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CONTOUR INTEGRATION UNDERLIES FUNDAMENTAL BERNOULLI NUMBER RECURRENCE

Abstract

One solution to a relatively recent American Mathematical Monthly problem [6], requesting the evaluation of a real definite integral, could be couched in terms of a contour integral which vanishes *a priori.* While the required real integral emerged on setting to zero the real part of the contour quadrature, the obligatory, simultaneous vanishing of the imaginary part alluded to still another pair of real integrals forming the first two entries in the infinite log-sine sequence, known in its entirety. It turns out that identical reasoning, utilizing the same contour but a slightly different analytic function thereon, sufficed not only to evaluate that sequence anew, on the basis of a vanishing real part, but also, in setting to zero its conjugate imaginary part, to recover the fundamental Bernoulli number recurrence. The even order Bernoulli numbers B_{2k} entering therein were revealed on the basis of their celebrated connection to Riemann's zeta function $\zeta(2k)$. Conversely, by permitting the related Bernoulli polynomials to participate as integrand factors, Euler's connection itself received an independent demonstration, accompanied anew by an elegant log-sine evaluation, alternative to that already given. And, while the Bernoulli recurrence is intended to enjoy here the pride of place, this note ends on a gloss wherein all the motivating real integrals are recovered yet again, and in quite elementary terms, from the Fourier series into which the Taylor development for $\mathrm{Log}(1-z)$ blends when its argument z is restricted to the unit circle.

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1 Introduction

An American Mathematical Monthly problem posed within relatively recent memory [6] sought the evaluation

$$\int_0^{\pi/2} \Big\{ \log(2\sin(x)) \Big\}^2 \, dx = \frac{\pi^3}{24}. \tag{1}$$

One mode of solution depended upon integration of an analytic function around the periphery Ω of a semi-infinite vertical strip with no singularities enclosed, the quadrature having thus a null outcome[1] known in advance on the strength of Cauchy's theorem. Evaluation (1) emerged automatically by setting to zero the real part of that integral,[2] whereas the complementary requirement that the imaginary part likewise vanish brought into play, and successfully so, both known quadratures

$$\int_0^{\pi} \log(\sin(x)) \, dx = -\pi \log(2) \tag{2}$$

and

$$\int_0^{\pi} x \log(\sin(x)) \, dx = -\frac{\pi^2}{2} \log(2). \tag{3}$$

With (2) and (3) in plain view, a temptation arose to provide for them, too, an *ab initio* verification, and, more even than that, to evaluate the entire

[1]Both contour Ω, a vertical rectangle of unlimited height, and the notion of integrating an analytic function thereon so as to obtain a null result, imitate a similar ploy utilized in [1, Section 5.3, Example 5] on behalf of (2) and still further attributed there to Ernst Lindelöf.

[2]Two solutions for (1) were submitted by the undersigned, one involving contour integration in the manner suggested, and the other based upon a Fourier series. The Bernoulli recurrence (5) emerged as a spontaneous by-product of an ancillary, null-quadrature calculation upon that same contour Ω, initially aimed only at evaluating the log-sine integrals (4). This note embodies the content of that collateral calculation, slightly rephrased so as to highlight the newly recovered Bernoulli number sum identity.

hierarchy of log-sine integrals[3]

$$I_n = \int_0^\pi x^n \log(\sin(x))\, dx \tag{4}$$

as the power of x roams over all non-negative integers $n \geq 0$. Not only was this fresh ambition, digressive and self-indulgent though it may have been, easy to satisfy via quadrature on the same contour as before, but it also exposed to view once more the fundamental Bernoulli number recurrence

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \tag{5}$$

which is valid for $n \geq 2$ and, together with the initial condition $B_0 = 1$ and the self-consistent choice $B_1 = -1/2$, is adequate to populate the entire Bernoulli ladder, complete with null entries at all odd indices beyond $k = 1$, *viz.*, $B_{2l+1} = 0$ whenever $l \geq 1$. Source material on the Bernoulli numbers and the related Bernoulli polynomials is ubiquitous, and can be sampled, for example, in [2, 10, 12]. References [3, 7] provide a valuable overview all at once of their mathematical properties and historical genesis in computing sums of finite progressions of successive integers raised to fixed positive powers. Equally valuable is online Reference [11], which cites a rich literature and covers besides a vast panorama of diverse mathematical knowledge.

