y)$ and $c,d \in [a,b]$, $c < d$, the semivariation of F on a half-closed interval $[c,d)$ is given by

$$\mathrm{SV}_{[c,d)}(F) = \lim_{t \to d-} \mathrm{SV}_c^t(F) = \sup_{t \in [c,d)} \mathrm{SV}_c^t(F).$$

In analogous way, we define the semivariation on the half-closed interval $(c,d]$ by

$$\mathrm{SV}_{(c,d]}(F) = \lim_{t \to c+} \mathrm{SV}_t^d(F) = \sup_{t \in (c,d]} \mathrm{SV}_t^d(F).$$

Theorems 11 and 12 guarantee that the semivariation over half-closed subintervals of $[a,b]$ is finite for every function from $SV([a,b], L(X,Y))$.

In what follows we show that a function of bounded semivariation is regulated provided some conditions on the semivariation over half-closed intervals are satisfied.

Theorem 25. *Let $F \in SV([a,b], L(X,Y))$ be such that*

$$\lim_{\delta \to 0+} \mathrm{SV}_{[t-\delta,t)}(F) = 0 \quad \textit{for every} \;\; t \in (a,b], \tag{15a}$$

$$\lim_{\delta \to 0+} \mathrm{SV}_{(t,t+\delta]}(F) = 0 \quad \textit{for every} \;\; t \in [a,b). \tag{15b}$$

Then F is a regulated function on $[a,b]$.

Proof. Given $t \in (a,b]$ we will prove that $F(t-) \in L(X,Y)$ exists. To this aim, consider an increasing sequence $\{t_n\}_n$ in (a,t) converging to t.

Let $\varepsilon > 0$ be given. By (15a) there exists $\delta > 0$ such that

$$\mathrm{SV}_{[t-\delta,t)}(F) < \varepsilon$$

Moreover, there is $N \in \mathbb{N}$ so that $t_n > t - \delta$ for every $n \geq N$. Thus, for $m > n > N$ and $x \in X$ with $\|x\|_X \leq 1$ we obtain

$$\|[F(t_m) - F(t_n)]x\|_Y \leq \mathrm{SV}_{t-\delta}^{t_m}(F) \leq \mathrm{SV}_{[t-\delta,t)}(F) < \varepsilon$$

which implies that $F(t-)$ exists. Analogously, using (15b), we can show the existence of $F(t+)$ for every $t \in [a,b)$. □

Remark 26. It is not hard to see that, replacing (15a) and (15b) by

$$\lim_{\delta \to 0+} \mathrm{SV}_{t-\delta}^t(F) = 0 \quad \text{and} \quad \lim_{\delta \to 0+} \mathrm{SV}_t^{t+\delta}(F) = 0 \;\; \text{for every} \;\; t \in [a,b],$$

it follows that F is continuous on $[a,b]$.

The next lemma is the analogue of [21, Proposition 4.13] and provides a condition ensuring that (15a) and (15b) hold.

Lemma 27. *If Y is a weakly sequentially complete Banach space, then* (15a) *and* (15b) *are satisfied for every $F \in SV([a,b], L(X,Y))$.*

Proof. By contradiction assume that there exists $F \in SV([a,b], L(X,Y))$ such that for some $t \in (a,b]$ we have $\lim_{\delta\to 0+} \mathrm{SV}_{[t-\delta,t)}(F) = M > 0$. Hence, there is $\delta_1 > 0$ such that

$$\sup_{s\in[t-\delta,t)} \mathrm{SV}_{t-\delta}^{s}(F) = \mathrm{SV}_{[t-\delta,t)}(F) > \frac{M}{2} \quad \text{for } \ 0 < \delta \le \delta_1.$$

Put $s_1 = t - \delta_1$. In view of the inequality above, there exists $s_2 \in (s_1, t)$ so that

$$\mathrm{SV}_{s_1}^{s_2}(F) > \frac{M}{2}.$$

Moreover, $\mathrm{SV}_{[s_2,t)}(F) > \frac{M}{2}$. Thus, we can choose $s_3 \in (s_2, t)$ with

$$\mathrm{SV}_{s_2}^{s_3}(F) > \frac{M}{2} \quad \text{and} \quad \mathrm{SV}_{[s_3,t)}(F) > \frac{M}{2}.$$

If we proceed in this way, we obtain an increasing sequence $\{s_n\}_n$ in (a,t) such that

$$\lim_{n\to\infty} s_n = t \quad \text{and} \quad \mathrm{SV}_{s_n}^{s_{n+1}}(F) > \frac{M}{2}, \quad n \in \mathbb{N}.$$

Having this in mind, for each $n \in \mathbb{N}$, we can find a division D_n of $[s_n, s_{n+1}]$, $D_n = \{\alpha_0^{(n)}, \alpha_1^{(n)}, \dots, \alpha_{\nu_n}^{(n)}\}$, and $x_j^{(n)} \in X$, $j = 1, \dots, \nu_n$ with $\|x_j^{(n)}\|_X \le 1$ such that

$$\Big\| \sum_{j=1}^{\nu_n} [F(\alpha_j^{(n)}) - F(\alpha_{j-1}^{(n)})] x_j^{(n)} \Big\|_Y > \frac{M}{2}.$$

Let

$$y_n = \sum_{j=1}^{\nu_n} [F(\alpha_j^{(n)}) - F(\alpha_{j-1}^{(n)})] x_j^{(n)} \quad \text{for } \ n \in \mathbb{N}.$$

We claim that $\sum_{n=1}^{\infty} |y^*(y_n)| < \infty$ for every $y^* \in Y^*$ with $\|y^*\|_{Y^*} \le 1$. Indeed, given $N \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{n=1}^{N} |y^*(y_n)| &= \sum_{n=1}^{N} \Big| \sum_{j=1}^{\nu_n} y^*\Big([F(\alpha_j^{(n)}) - F(\alpha_{j-1}^{(n)})]x_j^{(n)}\Big)\Big| \\ &\le \sum_{n=1}^{N} \sum_{j=1}^{\nu_n} \big\|y^* \circ F(\alpha_j^{(n)}) - y^* \circ F(\alpha_{j-1}^{(n)})\big\|_{X^*} \\ &\le \sum_{n=1}^{N} \operatorname{var}_{s_n}^{s_{n+1}}(y^* \circ F) = \operatorname{var}_{s_1}^{s_{N+1}}(y^* \circ F),\end{aligned}$$

which together with Theorem 16 leads to

$$\sum_{n=1}^{N} |y^*(y_n)| \le \mathrm{SV}_{s_1}^{s_{N+1}}(F) \le \mathrm{SV}_{[s_1,t)}(F) < \infty.$$

Thus, we conclude that the series $\sum_{n=1}^{\infty} y_n$ is weakly (unconditionally) convergent. Since Y is weakly sequentially complete, from Theorem A10 in the Appendix it follows that $\sum_{n=1}^{\infty} y_n$ converges in Y. This contradicts the fact that $\|y_n\|_Y > \frac{M}{2} > 0$ for every $n \in \mathbb{N}$. In summary, we conclude that (15a) holds for every function from $SV([a,b], L(X,Y))$. Analogously we can show that (15b) is also true. $\square$

The results above, together with [22, Corollary I.3.2], lead to the following conclusion about the continuity of a function of bounded semivariation.

Corollary 28. *Let $F \in SV([a,b], L(X,Y))$. If Y is a weakly sequentially complete Banach space, then F is continuous on $[a,b]$ except at a countable set.*

Remark 29. Recalling that reflexive spaces are weakly sequentially complete, cf. [20, Theorem 2.10.3], both Lemma 27 and Corollary 28 remain valid whenever Y is reflexive.

