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APPROXIMATELY CONTINUOUS FUNCTIONS HAVE APPROXIMATE EXTREMA, A NEW PROOF

Abstract

In 1975, Richard O'Malley proved that every approximately continuous function has approximate extrema, and this result provides an immediate solution to **Scottish Book Problem 157**. The purpose of this paper is to provide an additional proof of O'Malley's result.

1 Introduction

All sets and functions considered here will be assumed to be measurable with respect to λ, Lebesgue measure on $\mathbb{R}$. Suppose $E \subset \mathbb{R}$ and J is a given interval with length $|J|$. Then the density (or relative measure) of E in J is $\Delta(E, J) = \lambda(E \cap J)/|J|$. The upper density of E at a point $x \in \mathbb{R}$ is defined as $\limsup_{r \to 0^+} \Delta(E, (x-r, x+r))$ and is denoted by $\overline{\delta}(E, x)$. The lower density at x, $\underline{\delta}(E, x)$ is defined similarly where lim inf replaces lim sup. If these two are equal at x, their common value is called the density of E at x and is denoted $\delta(E, x)$. If $\delta(E, x) = 1$, then x is called a density point of E. We also use the one-sided versions of the upper and lower densities of E at x using traditional notation; so for example, the upper right density of E at a point $x \in \mathbb{R}$ is

$$\overline{\delta}^+(E, x) = \limsup_{r \to 0^+} \Delta(E, (x, x+r)).$$

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A function $f : [a, b] \to \mathbb{R}$ is approximately continuous at $x_0 \in (a, b)$ if there is a set E with $\delta(E, x_o) = 1$, so that

$$\lim_{x \in E \cap [a,b],\ x \to x_0} f(x) = f(x_0). \tag{1}$$

Approximate continuity at the endpoints is defined similarly. For example, f is approximately continuous t a if there is a set E with $\delta^+(E, a) = 1$, so that

$$\lim_{x \in E \cap [a,b],\ x \to a^+} f(x) = f(a). \tag{2}$$

Finally, a function f has an approximate maximum at x_0 if $\{x : f(x) > y\}$ has density 0 at x_0.

In [2, Theorem 1], O'Malley proves that approximately continuous functions always have approximate extrema. We state this for the maxima case below as Theorem 1.

Theorem 1 (O'Malley)**.** *Let $f : [a, b] \to \mathbb{R}$ be approximately continuous. Then f attains an approximate maximum at some point $x_0 \in [a, b]$.*

2 Proof of Theorem 1

Suppose $f : [a, b] \to \mathbb{R}$ is approximately continuous. If any level set of f has positive measure, then by the Lebesgue Density Theorem, that level set has a density point and that density point is an approximate maximum. So we may assume that every level set of f is of measure zero. Moreover, since f is approximately continuous, if either $f(a)$ or $f(b)$ is an absolute maximum then that value is an approximate maximum. So we assume that this is not the case and that there is an $x_1 \in (a, b)$ with $f(a) < y_1 = f(x_1) > f(b)$ and that $f^{-1}((-\infty, y_1])$ does not have full measure. Set $a_1 = a$ and $b_1 = b$.

As in O'Malley's original proof, we inductively choose $y_1 < y_2 < \dots$ and simultaneously choose nested intervals $(a_1, b_1) \supset (a_2, b_2) \supset \dots$ in such a way that

1. $\bigcap_{k=1}^{\infty} (a_k, b_k) = \{c\}$, and
2. $f(c) = \lim y_k$ is an approximate maximum of f.

To simplify notation, define

$$E_y = f^{-1}((-\infty, y)),\ E^*_y = f^{-1}((-\infty, y]) \text{ and } A_y = \{x : \underline{\delta}(E_y, x) > 0\}.$$

Then, for each y, A_y is an F_σ set since

$$A_y = \bigcup_{n\in\mathbb{N}} \left\{ x : \Delta(E_y, (x-r, x+r)) \geq \frac{1}{n} \text{ for all } r \in \left(0, \frac{1}{n}\right) \right\}.$$

Also, since f is approximately continuous,

$$E_y \subset A_y \subset E_y^*. \tag{3}$$

We use some notation and elementary facts later that are best defined here. For $n \in \mathbb{N}$, $a, b \in \mathbb{R}$ and $A \subset \mathbb{R}$ define

$$R_n(a, A) = \{x > a : \Delta(A, (z, x)) \geq 1 - \frac{1}{n} \text{ for all } a \leq z < x\}$$
$$L_n(b, A) = \{x < b : \Delta(A, (x, z)) \geq 1 - \frac{1}{n} \text{ for all } x < z \leq b\}.$$

Lemma 1.