Bernoulli identity (5), which is the principal object of our present concern, emerges thus by setting to zero the imaginary part of the analytic quadrature (6), below, around contour Ω, with the corresponding null value requirement on its real part providing an evaluation of the general term from sequence (4), listed in (14). No claim whatsoever is made here as to any ultimate novelty in outcome (14), which is available in symbolic form at any desired index n through routine demand from *Mathematica*. Outcome (14), expressed here as a finite sum of Riemann zeta functions at odd integer arguments, continues to attract the attention of contemporary research focused upon polylogarithms

[3]If that were the only goal, then we should assuredly stop dead in our tracks, simply because, on the one hand, *Mathematica* provides all such evaluations on demand, with great aplomb, and this even in its symbolic mode, while, on the other, a relatively painless derivation (Eq. (51)) can be based upon a Fourier series, one which emerges in its turn from the power series for $\operatorname{Log}(1-z)$ when argument z is forced to lie upon the unit circle. This Fourier series underlies in addition an essentially zinger verification of (1). All such manifold benefits of the Fourier option are sketched in an Appendix. And, prior even to that, when Bernoulli polynomials are admitted as integrand factors during contour integration (Section 4), the log-sine series is produced once more in (43). It goes without saying that contour-based (Eqs. (14) and (43)) and Fourier-based (Eq. (51)) evaluations of (4), even though they may be of secondary interest in the present context, do stand in complete agreement.

[4, 5, 8, 9]. But the formulae thus made available are subordinated in [4, 5] and elsewhere to the task of evaluating a variety of dissimilar quantities, and appear to be tangled in thickets of notation. From this standpoint, formula (14) (and its identical twin (51) derived in an even more elementary fashion) may perhaps still provide the modest service of a stand-alone, encapsulated result, easily derived and easily surveyed. In particular, the canonical method of derivation evolved in [8] and repeatedly alluded to in [4, 5] requires rather strenuous differentiations of Gamma function ratios, and results finally in a recurrence on the individual I_n (or else an equivalent generating function). To be sure, while the work in [8] is immensely elegant, it is at the same time immensely more intricate than either of our independent derivations culminating in (14) and (51).

On the other hand, it does appear to have escaped previous notice that the Bernoulli recurrence (5), which is ancient and foundational in its own right, should likewise re-emerge (via (16)) from the same quadrature around contour Ω when one insists that the corresponding imaginary part also vanish. And, just as is the case with (14), formula (16), too, emerges from a contact with Riemann zeta functions, but evaluated this time at even integer arguments, which latter circumstance, by virtue of the celebrated Euler connection, opens the portal to entry by the similarly indexed Bernoulli numbers. It is of course none of our purpose here to compete with, let alone to supplant in any way the standard derivations of (5). Rather, we seek merely to highlight its re-emergence in what surely must be conceded to be an unexpected setting.

In Section 4 we augment the discussion by admitting the full-fledged Bernoulli polynomials $B_n(z)$ as integrand factors during contour integration (Eq. (26)). And, while this route will no longer lead us directly to recurrence (5), it will underwrite two key ingredients upon which its demonstration in Sections 2 and 3 pivots, to wit, an *ab initio* derivation of the Euler link (17) between Riemann's zeta $\zeta(2m)$ and Bernoulli number B_{2m} at all even indices $2m \geq 2$, and the odd index nullity $B_{2m+1} = 0 \ \forall m \geq 1$ invoked during passage from Eq. (19) to Eq. (20) below. Moreover, the toolkit of Bernoulli polynomial identities will provide, in Eq. (43), a fresh derivation of great elegance, as if one were still needed, of the log-sine evaluation (14).

We round out this note with an appendix wherein contour integration cedes place to the more elementary setting of a Fourier series on whose basis (14) is recovered yet again (as Eq. (51)) through repeated integration by parts. That same Fourier series provides moreover an exceedingly short and simple confirmation of (1), complementary to the contour integral method, an option to which allusion has already been made in Footnote 3. Of course, at this point, no further light can, nor need be shed upon (5) *per se*.