According to the characterization given in Theorem 16, for every $F \in SV([a,b], L(X,Y))$ and $y^* \in Y^*$, the function $y^* \circ F : [a,b] \to X^*$ is of bounded variation on $[a,b]$. This implies that for each $t \in [a,b]$ both limits

$$\lim_{\delta \to 0+} y^* \circ F(t-\delta) \quad \text{and} \quad \lim_{\delta \to 0+} y^* \circ F(t+\delta)$$

exist in X^*. Such limits can be described by means of an operator mapping X into the second dual Y^{**} of Y. Now, we need to fix some notation to make our statement more precise.

Given $U \in L(X, Y^{**})$ and $y^* \in Y^*$, we can define a linear functional $y^* \bullet U : X \to \mathbb{R}$ by setting $(y^* \bullet U)(x) = \big(U(x)\big)(y^*)$ for every $x \in X$.

Theorem 30. *Let $F \in SV([a,b], L(X,Y)$. Then, for each $t \in (a,b]$ and $s \in [a,b)$, there exist $F(t-)$, $F(s+) \in L(X, Y^{**})$ such that, for every $y^* \in Y^*$,*

$$\lim_{\delta \to 0+} y^* \circ F(t-\delta) = y^* \bullet F(t-) \quad \textit{and} \quad \lim_{\delta \to 0+} y^* \circ F(s+\delta) = y^* \bullet F(s+)$$

where $y^* \circ F$ is as in (9).

Proof. Without loss of generality, let us assume $F(a) = 0$. Given $t \in (a,b]$, for each $y^* \in Y^*$ there exists $T_{y^*} \in X^*$ such that

$$\lim_{\delta \to 0+} y^* \circ F(t-\delta) = T_{y^*}.$$

Let $T : Y^* \to X^*$ be defined by $T(y^*) = T_{y^*}$, $y^* \in Y^*$. Clearly T is linear. Moreover, by Theorem 16, for $\delta > 0$ we have

$$\|y^* \circ F(t-\delta)\|_{X^*} \leq \|y^*\|_{Y^*} \, \mathrm{SV}_a^b(F) \ \text{ for every } \ y^* \in Y^*.$$

Hence $T \in L(Y^*, X^*)$ with $\|T\|_{L(Y^*,X^*)} \leq \mathrm{SV}_a^b(F)$.

Let $T^\times : X \to Y^{**}$ be the mapping which associates to each $x \in X$ the linear functional $x^\times : Y^* \to \mathbb{R}$ given by $x^\times(y^*) = T_{y^*}x$ for $y^* \in Y^*$. Note that, for every $x \in X$ and $y^* \in Y^*$, we have

$$\lim_{\delta \to 0+} (y^* \circ F(t-\delta))x = T_{y^*}x = x^\times(y^*) = (y^* \bullet T^\times)(x).$$

Therefore, $F(t-) = T^\times \in L(X, Y^{**})$ is the desired operator. Similarly, we can construct $F(s+) \in L(X, Y^{**})$ for $s \in [a,b)$. $\square$

The theorem above suggests that functions of bounded semivariation are regulated in some weak sense. For operator-valued functions a more general notion of regulated function can be defined.

Definition 31. Given $F : [a,b] \to L(X,Y)$, we say F is simply regulated on $[a,b]$ if, for each $x \in X$, the function $t \in [a,b] \longmapsto F(t)\,x \in Y$ is regulated. We will denote the set of such functions by $SG([a,b], L(X,Y))$.

From the Banach-Steinhaus Theorem, cf. [20, Theorem 2.11.4], given a function $F \in SG([a,b], L(X,Y))$, for each $t \in (a,b]$ there exists $F(t\dot{-}) \in L(X,Y)$ such that

$$\lim_{s \to t-} F(s)x = F(t\dot{-})x \ \text{ for every } \ x \in X.$$

Analogously, for $t \in [a, b)$, there exists $F(t\dot{+}) \in L(X, Y)$ satisfying

$$\lim_{s \to t+} F(s)x = F(t\dot{+})x \quad \text{for every} \quad x \in X.$$

The concept of simply regulated function appears in the literature under different nomeclatures, for instance, weakly regulated or ($\mathcal{B}$)-regulated with respect to the bilinear triple $\mathcal{B} = (L(X, Y), X, Y)$; see [22] and [34]. Our choice follows the work of Hönig in [23], among other of his publications and followers; cf. [3]. In some sense, such a terminology could be seen as reference to the notion of regulated function in the weak* topology - also known as simple topology.

By the Definition 31, it is clear

$$G([a, b], L(X, Y)) \subset SG([a, b], L(X, Y));$$

for details, see [34, Proposition 3].

Recalling that $BV([a, b], X) \subset G([a, b], X)$, we could expect that a similar relation would hold in the context of functions of bounded semivarition relative to the notion of simply regulated functions defined above. The following example, inspired by [4], shows that this is not the case.

Example 32. Let ℓ_∞ be the Banach space of bounded sequences $x = \{x_n\}_n$ in $\mathbb{R}$, endowed with the usual supremum norm

$$\|x\|_\infty = \sup\{\, |x_n| : n \in \mathbb{N}\}.$$

Denote by e_k, $k \in \mathbb{N}$, the canonical basis of ℓ_∞, where e_k is the sequence which is 1 in the k-th coordinate and null elsewhere. Consider the function $F : [0, 1] \to L(\ell_\infty)$ given by

$$\big(F(t)\big)\, x = \begin{cases} x_1\, e_n & \text{if} \;\; t \in (\frac{1}{n+1}, \frac{1}{n}],\; n \in \mathbb{N}, \\[2ex] 0 & \text{if} \;\; t = 0 \end{cases}$$

for $t \in [0, 1]$ and $x = \{x_n\}_n \in \ell_\infty$.

Note that, for every $k \in \mathbb{N}$,

$$\|[F(\tfrac{1}{k}) - F(\tfrac{1}{k+1})]\, e_1\|_\infty = \|e_k - e_{k+1}\|_\infty = 1.$$

Hence, the limit $\lim_{k\to\infty} \big(F(\frac{1}{k})\big)e_1$ does not exist and, consequently, neither does $F(0+)$. This shows that F is not simply regulated.

Let us prove that $F \in SV([0,1], L(\ell_\infty))$. Given $D \in \mathcal{D}[0,1]$, with $D = \{\alpha_0, \alpha_1, \ldots, \alpha_{\nu(D)}\}$, let $k_j = \max\{k \in \mathbb{N}\,;\, k\alpha_j \leq 1\}$ for $j = 1, \ldots, \nu(D)$. Considering $x_j \in \ell_\infty$, $x_j = \{x_n^{(j)}\}_n$, $j = 1, \ldots, \nu(D)$ with $\|x_j\|_\infty \leq 1$, we have

$$\sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]x_j = x_1^{(1)}\, e_{k_1} + \sum_{j=2}^{\nu(D)} \big[x_1^{(j)}\, e_{k_j} - x_1^{(j)}\, e_{k_{j-1}}\big].$$

Taking $\Lambda = \{j \,:\, k_j \neq k_{j-1}\} \subset \{2, \ldots, \nu(D) - 1\}$, we can write

$$\begin{aligned}
&\sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]x_j \\
&= x_1^{(1)}\, e_{k_1} + \sum_{j\in\Lambda} \big[x_1^{(j)}\, e_{k_j} - x_1^{(j)}\, e_{k_{j-1}}\big] + x_1^{(\nu(D))}\, e_{k_{\nu(D)}} \\
&= \sum_{j\in\Lambda\cup\{1\}} \lambda_j\, e_{k_j} + x_1^{(\nu(D))}\, e_{k_{\nu(D)}}
\end{aligned}$$

where, for each $j \in \Lambda \cup \{1\}$, λ_j corresponds to the difference between two elements of the set $\{x_1^{(i)} \,:\, i = 1, \ldots, \nu(D) - 1\}$. Clearly $|\lambda_j| \leq 2$, and thus

$$\Big\| \sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})]x_j \Big\|_\infty \leq 2$$

which implies that $\mathrm{SV}_0^1(F) < \infty$.