1. *If* $\underline{\delta}^+(A, a) > 1 - \frac{1}{n}$, *then for every* $x > a$, $R_n(a, A) \cap (a, x] \neq \emptyset$.
2. *If* $\underline{\delta}^-(A, b) > 1 - \frac{1}{n}$, *then for every* $x < b$, $L_n(b, A) \cap [x, b) \neq \emptyset$.

Proof. Suppose $x > a$, $\Delta(A, (a, x)) > 1 - \frac{1}{n}$ and yet $R_n(a, A) \cap (a, x] = \emptyset$. In particular, $x \notin R_n(a, A)$ so there is an $a \leq z < x$ with $\Delta(A, (z, x)) < 1 - \frac{1}{n}$. Let

$$z^* = \inf\left\{ z \geq a : \Delta(A, (z, x)) < 1 - \frac{1}{n} \right\}.$$

If $z^* > a$, then since $z^* \notin R_n(a, A)$, there is a $z^{**} < z^*$ with $\Delta(A, (z^{**}, z^*) < 1 - \frac{1}{n}$. But then $\Delta(A, (z^{**}, x) < 1 - \frac{1}{n}$ contradicting the minimality of z^*. Hence, $z^* = a$. However, in that case, $\Delta(A, (a, x)) \leq 1 - \frac{1}{n}$ contrary to the choice of x. The proof of part 2 is similar. □

Assume now that numbers $y_1 < y_2 < \cdots < y_k$ and intervals (a_i, b_i) have been defined for $i \leq k$ in such a way that

$$a_1 < a_2 < \cdots < a_k < b_k < \cdots < b_2 < b_1$$

and for all $i \leq k$, $\{a_i, b_i\} \subset A_{y_i}$, but $\Delta(A_{y_i}, (a_i, b_i)) < 1$. We show how to determine $y_{k+1}, a_{k+1}, b_{k+1}$ along with an important natural number that we call n_{k+1}. Let $e : \mathbb{N}^2 \to \mathbb{N}$ be a fixed injection.

First, choose $y_{k+1} > y_k$ such that $1 - \frac{1}{2n_k} < \Delta(A_{y_{k+1}}, (a_k, b_k)) < 1$. Since no level set of f has positive measure, $\lambda(E_y)$ is a continuous function of y. Too, $\Delta(A_{y_{k+1}}, (a_k, b_k)) < 1$ and so $(a_k, b_k)\backslash A_{y_{k+1}} \neq \emptyset$. Fix $d \in (a_k, b_k)\backslash A_{y_{k+1}}$.

For $i \leq k+1$ the A_{y_i} are nested as $A_{y_1} \subset A_{y_2} \subset \cdots \subset A_{y_{k+1}}$ and each such set is an $\mathcal{F}_\sigma$. So, for each $i \leq k+1$ we write

$$A_{y_i} = \bigcup_{n=1}^{\infty} F_{e(i,n)}, \tag{4}$$

where $F_{e(i,n)}$ is closed for every $i \leq k+1$ and every $n \in \mathbb{N}$. Define

$$n_{k+1} = \min\{m = e(i,n) : i \leq k+1 \text{ and } F_m \cap (a_k, b_k) \neq \emptyset\}.$$

This exists because, at the very least, $A_{y_{k+1}} \cap (a_k, b_k) \neq \emptyset$. Finally, define

$$a_{k+1} = \sup R_{n_k}(a_k, A_{y_{k+1}}) \cap (a_k, d), \text{ and}$$
$$b_{k+1} = \inf L_{n_k}(b_k, A_{y_{k+1}}) \cap (d, b_k).$$

This completes the inductive definition of the numbers y_k and n_k as well as the intervals (a_k, b_k). What remains is to use these to locate an approximate maximum of f.

Since a_k and b_k are in A_{y_k}, it follows by (3) and approximate continuity that they are density points of $A_{y_{k+1}}$. It also follows from the definition of $A_{y_{k+1}}$ that both a_{k+1} and b_{k+1} are in $A_{y_{k+1}}$. Hence, by Lemma 1 and the choice of d we have $a_k < a_{k+1} < d < b_{k+1} < b_k$. The following lemma will be used to show $b_{k+1} - a_{k+1} < \frac{1}{2}(b_k - a_k)$.