2 Null quadratures on contour Ω

Guided by the cited Example 5 in [1, Section 5.3], we consider for $n \geq 0$ the sequence of numbers

$$K_n = \int_{\Omega} z^n \mathrm{Log}\left(1 - e^{2iz}\right)\, dz = 0, \tag{6}$$

all of them annulled by virtue of closed contour Ω being required to lie within a domain of analyticity for $\mathrm{Log}\left(1 - e^{2iz}\right)$ in the plane of complex $z = x + iy$. Save for quarter-circle indentations of vanishing radius δ around $z = 0$ and $z = \pi$, contour Ω bounds a semi-infinite vertical strip, with a left leg having $x = 0$ fixed and descending from $y = \infty$ to $y = \delta$ (quadrature contribution L_n), and a right leg at a fixed $x = \pi$ ascending from $y = \delta$ to $y = \infty$ (quadrature contribution R_n), linked at their bottom by a horizontal segment with $y = 0$ and $\delta \leq x \leq \pi - \delta$ (quadrature contribution H_n). In what follows it will be readily apparent that the limit $\delta \downarrow 0+$ may be enforced with full impunity, a gesture whose *fait accompli* status will be taken for granted. Likewise passed over without additional comment will be the fact that no contribution is to be sought from contour completion by a retrograde horizontal segment $\pi \geq x \geq 0$ at infinite remove, $y \to \infty$.

We now find

$$L_n = -i^{n+1} \int_0^{\infty} y^n \log\left(1 - e^{-2y}\right)\, dy, \tag{7}$$

$$R_n = +i \int_0^{\infty} (\pi + iy)^n \log\left(1 - e^{-2y}\right)\, dy, \tag{8}$$

and

$$\begin{aligned} H_n &= \int_0^{\pi} x^n \left[\log(2) - \frac{i\pi}{2} + ix + \log(\sin(x))\right] dx \\ &= \frac{\pi^{n+1}}{n+1}\log(2) - i\frac{\pi^{n+2}}{2(n+1)} + i\frac{\pi^{n+2}}{n+2} + \int_0^{\pi} x^n \log(\sin(x))\, dx. \end{aligned} \tag{9}$$

Series expansion of the logarithm further gives

$$L_n = +i^{n+1} \sum_{l=1}^{\infty} \frac{1}{l} \int_0^{\infty} y^n e^{-2ly} dy = +i^{n+1} \frac{n!}{2^{n+1}} \sum_{l=1}^{\infty} \frac{1}{l^{n+2}}, \tag{10}$$

the interchange in summation and integration being legitimated by Beppo Levi's monotone convergence theorem, and similarly

$$R_n = -i \sum_{k=0}^{n} \binom{n}{k} \pi^{n-k} i^k \frac{k!}{2^{k+1}} \sum_{l=1}^{\infty} \frac{1}{l^{k+2}}, \tag{11}$$

in both of which there insinuates itself the Riemann zeta function

$$\zeta(s)=\sum_{l=1}^{\infty}\frac{1}{l^s} \tag{12}$$

at a variety of its argument values s.[4] So armed, we proceed next to set

$$K_n=L_n+H_n+R_n=0 \tag{13}$$

and remark that, regardless of the parity of index n, L_n *per se* is always absorbed by the contribution from the highest power y^n within the integrand for R_n. This circumstance accounts for the imminent appearance of the floor function affecting the highest value of summation index k in Eqs. (14)-(16) and (19) below.

A requirement that the real part of (13) vanish provides now the following string of valuable log-sine quadrature formulae

$$\begin{aligned}\int_0^{\pi} x^n \log(\sin(x))\,dx &= -\frac{\pi^{n+1}}{n+1}\log(2)\\ &\quad+\frac{n!}{2^{n+1}}\sum_{k=1}^{\lfloor n/2\rfloor}(-1)^k\frac{(2\pi)^{n-2k+1}}{(n-2k+1)!}\zeta(2k+1),\end{aligned} \tag{14}$$

of which the first two, at $n=0$ and $n=1$, with the sum on the right missing, validate (2) and (3), both of them being in any event widely tabulated. And again, as was first stated in Footnote 3, Eq. (14) is consistently reaffirmed by *Mathematica*, even when harnessed in its symbolic mode. We note in passing the self-evident fact that, unlike the corresponding prescriptions found in [8, 9], formula (14) is fully explicit, needing to rely neither upon a generating function nor a recurrence, even though, naturally, such recurrence arrives at a final rendezvous with identically the same result.