In view of this, a quite natural question arises: Under which conditions is the space $SV([a,b], L(X))$ contained in the set of simply regulated functions?

In [3, Theorem 1] it was proved that the inclusion holds whenever X is a uniformly convex Banach space. Later, a final answer was given in [4], where a necessary and sufficient condition was established.

Aiming to present such result here, we have to consider a very special class of spaces, namely, all Banach spaces which do not contain an isomorphic copy of c_0. Here c_0 denotes the space of sequences in $\mathbb{R}$ converging to zero with respect to the supremum norm. By the Theorem of Bessaga and Pelczinsky, Theorem A9 in the Appendix, the fact that a Banach space X does not contain a copy of c_0 is equivalent to the following property:

(BP) *All series $\sum x_n$ in X such that $\sum |x^*(x_n)| < \infty$ for every $x^* \in X^*$ are unconditionally convergent.*

Using this characterization, we will present the details of relation between the sets $SV([a,b],L(X))$ and $SG([a,b],L(X))$ as described in [4, Theorem 5].

Theorem 33. *The following assertions are equivalent:*

(i) the space X does not contain an isomorphic copy of c_0

(ii) every function $F : [a,b] \to L(X)$ of bounded semivariation is simply regulated.

The proof of this theorem follows from the next two lemmas.

Lemma 34. *If X does not contain an isomorphic copy of c_0, then*

$$SV([a,b],L(X)) \subset SG([a,b],L(X)).$$

Proof. Given $F \in SV([a,b],L(X))$ and $x \in X$, $x \neq 0$, let $F_x : [a,b] \to X$ be the function given by

$$F_x(t) = F(t)\,x \quad \text{for} \quad t \in [a,b].$$

Fix an arbitrary $t \in (a,b]$. To show that the left-sided limit $F_x(t-)$ exists, consider an increasing sequence $\{t_n\}_n$ in (a,t) converging to t.

Let $x^* \in X^*$, and for $N \in \mathbb{N}$, define the division $D_N = \{t_0, t_1, \ldots, t_N, b\}$ of $[a,b]$ where $t_0 = a$ and t_n's are from the given sequence. Then,

$$\sum_{j=1}^{N} |x^*(F_x(t_j) - F_x(t_{j-1}))| = x^* \left(\sum_{j=1}^{N} [F(t_j) - F(t_{j-1})]\,\lambda_j\,x \right)$$

where $\lambda_j = \operatorname{sgn}(x^*(F_x(t_j) - F_x(t_{j-1})))$ for $j = 1, \ldots, N$. If we put $x_j = \frac{\lambda_j\,x}{\|x\|_X}$, we get

$$\begin{aligned}
\sum_{j=1}^{N} |x^*\big(F_x(t_j) - F_x(t_{j-1})\big)| &\leq \|x^*\|_{X^*} \|x\|_X \left\| \sum_{j=1}^{N} [F(t_j) - F(t_{j-1})]\,x_j \right\|_X \\
&\leq \|x^*\|_{X^*}\,\|x\|_X\,\mathrm{SV}_a^b(F).
\end{aligned}$$

Since this inequality is valid for every $N \in \mathbb{N}$, we conclude that

$$\sum_{n=1}^{\infty} |x^*(F_x(t_n) - F_x(t_{n-1}))| < \infty, \quad x^* \in X^*.$$

By the property (BP) of the space X, the series $\sum_{n=1}^{\infty}\big(F_x(t_n)-F_x(t_{n-1})\big)$ converges to some $z\in X$ and, consequently,

$$\lim_{n\to\infty} F(t_n)\,x=\lim_{n\to\infty}\sum_{k=1}^{n}\big(F_x(t_k)-F_x(t_{k-1})\big)+F_x(a)=z+F(a)\,x.$$

It remains to show that the limit does not depend on the choice of the sequence $\{t_n\}_n$. To this aim, let $\{s_n\}_n$ be another increasing sequence with $\lim_{n\to\infty} s_n=t$. By the same argument used above, there exists $\tilde{z}\in X$ such that

$$\tilde{z}=\sum_{n=1}^{\infty}\big(F_x(s_n)-F_x(s_{n-1})\big)\quad\text{and}\quad\lim_{n\to\infty}F_x(s_n)=\tilde{z}+F(a)\,x.$$

Ordering the set $\{t_n : n\in\mathbb{N}\}\cup\{s_n : n\in\mathbb{N}\}$ we obtain an increasing sequence $\{r_n\}_n$ converging to t whose series $\sum_{n=1}^{\infty}\big(F_x(r_n)-F_x(r_{n-1})\big)$ also converges. Moreover, the limit $\lim_{n\to\infty}F_x(r_n)$ exists. Since $\{F_x(t_n)\}_n$ and $\{F_x(s_n)\}_n$ are convergent subsequences of $\{F_x(r_n)\}_n$, we have

$$\lim_{n\to\infty}F_x(t_n)=\lim_{n\to\infty}F_x(r_n)=\lim_{n\to\infty}F_x(s_n).$$

This proves that $\tilde{z}=z$ and $F_x(t-)=z+F(a)\,x$.

Similarly, we can show that the right-sided limit of F_x exists for every $t\in[a,b)$. □

The second lemma gives the reverse implication from Theorem 33. Roughly speaking, we will show that if c_0 is isomorphically embedded into the space Y, one can construct a function $F:[a,b]\to L(X)$ of bounded semivariation which is not simply regulated.

Lemma 35. *If $SV([a,b],L(X))\subset SG([a,b],L(X))$, then X does not contain a copy of c_0.*

Proof. To the contrary, assume that X contains an isomorphic copy of c_0 and denote it by Z. Let $\psi : c_0\to Z$ be an isomorphism and put $z_k:=\psi(e_k)$ where e_k, $k\in\mathbb{N}$, stands for the canonical Schauder basis of c_0.

It is known that there exist positive constants C_1 and C_2 such that, for $N\in\mathbb{N}$, and taking $\lambda_j\in\mathbb{R}$, $j=1,\dots,N$, we have

$$C_1\sup_{1\le j\le N}|\lambda_j|\le\Big\|\sum_{j=1}^{N}\lambda_j\,e_j\Big\|_\infty\le C_2\sup_{1\le j\le N}|\lambda_j|.$$

See [25, Theorem 6.3.1] or [8, Theorem V.6]. Thus, by the fact that Z and c_0 are isomorphic,

$$C_1 \sup_{1\le j\le N} |\lambda_j| \le \Big\| \sum_{j=1}^{N} \lambda_j\, z_j \Big\|_X \le C_2 \sup_{1\le j\le N} |\lambda_j|, \tag{16}$$

for $N \in \mathbb{N}$ and $\lambda_j \in \mathbb{R}$, $j = 1, \dots, N$.