Lemma 2. *Suppose $a < r < s$ and $(r,s) \cap R_n(a,A) = \emptyset$. Then there is an $a \leq z \leq r$ with $\Delta(A,(z,s)) \leq 1 - \frac{1}{n}$. A similar statement holds for $L_n(b,A)$.*

PROOF. Let $\{s_i\} \subset (r,s)$ be any sequence converging to s and set $z_i = \inf\{z \geq a : \Delta(A,(z_i,s_i)) \leq 1 - \frac{1}{n}\}$. Since $s_i \notin R_n(a,A)$ it follows that such a z_i exists and, as $(r,s) \cap R_n(a,A) = \emptyset$, that $z_i \leq r$. If z is any accumulation point of $\{z_i\}$, then $z \in [a,r]$ and $\Delta(A,(z,s)) \leq 1 - \frac{1}{n}$ as promised. □

Suppose that $\frac{b_{k+1}-a_{k+1}}{b_k - a_k} \geq 1/2$. Since $(a_{k+1}, d) \cap R_{n_k}(a_k, A_{y_{k+1}}) = \emptyset$, we can apply Lemma 2 to find a z_1 with $a_k \leq z_1 \leq a_{k+1}$ and $\Delta(A_{y_{k+1}}, (z_1, d)) \leq 1 - \frac{1}{n_k}$. Similarly, there is a z_2 such that $b_k \geq z_2 \geq b_{k+1}$ and $\Delta(A_{y_{k+1}}, (d, z_2)) \leq 1 - \frac{1}{n_k}$. But then $\Delta(A_{y_{k+1}}, (z_1, z_2)) \leq 1 - \frac{1}{n_k}$ as well, so that

$$\begin{aligned}\Delta(A_{y_{k+1}}, (a_k, b_k)) &\leq \frac{(1 - \frac{1}{n_k})(b_{k+1} - a_{k+1})}{b_k - a_k} + 1 - \frac{b_{k+1} - a_{k+1}}{b_k - a_k} \\ &= 1 - \frac{b_{k+1} - a_{k+1}}{n_k(b_k - a_k)}.\end{aligned}$$

However, y_{k+1} was initially chosen so that $\Delta(A_{y_{k+1}},(a_k,b_k)) > 1-\frac{1}{2n_k}$. So $b_{k+1}-a_{k+1} < (b_k-a_k)/2$ as claimed. Hence, there is a unique point $c \in (a,b)$ with

$$\bigcap_{k=1}^{\infty}(a_k,b_k) = \{c\}.$$

Also, $\{n_k\} \to \infty$ for if not, then there is a minimal $m \in \mathbb{N}$ such that $n_i = m$ infinitely often. Let k be such that $m = e(k,n)$. As F_m is closed, $c \in F_m \subset A_{y_k}$ and so $f(c) \leq y_k$. But f is approximately continuous at c (and on land), and so by (2) for large indices i, $c \in R_m(a_i, A_{y_{i+1}}) \cap L_m(b_i, A_{y_{i+1}})$. If $n_i = m$, then c is excluded from (a_{i+1}, b_{i+1}).

There are two consequences to $n_k \to \infty$. The first is that c avoids all of the closed sets $F_{e(i,n)}$ and so by (2) we get $f(c) \geq \lim y_i$. Secondly, by choice of a_{k+1} for every $z \in [a_k, a_{k+1})$ we have $\Delta(A_{y_{k+1}},(z,a_{k+1})) \geq 1-\frac{1}{n_k}$. And so using (2), $E_{f(c)}$ has left density 1 at c. Similarly, $E_{f(c)}$ has right density 1 at c and so c is an approximate maximum. This then, completes the proof of Theorem 1.

References

[1] K. Beanland, P. D. Humke, and T. Richards, *On Scottish Book Problem 157*, Real Anal. Exchange, **41(2)** (2016), 331–346.

[2] R. J. O'Malley, *Approximate maxima*, Fund. Math., **94(1)** (1977), 75–81.

[3] R. D. Mauldin, ed., *The Scottish Book: Mathematics Problems from the Scottish Cafe*, Birkhauser, 1981.