A close prelude to identity (5) follows next from the co-existing requirement that the imaginary part of (13) vanish. This requirement takes the initial form

$$-\frac{\pi^{n+2}}{2(n+1)}+\frac{\pi^{n+2}}{n+2}=\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k}\pi^{n-2k}(-1)^k\frac{(2k)!}{2^{2k+1}}\zeta(2k+2) \tag{15}$$

and is subsequently moulded into the shape

$$\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2k}\frac{B_{2k+2}}{(k+1)(2k+1)}=\frac{n}{(n+1)(n+2)} \tag{16}$$

[4] This canonical definition implies a guarantee of series convergence, assured by the requirement that $\Re s>1$. A robust arsenal of knowledge exists for continuing $\zeta(s)$ across the entire plane of complex variable $s=\sigma+it$, with a simple pole emerging at $s=1$.

on taking note of Euler's celebrated connection [2, 3, 7, 10, 11, 12]

$$\zeta(2k) = (-1)^{k+1}(2\pi)^{2k}\frac{B_{2k}}{2(2k)!} \quad \forall k \geq 1, \tag{17}$$

allowing us to displace attention from the even-argument values of Riemann's zeta to the correspondingly indexed Bernoulli numbers B_{2k}.

3 Recurrence reduction

Recurrence (16) is not quite yet in the desired form (5), but it is easily steered toward this goal. That process begins by noting that

$$\binom{n}{2k}\frac{1}{(k+1)(2k+1)} = \binom{n+2}{2k+2}\frac{2}{(n+1)(n+2)}, \tag{18}$$

whereupon (16) becomes

$$\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n+2}{2k+2}B_{2k+2} = \frac{n}{2}. \tag{19}$$

Now the advance of $2k$ in steps of two means that it reaches a maximum value $M = n-1$ when n is odd, and one offset instead by two below n, $M = n-2$, when n is even. At the same time the accepted null value of odd-index Bernoulli numbers starting with $B_3 = 0$ means that we are free, and self-consistently so, to intercalate all odd indices missing from the progression $2k+2$ in order to attain an index advance in steps of one and to entertain a common index maximum of $n+1$, regardless of the parity of n. Altogether then, (19) re-emerges as

$$\sum_{k=2}^{n+1}\binom{n+2}{k}B_k = \frac{n}{2}, \tag{20}$$

or else

$$\sum_{k=0}^{n+1}\binom{n+2}{k}B_k = \frac{n}{2} + \left\{\binom{n+2}{0}B_0 + \binom{n+2}{1}B_1\right\}. \tag{21}$$

But now we find that

$$\binom{n+2}{0}B_0 + \binom{n+2}{1}B_1 = 1 - \frac{n+2}{2} = -\frac{n}{2}, \tag{22}$$

with the effect of reducing (21) to just

$$\sum_{k=0}^{n+1} \binom{n+2}{k} B_k = 0, \tag{23}$$

which is nothing other than (5).

4 Bernoulli polynomials as integrand factors on contour $\tilde{\Omega}$

Added perspective accrues when power multiplier z^n in the integrand from (6) is replaced by the Bernoulli polynomial $B_n(z)$. By following this route we will, on the one hand, forfeit a direct access to recurrence (5), but, by way of compensation, we will recover Euler's connection (17) linking even index Bernoulli number B_{2m} values to their Riemann $\zeta(2m)$ counterparts, and will similarly confirm the vanishing $B_{2m+1} = 0$ of all odd index Bernoulli numbers beginning with index 3. More even than that, a continuing interplay between contour integration around a half strip as presently considered, and one among the tangle of Bernoulli polynomial identities, will disclose once more the log-sine formula (14).

All required Bernoulli polynomial relations will be drawn without further comment from [7], which is readily accessed electronically. And so, bowing to the notational convention adopted there, we denote the Bernoulli number symbol from capital B_n to lower case b_n, upper case $B_n(z)$ being reserved for the polynomials so named.[5] The polynomials themselves are uniquely defined by

$$\begin{cases} B_0(z) = 1 \\ B_n(z+1) - B_n(z) = nz^{n-1}, & \forall n \geq 1 \\ \int_0^1 B_n(t)\,dt = 0, & \forall n \geq 1 \end{cases} \tag{24}$$