Using the sequence z_n, $n \in \mathbb{N}$, mentioned above and its properties we will construct a function $F : [a, b] \to L(X)$ in a few steps.

Step 1. Clearly, z_n, $n \in \mathbb{N}$, defines a basis for Z and, for each $k \in \mathbb{N}$, the projection $\pi_k : Z \to \mathbb{R}$, given by $\pi_k(\sum_n \lambda_n\, z_n) = \lambda_k$, is continuous. See [8, p. 32]. Since (16) implies that

$$\big|\pi_k\Big(\sum_{n=1}^{N} z_n\Big)\big| \le \frac{1}{C_1}\Big\| \sum_{n=1}^{N} z_n \Big\|_Z, \quad N \in \mathbb{N},$$

we have $\|\pi_k\|_{Z^*} \le \frac{1}{C_1}$ for every $k \in \mathbb{N}$.

Step 2. For $k \in \mathbb{N}$, let $S_k : Z \to Z$ be given by

$$S_k(x) = \sum_{n=1}^{k} \pi_n(x)\, z_{2^k+n}, \quad \text{for} \ \ x \in Z.$$

Note that S_k is a bounded linear operator on Z for every $k \in \mathbb{N}$. Indeed, given $x \in Z$, we can write

$$S_k(x) = \sum_{n=1}^{k} \pi_n(x)\, z_{2^k+n} = \sum_{j=1}^{2^k+k} \beta_j z_j$$

where $\beta_j = \pi_n(x)$ if $j = 2^k + n$ for some $n = 1, \dots, k$, and $\beta_j = 0$ otherwise. Thus, by (16),

$$\|S_k(x)\|_Z \le C_2 \sup_{1\le j\le 2^k+k} |\beta_j| = C_2 \sup_{1\le n\le k} |\pi_n(x)| \le C_2 \sup_{1\le n\le k} \|\pi_n\|_{Z^*}\|x\|_Z,$$

which implies that $\|S_k\|_{L(Z)} \le \frac{C_2}{C_1}$ for every $k \in \mathbb{N}$.

Step 3. Given $j, k \in \mathbb{N}$, put $f_{k,j} = \pi_j \circ S_k$. By the Hahn-Banach theorem, the functional $f_{k,j} \in Z^*$ can be extended to a continuous linear functional $\tilde{f}_{k,j}$ on X satisfying

$$\|\tilde{f}_{k,j}\|_{X^*} = \|f_{k,j}\|_{Z^*} \le \|\pi_j\|_{Z^*}\|S_k\|_{L(Z)} \le \frac{C_2}{(C_1)^2}. \tag{17}$$

Step 4. For $x \in X$ and $k \in \mathbb{N}$, let $T_k(x) = \sum_{j=1}^{k} \tilde{f}_{k,2^k+j}(x)\, z_{2^k+j}$. Clearly, $T_k \in L(X)$ and it follows from (16) and (17) that for each $x \in X$,

$$\|T_k(x)\|_X = \Big\| \sum_{j=1}^{k} \tilde{f}_{k,2^k+j}(x)\, z_{2^k+n} \Big\|_X \le C_2 \sup_{1\le j\le k} |\tilde{f}_{k,2^k+j}(x)| \le \Big(\frac{C_2}{C_1}\Big)^2 \|x\|_X.$$

That is, $\|T_k\|_{L(X)} \le \big(\frac{C_2}{C_1}\big)^2$ for all $k \in \mathbb{N}$.

We are now ready to define $F : [a,b] \to L(X)$. Given $x \in X$, let

$$F(t)x = T_k(x) \quad \text{for} \quad t \in (t_{k+1}, t_k]$$

where $t_k = a + \frac{(b-a)}{k}$, $k \in \mathbb{N}$.

It is not hard to see that F is not simply regulated. Indeed, for each $k \in \mathbb{N}$, and noting that $S_k(z_1) = z_{2^k+1}$, we get

$$F(t_k) z_1 = T_k(z_1) = \sum_{j=1}^{k} f_{k,2^k+j}(z_1)\, z_{2^k+j} = \sum_{j=1}^{k} \pi_{2^k+j}\big(S_k(z_1)\big) z_{2^k+j} = z_{2^k+1}.$$

Using (16) this leads to

$$\|[F(t_k) - F(t_{k+1})]\, z_1\|_X = \|z_{2^k+1} - z_{2^{k+1}+1}\|_X \ge C_1.$$

Hence the limit, $\lim_{t\to a+} F(t)\, z_1$ does not exist.

Now, we will show that $F \in SV([a,b], L(X))$. Considering $D \in \mathcal{D}[a,b]$, with $D = \{\alpha_0, \alpha_1, \ldots, \alpha_{\nu(D)}\}$, let $k_j \in \mathbb{N}$ be such that $\alpha_j \in (t_{k_j+1}, t_{k_j}]$, $j = 1, \ldots, \nu(D)$. For $x_j \in X$, $j = 1, \ldots, \nu(D)$ with $\|x_j\|_X \le 1$ we have

$$\sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})] x_j = T_{k_1}(x_1) + \sum_{j=2}^{\nu(D)} [T_{k_j}(x_j) - T_{k_{j-1}}(x_j)].$$

Taking $\Lambda = \{j : k_j \ne k_{j-1}\} \subset \{2, \ldots, \nu(D) - 1\}$, we can write

$$\sum_{j=1}^{\nu(D)} [F(\alpha_j) - F(\alpha_{j-1})] x_j = \sum_{j\in\Lambda\cup\{1\}} T_{k_j}(y_j) + T_{k_{\nu(D)}}(x_{\nu(D)}).$$

Here, for each $j \in \Lambda \cup \{1\}$, $y_j \in X$ corresponds to the difference between two elements of the set $\{x_i : i = 1, \ldots, \nu(D) - 1\}$. Noting that $\|y_j\|_X \le 2$, by

(16) and (17), it follows that

$$\Big\| \sum_{j\in\Lambda\cup\{1\}} T_{k_j}(y_j)\Big\|_X = \Big\| \sum_{j\in\Lambda\cup\{1\}} \sum_{n=1}^{k_j} \tilde{f}_{k_j,2^{k_j}+n}(y_j)\, z_{2^{k_j}+n}\Big\|_X$$
$$\le C_2 \sup_{\substack{1\le n\le k_j\\ j\in\Lambda\cup\{1\}}} |\tilde{f}_{k_j,2^{k_j}+n}(y_j)| \le 2\Big(\frac{C_2}{C_1}\Big)^2.$$

Therefore,

$$\Big\| \sum_{j=1}^{\nu(D)} [F(\alpha_j)-F(\alpha_{j-1})]\, x_j\Big\|_X \le 2\Big(\frac{C_2}{C_1}\Big)^2 + \|T_{k_{\nu(D)}}(x_{\nu(D)})\|_X \le 3\Big(\frac{C_2}{C_1}\Big)^2$$

wherefrom it follows that $\mathrm{SV}_a^b(F) < \infty$.

In summary, $F \in SV([a,b], L(X))$ and F is not simply regulated, which is a contradiction. Thus, the lemma is established. □

5 Semivariation and the Kurzweil integral

In the recent years non-absolute integrals have been increasingly investigated. Among them it is worth highlighting the one due to Kurzweil, [27] and [28], whose concept of integration has been the background of several papers related to differential and difference equations. See, for instance, [39], [16] and [29].

This section is dedicated to investigate the connection between the semivariation and the integral due to Kurzweil in two different aspects. First, we present a result by Hönig which generalizes the following fact:

> *Every function of bounded variation is a multiplier for Kurzweil integrable functions.*

Next, we apply the concept of semivariation to derive two convergence results for Stieltjes type integral, and we conclude the section by proving a new characterization of semivariation by the means of the abstract Kurzweil-Stieltjes integral.

In what follows we deal with special cases of the integral introduced by J. Kurzweil in [27] under the name "generalized Perron integral". For the reader's convenience, let us recall its definition.

As usual, a *partition of* $[a,b]$ is a tagged division $P = (\tau_j, [\alpha_{j-1}, \alpha_j])$ where the set $\{\alpha_0, \alpha_1, \dots, \alpha_{\nu(P)}\}$ is a division of $[a,b]$ and $\tau_j \in [\alpha_{j-1}, \alpha_j]$ for $j = 1, \dots, \nu(P)$. A *gauge on* $[a,b]$ is a positive function $\delta : [a,b] \to \mathbb{R}^+$.

Furthermore, given a gauge δ on $[a,b]$, a partition $P=(\tau_j,[\alpha_{j-1},\alpha_j])$ is called δ-fine if

$$[\alpha_{j-1},\alpha_j]\subset(\tau_j-\delta(\tau_j),\tau_j+\delta(\tau_j))\quad\text{for}\quad j=1,\dots,\nu(P).$$

Given an arbitrary gauge δ on $[a,b]$, the existence of (at least one) δ-fine partition is the so-called Cousin's Lemma; see, for instance, [19, Theorem 4.1] or [32, Lemma 1.4].

Definition 36. A function $U:[a,b]\times[a,b]\to X$ is Kurzweil integrable on $[a,b]$, if there exists $I\in X$ such that for every $\varepsilon>0$, there is a gauge δ on $[a,b]$ such that

$$\Big\|\sum_{j=1}^{\nu(P)}[U(\tau_j,\alpha_j)-U(\tau_j,\alpha_{j-1})]-I\Big\|_X<\varepsilon\;\text{ for all }\delta\text{-fine partitions of }[a,b].$$

In this case, we define the Kurzweil integral as $\int_a^b DU(\tau,t)=I$.

For a more comprehensive study of the properties of the Kurzweil integral we refer to the monograph [32] and the references therein.

Taking $U(\tau,t)=f(\tau)\,t$ for $t\in[a,b]$, where $f:[a,b]\to X$ is a given function, the definition above corresponds to an integration process based on Riemann-type sums, namely, the Henstock-Kurzweil integral. Such integral is known to extend the theory of Lebesgue integral. In what follows, when dealing with the Kurzweil-Henstock integral we will write simply $\int_a^b f(t)\mathrm{d}t$ instead of $\int_a^b D[f(\tau)t]$.

Secondly, we are interested in the abstract Kurzweil-Stieltjes integrals $\int_a^b F\,\mathrm{d}[g]$ and $\int_a^b \mathrm{d}[F]\,g$, where $F:[a,b]\to L(X)$ and $g:[a,b]\to X$; see [34]. These integrals are obtained by choosing $U(\tau,t)=F(\tau)\,g(t)$ and $U(\tau,t)=F(t)\,g(\tau)$ for $t,\tau\in[a,b]$, respectively.

In the sequel we state two existence results for the abstract Kurzweil-Stieltjes integral relative to functions of bounded semivariation. For the proof see [30, Thereom 3.3].

Theorem 37. *Let $F\in SV([a,b],L(X,Y))$.*

1. *If $g\in G([a,b],X)$, then the integral $\int_a^b F\,\mathrm{d}[g]$ exists and*

$$\Big\|\int_a^b F\,\mathrm{d}[g]\Big\|_X\le\big(\|F(a)\|_X+\|F(b)\|_X+\mathrm{SV}_a^b(F)\big)\,\|g\|_\infty.$$

2. *If $F \in SG([a,b],L(X))$ and $g \in G([a,b],X)$, then the integral $\int_a^b \mathrm{d}[F]g$ exists and*
$$\Big\| \int_a^b \mathrm{d}[F]\, g \Big\|_X \leq \mathrm{SV}_a^b(F)\, \|g\|_\infty.$$

Let us denote by $\mathcal{K}([a,b],X)$ the set of all Henstock-Kurzweil integrable functions, that is, all functions $f:[a,b]\to X$ whose integral $\int_a^b f(t)\mathrm{d}t$ exists. The linearity of the integral implies that $\mathcal{K}([a,b],X)$ is a linear space; cf. [17] or [32].

The following theorem, borrowed from [24, 1.15], shows that the functions of bounded semivariation are multipliers for the space $\mathcal{K}([a,b],X)$.

Theorem 38. *Let $g \in \mathcal{K}([a,b],X)$ and $F \in SV([a,b],L(X))$. Consider the function $Fg:[a,b]\to X$ given by $(Fg)(t)=F(t)g(t)$ for $t\in[a,b]$. Then $Fg \in \mathcal{K}([a,b],X)$ and*
$$\int_a^b F(t)g(t)\,\mathrm{d}t = \int_a^b F\,\mathrm{d}[\tilde{g}], \tag{18}$$
where $\tilde{g}(t)=\int_a^t g(s)\,\mathrm{d}s$ for $t\in[a,b]$.

Proof. First of all, the indefinite integral of g defines a continuous function on $[a,b]$, see [17, Theorem 9.12]. Hence by Theorem 37, $\int_a^b F\,\mathrm{d}\tilde{g}$ exists.

Given $\varepsilon>0$, let δ_1 and δ_2 be gauges on $[a,b]$ such that
$$\Big\| \sum_{j=1}^{\nu(P)} F(\tau_j)[\tilde{g}(\alpha_j)-\tilde{g}(\alpha_{j-1})] - \int_a^b F\,\mathrm{d}[\tilde{g}]\Big\|_X < \varepsilon \tag{19}$$
for all δ_1-fine partitions of $[a,b]$, and
$$\Big\| \sum_{j=1}^{\nu(P)} g(\tau_j)(\alpha_j-\alpha_{j-1}) - \int_a^b g(s)\,\mathrm{d}s\Big\|_X < \varepsilon$$
for all δ_2-fine partitions of $[a,b]$.

Due to the Saks-Henstock Lemma, [32, Lemma 1.13], for any δ_2-fine partition of $[a,b]$, $P=(\tau_j,[\alpha_{j-1},\alpha_j])$, we have
$$\Big\| \sum_{k=j}^{\nu(P)} \Big[g(\tau_k)(\alpha_k-\alpha_{k-1}) - \int_{\alpha_{k-1}}^{\alpha_k} g(s)\,\mathrm{d}s\Big]\Big\|_X < \varepsilon \tag{20}$$

for each $j = 1, 2, \ldots, \nu(P)$.