[7, Section 1, Corollary 1.3], and the Bernoulli numbers b_n then follow from

$$b_n = B_n(0) \quad \forall n \geq 0 \tag{25}$$

[7, Section 2, Definition 2.2]. Their recurrence (5) is found as entry *iv* to [7, Section 2, Proposition 2.3], with no mention whatsoever at that point of a Riemann ζ connection. Also disclosed there under entry *i* is the vanishing of all odd index numbers $b_{2n+1} = 0\ \forall n \geq 1$. The auxiliary fact that $\Im(B_n(z)) = 0$ without exception whenever $\Im(z) = 0$, and hence *a fortiori* that $\Im(b_n) = 0$,

[5]This shift evidently seeks to minimize visual confusion.

follows from the bijective mapping established between real polynomial bases in [7, Section 1], Lemma 1.1 and Bernoulli polynomial Definition 1.2, which latter underwrites the constructive recipe in (24).

Our imminent intent to capitalize on the second entry in (24) suggests that width π of contour Ω be compressed to just 1 under variable scaling $x = \pi u$, $y = \pi v$, acknowledged by the notational shift $\Omega \to \tilde{\Omega}$ in the plane of complex $\eta = u + iv$. And so we replace quantities K_n from (6) by similarly null analogs

$$\tilde{K}_n = \int_{\tilde{\Omega}} B_n(\eta) \mathrm{Log}\left(1 - e^{2i\pi\eta}\right)\, d\eta = 0. \tag{26}$$

The contour integration apparatus of Eqs. (7)-(13), marshalled out on behalf of K_n, carries over essentially verbatim, ceding place now to

$$\tilde{L}_n = -i \int_0^{\infty} B_n(iv) \log\left(1 - e^{-2\pi v}\right)\, dv, \tag{27}$$

$$\tilde{R}_n = +i \int_0^{\infty} B_n(1 + iv) \log\left(1 - e^{-2\pi v}\right)\, dv, \tag{28}$$

and

$$\begin{aligned} \tilde{H}_n &= \int_0^1 B_n(u) \left[\log(2) - \frac{i\pi}{2} + i\pi u + \log(\sin(\pi u))\right] du \\ &= \int_0^1 B_n(u) \left[\log(2) + \log(\sin(\pi u)) + i\pi B_1(u)\right] du, \end{aligned} \tag{29}$$

wherein we have identified $(u - 1/2)$ with $B_1(u)$ [7, Section 1, Corollary 1.3]. And, of course, we still have, $\forall n \geq 0$,

$$\tilde{K}_n = \tilde{L}_n + \tilde{H}_n + \tilde{R}_n = 0 \tag{30}$$

which, in particular, forces one to consider the sum[6]

$$
\begin{aligned}
\tilde{L}_n + \tilde{R}_n &= i\int_0^\infty \Big\{B_n(1+iv) - B_n(iv)\Big\} \log\left(1-e^{-2\pi v}\right)\, dv \\
&= ni^n \int_0^\infty v^{n-1} \log\left(1-e^{-2\pi v}\right)\, dv \\
&= \frac{ni^n}{(2\pi)^n}\int_0^\infty v^{n-1}\log\left(1-e^{-v}\right)\, dv \\
&= -\frac{ni^n}{(2\pi)^n}\sum_{l=1}^\infty \frac{1}{l}\int_0^\infty v^{n-1}e^{-lv}\, dv \\
&= -\frac{n!i^n}{(2\pi)^n}\sum_{l=1}^\infty \frac{1}{l^{n+1}} \\
&= -\frac{n!i^n}{(2\pi)^n}\zeta(n+1)
\end{aligned}
\tag{31}
$$

after invoking the second line from (24) and mimicking the series expansion in (10).

Turning attention once more to (29), we must again discriminate between the null index $n = 0$ case and all others with $n \geq 1$. Thus, when $n = 0$, reference to the first and third lines in (24) and Footnote 6 gives

$$
\tilde{K}_0 = \tilde{H}_0 = \log(2) + \int_0^1 \log(\sin(\pi u))\, du = 0 \tag{32}
$$

or else

$$
\int_0^\pi \log(\sin(x))\, dx = -\pi\log(2), \tag{33}
$$

which is Eq. (2).