Put $\delta(t) = \min\{\delta_1(t), \delta_2(t)\}$ for $t \in [a,b]$. Given a δ-fine partition $P = (\tau_j, [\alpha_{j-1}, \alpha_j])$ of $[a,b]$, by (19) we get

$$\begin{aligned}
&\Big\| \sum_{j=1}^{\nu(P)} F(\tau_j) g(\tau_j)(\alpha_j - \alpha_{j-1}) - \int_a^b F\,\mathrm{d}[\tilde{g}] \Big\|_X \\
&\quad \leq \Big\| \sum_{j=1}^{\nu(P)} F(\tau_j) g(\tau_j)(\alpha_j - \alpha_{j-1}) - \sum_{j=1}^{\nu(P)} F(\tau_j)[\tilde{g}(\alpha_j) - \tilde{g}(\alpha_{j-1})] \Big\|_X \\
&\qquad + \Big\| \sum_{j=1}^{\nu(P)} F(\tau_j)[\tilde{g}(\alpha_j) - \tilde{g}(\alpha_{j-1})] - \int_a^b F\,\mathrm{d}[\tilde{g}] \Big\|_X \\
&\quad < \Big\| \sum_{j=1}^{\nu(P)} F(\tau_j) \Big[g(\tau_j)(\alpha_j - \alpha_{j-1}) - \int_{\alpha_{j-1}}^{\alpha_j} g(s)\mathrm{d}s \Big] \Big\|_X + \varepsilon.
\end{aligned}$$

In order to estimate the other term in the last inequality, we will make use of the following equality mentioned in [24, 1.15]:

$$\sum_{j=1}^{m} A_j\, x_j = \sum_{j=1}^{m} [A_j - A_{j-1}] \left(\sum_{k=j}^{m} x_k \right) + A_0 \left(\sum_{k=1}^{m} x_k \right)$$

for all $A_j \in L(X)$ and all $x_j \in X$. Let us consider $m = \nu(P)$ and also

$$A_0 = F(a), \quad A_j = F(\tau_j), \quad x_j = g(\tau_j)(\alpha_j - \alpha_{j-1}) - \int_{\alpha_{j-1}}^{\alpha_j} g(s)\mathrm{d}s$$

for $j = 1, \ldots \nu(P)$. Note that, by (20) we have $\Big\| \sum_{k=j}^{\nu(P)} x_k \Big\|_X \leq \varepsilon$ for each $j = 1, \ldots \nu(P)$. Therefore,

$$\begin{aligned}
&\Big\| \sum_{j=1}^{\nu(P)} F(\tau_j) \Big[g(\tau_j)(\alpha_j - \alpha_{j-1}) - \int_{\alpha_{j-1}}^{\alpha_j} g(s)\mathrm{d}s \Big] \Big\|_X \\
&\qquad < \varepsilon \Big\| \sum_{j=1}^{\nu(P)} [F(\tau_j) - F(\tau_{j-1})] \, \frac{\sum_{k=j}^{\nu(P)} x_k}{\varepsilon} \Big\|_X + \Big\| F(a) \Big(\sum_{j=1}^{\nu(P)} x_j \Big) \Big\|_X \\
&\qquad < \varepsilon \left(\mathrm{SV}_a^b(F) + \|F(a)\|_{L(X)} \right) = \varepsilon \, \|F\|_{SV}
\end{aligned}$$

(where $\tau_0 = a$). Having all these in mind, we obtain

$$\Big\| \sum_{j=1}^{\nu(P)} F(\tau_j) g(\tau_j)(\alpha_j - \alpha_{j-1}) - \int_a^b F\,\mathrm{d}[\tilde{g}] \Big\|_X < \varepsilon \left(1 + \|F\|_{SV} \right)$$

for all δ-fine partitions of $[a,b]$. We conclude that $\int_a^b F(t)\,g(t)$ exists and the unicity of the integral leads to (18). $\square$

Remark 39. The theorem above can be regarded as a special case of substitution formulas for the Kurzweil integral, see, for instance, [14, Theorem 11]. In [24], this result is presented when integration by parts formulas for Henstock-Kurzweil integral are discussed. Indeed, taking into account the results from [36], see also [30, Corollary 3.6], the equality (18) can be rewritten as

$$\int_a^b F(t)g(t)\,\mathrm{d}t = F(b)\tilde{g}(b) - \int_a^b \mathrm{d}[F]\,\tilde{g}.$$

Moreover, due to the continuity of the function $\tilde{g}$, the Stieltjes-type integral in the formula above and also in (18) can be read as a Riemann-Stieltjes integral defined in the Banach space setting, see [24, 1.13].

We also remark that the result in Theorem 38 remains valid if we replace the function $g : [a,b] \to X$ by Henstock-Kurzweil integrable functions defined in $[a,b]$ and taking values in $L(X)$.

Now we turn our attention to the connection between semivariation and the abstract Kurzweil-Stieltjes integral. First, we focus on Helly type results, that is, convergence results for the integral based on assumptions similar to those presented in Lemma 8. The theorem in the sequel, to our knowlegde, is not available in literature in the present formulation.

Theorem 40. *Let $F : [a,b] \to L(X)$. Assume that a sequence $\{F_n\}_n$ in $SV([a,b], L(X))$ and a constant $M > 0$ are such that*

$$\mathrm{SV}_a^b(F_n) \le M \quad \textit{for every} \;\; n \in \mathbb{N},$$

and

$$\lim_{n\to\infty} \|F_n(t) - F(t)\|_{L(X)} = 0 \quad \textit{for every} \;\; t \in [a,b].$$

If $g \in G([a,b], X)$, then the integrals $\int_a^b F\,\mathrm{d}[g]$ and $\int_a^b F_n\,\mathrm{d}[g]$, $n \in \mathbb{N}$, exist and

$$\lim_{n\to\infty} \int_a^b F_n\,\mathrm{d}[g] = \int_a^b F\,\mathrm{d}[g]. \tag{21}$$

Proof. By Lemma 8 we know that $F \in SV([a,b], L(X))$, thus the existence of the integrals is guaranteed by Theorem 37.

To prove the convergence, we first consider the case in which g is a finite step function. Due to the linearity of the integral, it is enough to show that

(21) holds for functions of the form $\chi_{[a,\tau]}x$, $\chi_{[\tau,b]}x$, $\chi_{[a]}x$ and $\chi_{[b]}x$, where $\tau \in (a,b)$ and $x \in X$.

Given $\tau \in [a,b)$ and $x \in X$, using an obvious extension of [42, Proposition 2.3.3] to Banach spaces-valued functions, we have

$$\int_a^b (F_n - F)\,\mathrm{d}[\chi_{[a,\tau]}x] = F(\tau)x - F_n(\tau)x.$$

Hence (21) follows. Similarly, one can prove the equality for $\chi_{[\tau,b]}x$, $\chi_{[a]}x$ and $\chi_{[b]}x$.

Now, assuming $g \in G([a,b],X)$ and given $\varepsilon > 0$, there exists a finite step function $\varphi : [a,b] \to X$ such that $\|g-\varphi\|_\infty < \varepsilon$, see [22, Theorem I.3.1]. Let $n_0 \in \mathbb{N}$ be such that

$$\|(F_n - F)(a)\| + \|(F_n - F)(b)\| < M \quad \text{and} \quad \Big\| \int_a^b (F_n - F)\, d[\varphi] \Big\| < \varepsilon$$

for $n > n_0$. These inequalities, together with (2) and Theorem 37, imply that

$$\begin{aligned}
&\Big\| \int_a^b (F_n - F)\,\mathrm{d}[g] \Big\|_X \\
&\le \Big\| \int_a^b (F_n - F)\,\mathrm{d}[g-\varphi] \Big\|_X + \Big\| \int_a^b (F_n - F)\,\mathrm{d}[\varphi] \Big\|_X \\
&\le \big(\|(F_n - F)(a)\|_X + \|(F_n - F)(b)\|_X + \mathrm{SV}_a^b(F_n - F) \big)\, \|g-\varphi\|_\infty + \varepsilon \\
&< \big(M + \mathrm{SV}_a^b(F_n) + \mathrm{SV}_a^b(F) \big)\,\varepsilon + \varepsilon < \varepsilon(3\,M + 1)
\end{aligned}$$

for every $n > n_0$, which proves (21). □

We remark that in [33, Proposition 2] the convergence result above is proved for real-valued functions of bounded variation.

Still a Helly type result, the following theorem concerns integrals of the form $\int_a^b \mathrm{d}[F]\, g$.

Theorem 41. *Let $F : [a,b] \to L(X)$. Assume that a sequence $\{F_n\}_n$ in $SV([a,b], L(X)) \cap SG([a,b], L(X))$ and a constant $M > 0$ are such that*

$$\mathrm{SV}_a^b(F_n) \le M \quad \textit{for every} \;\; n \in \mathbb{N},$$

and

$$\lim_{n\to\infty} \Big(\sup_{t\in[a,b]} \|F_n(t)x - F(t)x\|_X \Big) = 0 \quad \textit{for every} \;\; x \in X. \tag{22}$$

If $g \in G([a,b],X)$, then the integrals $\int_a^b \mathrm{d}[F]\,g$ and $\int_a^b \mathrm{d}[F_n]\,g$, $n \in \mathbb{N}$, exist and

$$\lim_{n\to\infty} \int_a^b \mathrm{d}[F_n]\,g = \int_a^b \mathrm{d}[F]\,g. \tag{23}$$

Proof. Given $x \in X$, for each $n \in \mathbb{N}$ put $(F_n)_x : t \in [a,b] \longmapsto F_n(t)x \in X$. By (22) it follows that the sequence of regulated functions $\{(F_n)_x\}_n$ converges uniformly in $[a,b]$ to $F_x : t \in [a,b] \longmapsto F(t)x \in X$. Hence, by [22, I.3.5] the function F_x is regulated and, since this holds for each $x \in X$, we have $F \in SG([a,b],L(X))$. Due to Lemma 8, $F \in SV([a,b],L(X))$, thus the existence of the integral $\int_a^b \mathrm{d}[F]\,g$, as well as the existence of $\int_a^b \mathrm{d}[F_n]\,g$, $n \in \mathbb{N}$, follows from Theorem 37.

In order to prove (23), we first consider the case when $g(t) = \chi_{[a,\tau]}(t)\tilde{x}$ for $t \in [a,b]$, where $\tau \in (a,b)$ and $\tilde{x} \in X$ are arbitrarily fixed. For each $n \in \mathbb{N}$, by [34, Proposition 14] we have

$$\int_a^b \mathrm{d}[F_n - F]g = \lim_{s\to\tau-} F_n(s)\tilde{x} - \lim_{s\to\tau-} F(s)\tilde{x} - [F_n(a) - F(a)]\tilde{x},$$

or equivalently,

$$\int_a^b \mathrm{d}[F_n - F]g = F_n(\tau\dot{-})\tilde{x} - F(\tau\dot{-})\tilde{x} - [F_n(a) - F(a)]\tilde{x}. \tag{24}$$

Here $F_n(\tau\dot{-})$, $F(\tau\dot{-}) \in L(X)$ are operators satisfying

$$\lim_{s\to\tau-} F_n(s)x = F_n(\tau\dot{-})x \quad \text{and} \quad \lim_{s\to\tau-} F(s)x = F(\tau\dot{-})x$$

for every $x \in X$. Given $\varepsilon > 0$, by (22) there exists $n_0 \in \mathbb{N}$ such that

$$\|[F_n(t) - F(t)]\tilde{x}\|_X < \frac{\varepsilon}{3} \quad \text{for} \quad n \ge n_0 \quad \text{and} \quad t \in [a,b]. \tag{25}$$

Let $n \ge n_0$ and choose $\delta > 0$ such that

$$\|F_n(\tau\dot{-})\tilde{x} - F_n(s)\tilde{x}\|_X \le \frac{\varepsilon}{3} \quad \text{and} \quad \|F_n(\tau\dot{-})\tilde{x} - F_n(s)\tilde{x}\|_X \le \frac{\varepsilon}{3}. \tag{26}$$

Fixed an arbitrary $s \in (\tau - \delta, \tau)$, it follows from (25) and (26) that

$$\begin{aligned}\|F_n(\tau\dot{-})\tilde{x} - F(\tau\dot{-})\tilde{x}\|_X &\le \|F_n(\tau\dot{-})\tilde{x} - F_n(s)\tilde{x}\|_X \\ &\quad + \|F_n(s)\tilde{x} - F(s)\tilde{x}\|_X + \|F(s)\tilde{x} - F(\tau\dot{-})\tilde{x}\|_X < \varepsilon,\end{aligned}$$

which together with (25) applied to $t = a$, shows that the integral in (24) tends to zero. With similar argument we can prove that (23) holds when g is a function of the form $\chi_{[\tau,b]}x$, $\chi_{[a]}x$ and $\chi_{[b]}x$ for $\tau \in (a,b)$ and $x \in X$. As a consequence of the linearity of the integral we conclude that (23) is valid if g is a step function.

Now, assuming that $g \in G([a,b],X)$ and given $\varepsilon > 0$, let $\varphi : [a,b] \to X$ be a finite step function such that $\|g-\varphi\|_\infty < \varepsilon$; see [22, Theorem I.3.1]. Thus, by Theorem 37 we have

$$\begin{aligned}\Big\| \int_a^b \mathrm{d}[F_n - F]\, g \Big\|_X &\le \Big\| \int_a^b \mathrm{d}[F_n - F]\, (g-\varphi) \Big\|_X + \Big\| \int_a^b \mathrm{d}[F_n - F]\, \varphi \Big\|_X \\ &\le \mathrm{SV}_a^b(F_n - F)\|g-\varphi\|_\infty + \Big\| \int_a^b \mathrm{d}[F_n - F]\, \varphi \Big\|_X \\ &\le 2\, M\varepsilon + \Big\| \int_a^b \mathrm{d}[F_n - F]\, \varphi \Big\|_X .\end{aligned}$$

Since φ is a step function, the result now follows from first part of the proof. $\square$

Similar convergence results have been proved in [31] and [29] in the context of functions of bounded variation.

In the literature various notions of variation are sometimes described by the means of different integrals of the Stieltjes type. In [5], using the Young integral on Hilbert spaces, not only is a characterization for the norm $\|\cdot\|_{BV}$ presented but also the notion of essential variation is treated. This is also the case with the semivariation and the interior integral, sometimes referred to as the Dushnik integral, and in this regard it is worth highlighting [22, Corollary I.5.2].

Inspired by those results, we present here a characterization of semivariation via the abstract Kurzweil-Stieltjes integral. To this end, we will need the following estimates whose proofs are quite similar to [30, Lemma 3.1].

Lemma 42. *Let $F : [a,b] \to L(X)$ and $g : [a,b] \to X$ be given. For every partition $P = (\tau_j, [\alpha_{j-1}, \alpha_j])$ of $[a,b]$ we have*

$$\Bigg\| F(b)\, g(b) - \sum_{j=1}^{\nu(P)} F(\tau_j)[g(\alpha_j) - g(\alpha_{j-1})] \Bigg\|_X \le \|F(a)\, g(a)\|_X + \|g\|_\infty\, \mathrm{SV}_a^b(F).$$

Furthermore, if $\int_a^b F\,\mathrm{d}[g]$ exists then

$$\Big\| F(b)\, g(b) - \int_a^b F\,\mathrm{d}[g] \Big\|_X \le \|F(a)\, g(a)\|_X + \|g\|_\infty\, \mathrm{SV}_a^b(F). \tag{27}$$

Now we present the main result of this section. In what follows, we use $S_L([a,b],X)$ to denote the set of all finite step functions $g:[a,b]\to X$ which are left-continuous on $(a,b]$ and such that $g(a)=0$.

Theorem 43. *If $F\in SV([a,b],L(X))$, then*

$$\mathrm{SV}_a^b(F)=\sup\left\{\left\|F(b)\,g(b)-\int_a^b F\,\mathrm{d}[g]\right\|_X;\ g\in S_L([a,b],X),\ \|g\|_\infty\le 1\right\}.$$

Proof. At first, note that by (27), $\mathrm{SV}_a^b(F)$ is an upper bound of the set

$$\mathcal{A}:=\left\{\left\|F(b)\,g(b)-\int_a^b F\,\mathrm{d}[g]\right\|_X;\ g\in S_L([a,b],X),\ \|g\|_\infty\le 1\right\}.$$

To conclude the proof it is enough to show that $\mathrm{SV}_a^b(F)\le\sup\mathcal{A}$.

Let $\varepsilon>0$ be given. Then, there exist a division $D=\{\alpha_0,\alpha_1,\dots,\alpha_m\}$ of $[a,b]$ and $x_j\in X$, $j=1,\dots,m$ with $\|x_j\|\le 1$ such that

$$\mathrm{SV}_a^b(F)-\varepsilon<\left\|\sum_{j=1}^m[F(\alpha_j)-F(\alpha_{j-1})]\,x_j\right\|_X.$$

Let $\tilde g:[a,b]\to X$ be the function given by

$$\tilde g(t)=\sum_{j=1}^m\chi_{(\alpha_{j-1},\alpha_j]}(t)\,x_j\quad\text{for}\quad t\in[a,b].$$

Thus, $\tilde g$ is a left continuous step function with $\tilde g(a)=0$ and $\|\tilde g\|_\infty\le 1$, that is, $\tilde g\in S_L([a,b],X)$. Using [34, Proposition 14] we calculate that

$$\int_a^b F\,\mathrm{d}[\tilde g]=-\sum_{j=1}^{m-1}[F(\alpha_j)-F(\alpha_{j-1})]\,x_j+F(\alpha_{m-1})\,x_m.$$

Therefore,

$$\begin{aligned}\mathrm{SV}_a^b(F)-\varepsilon&<\left\|\sum_{j=1}^m[F(\alpha_j)-F(\alpha_{j-1})]\,x_j\right\|_X\\&=\left\|F(b)\,\tilde g(b)-\int_a^b F\,\mathrm{d}[\tilde g]\right\|_X\le\sup\mathcal{A}.\end{aligned}$$

Since $\varepsilon>0$ is arbitrary, the result follows. □

6 Concluding remarks

This paper provides an accessible summary of the notion of semivariation by presenting its properties together with a few illustrative examples. Based on Thorp's paper [40], the relation between the notion of semivariation and variation in the sense of Jordan was described in detail. In addition, following [21] and [4], we have investigated continuity properties inherent to functions of bounded semivariation. In closing the survey, we have shown the applicability of the concept of semivariation in the integration theory of Kurzweil, where, among other results, a new characterization is given.

Having in mind how well developed the theory of functions of bounded variation and its generalizations (see [6], [2] and the references in therein) are, it becomes clear that there is a great deal to be done concerning semivariation. An interesting point to consider is how some results are interconnected to specific characteristics of the codomain; that is, characteristics of the space Y in $L(X,Y)$. This is the case, for instance, in Corollary 28 and Theorem 33.

This suggests the following open problem:

> *Which other properties of the functions of bounded semivariation are directly connected to the nature of the space Y?*

A Appendix: Series in Banach space

Definition A1. Let $x_n \in X$ for $n \in \mathbb{N}$. We say that:

1. The series $\sum_{n=1}^{\infty} x_n$ is convergent if the sequence of its partial sums $s_n = \sum_{k=1}^{n} x_k$ converges in X.
2. The series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent if $\sum_{n=1}^{\infty} \|x_n\|_X < \infty$.
3. The series $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent if the series $\sum_{n=1}^{\infty} x_{\pi(n)}$ converges in X for any permutation π of $\mathbb{N}$.

Theorem A2. *If $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent, then all rearrangements have the same sum.* See [25, Theorem 1.3.1].

Theorem A3. *For the series* $\sum_{n=1}^{\infty} x_n$ *in* X *the following conditions are equivalent:*

(i) *the series is unconditionally convergent;*

(ii) *for any bounded sequence* $\{\alpha_n\}_n$ *in* $\mathbb{R}$, $\sum_{n=1}^{\infty} \alpha_n x_n$ *converges in* X.

See [25, Theorem 1.3.2] or [1, Proposition 2.4.9].

Lemma A4. *If the series* $\sum_{n=1}^{\infty} x_n$ *is unconditionally convergent in* X, *then*

$$\left\{ \sum_{n=1}^{\infty} \alpha_n \, x_n \, : \, \alpha_n \in \mathbb{R} \quad \textit{with} \quad |\alpha_n| \leq 1, \, n \in \mathbb{N} \right\}$$

is a bounded subset of X. See [40, Lemma 1].

It is clear that absolute convergence implies unconditional convergence. However, the converse is not true in general.

Theorem A5. (Dvoretzky-Rogers) *If every unconditionally convergent series in an Banach space* X *is absolutely convergent, then the dimension of* X *is finite.* See [12], [1, Theorem 8.2.14] or [8, Chapter VI].

Corollary A6. *In every infinite-dimensional Banach space there exists an unconditionally convergent series that is not absolutely convergent.*

In the sequel we recall some aspects of convergence of series involving weak topology.

Definition A7. Let $x_n \in X$ for $n \in \mathbb{N}$. We say that:

1. The sequence $\{x_n\}_n$ is a weakly Cauchy sequence if for every $x^* \in X^*$ the sequence $\{x^*(x_n)\}_n$ converges in $\mathbb{R}$.
2. The series $\sum_{n=1}^{\infty} x_n$ is weakly convergent if there exists $z \in X$ such that the series $\sum_{n=1}^{\infty} x^*(x_n)$ converges to $x^*(z)$ for every $x^* \in X^*$.

3. The series $\sum_{n=1}^{\infty} x_n$ is weakly absolutely convergent if $\sum_{n=1}^{\infty} |x^*(x_n)| < \infty$ for every $x^* \in X^*$.

Proposition A8. *If the series* $\sum_{n=1}^{\infty} x_n$ *is unconditionally convergent, then it is weakly absolutely convergent.* See [1, Proposition 2.4.4 (*iii*)].

The converse of Proposition A8 charaterizes in an important class of Banach spaces.

Theorem A9. (Bessaga-Pelczynski) *A Banach space X does not contain an isomorphic copy of c_0 if and only if every weakly absolutely convergent series in X is unconditionally convergent.* See [8, Theorem V.8], [25, Theorem 6.4.3] or [1, Theorem 2.4.11].

Another important class of spaces which is worth mentioning is the class of weakly sequentially complete Banach spaces. Recall that X is weakly sequentially complete if every weakly Cauchy sequence is weakly convergent in X. We have the following result for series in such spaces.

Theorem A10. *If X is weakly sequentially complete, then every weakly unconditionally convergent series is unconditionally convergent in X.* See [20, Theorem 3.2.3] or [1, Corollary 2.4.15].

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