By contrast, for $n \geq 1$, recourse to [7, Section 3, Corollary 3.3] provides, when integer indices p and q are both in excess of zero,[7]

$$
\int_0^1 B_p(u)B_q(u)\, du = \frac{(-1)^{q-1}}{\binom{p+q}{q}} b_{p+q} \tag{34}
$$

---

[6] Result (31) holds only for $n \geq 1$. When $n = 0$ the sum $\tilde{L}_0 + \tilde{R}_0$ is trivially null by virtue of the first line in (24). The parent quantities $\tilde{K}_n$, by contrast, are null $\forall n \geq 0$ without restriction.

[7] Both sides of (34) are symmetric in indices p and q, on its left by inspection while on its right because of the fact, soon to be confirmed (Eqs. (38) and (40) below), that b_{p+q} vanishes unless $p + q$ is even.

and thus causes (29) to read

$$\tilde{H}_n = \int_0^1 B_n(u)\log(\sin(\pi u))\,du + \frac{i\pi}{n+1}b_{n+1}. \tag{35}$$

And so, on putting (30), (31), and (35) together we arrive at

$$\int_0^1 B_n(u)\log(\sin(\pi u))\,du + \frac{i\pi}{n+1}b_{n+1} = \frac{n!i^n}{(2\pi)^n}\zeta(n+1) \tag{36}$$

whenever $n \geq 1$. On sorting (36) out according to its real and imaginary components and the parity of n, we thus find, when $n = 2m$,

$$\begin{aligned} \int_0^1 B_{2m}(u)\log(\sin(\pi u))\,du &= \frac{(2m)!(-1)^m}{(2\pi)^{2m}}\zeta(2m+1) && (37)\\ b_{2m+1} &= 0 && (38) \end{aligned}$$

and if instead $n = 2m-1$,

$$\begin{aligned} \int_0^1 B_{2m-1}(u)\log(\sin(\pi u))\,du &= 0 && (39)\\ b_{2m} &= \frac{2(2m)!(-1)^{m+1}}{(2\pi)^{2m}}\zeta(2m) && (40) \end{aligned}$$

$\forall m \geq 1$. Odd index vanishing of b_{2m+1}, $m \geq 1$ is thus vindicated, as is also the full content of Euler's connection (17) at all even indices b_{2m} beginning with $m = 1$.[8]

There exists one further, polynomial basis inversion identity, namely

$$u^n = \frac{1}{n+1}\sum_{m=0}^{n}\binom{n+1}{m}B_m(u) \quad \forall n \geq 0 \tag{41}$$

[7, Section 2, Application 1], which enables us to consolidate the fragmented information scattered among the quadratures (37) into a single, compact form. Thus, with (32) adjoined,

$$\begin{aligned} \int_0^1 u^n \log(\sin(\pi u))\,du &= -\frac{1}{n+1}\log(2) \\ &\quad + n!\sum_{m=1}^{\lfloor n/2\rfloor}\frac{(-1)^m}{(n-2m+1)!(2\pi)^{2m}}\zeta(2m+1) && (42) \end{aligned}$$

[8]The missing start-up values $b_0 = 1$ and $b_1 = -1/2$ follow respectively from $b_0 = B_0(0) = 1$ and $b_1 = B_1(0) = (u-1/2)|_{u=0} = -1/2$.

or else

$$\int_0^\pi x^n \log(\sin(x))\,dx = -\frac{\pi^{n+1}}{n+1}\log(2) + \frac{n!}{2^{n+1}}\sum_{m=1}^{\lfloor n/2\rfloor}(-1)^m\frac{(2\pi)^{n-2m+1}}{(n-2m+1)!}\zeta(2m+1) \tag{43}$$

which is (14) once more.

The null quadratures in (39) are a consequence of the antisymmetry

$$B_{2m-1}(1/2-\Delta u) = -B_{2m-1}(1/2+\Delta u) \tag{44}$$

around $u = 1/2$ which follows readily from the fact that[9]

$$B_n(1-u) = (-1)^n B_n(u) \quad \forall n \geq 0 \tag{45}$$

[7, Section 2, Proposition 2.1*ii* (amended so as to include $n = 0$)]. When viewed at $m = 1$ and with reference to (32), Eq. (39) leads us thus to confront

$$\int_0^1 u\log(\sin(\pi u))\,du - \frac{1}{2}\int_0^1 \log(\sin(\pi u))\,du = \int_0^1 u\log(\sin(\pi u))\,du + \frac{1}{2}\log(2) = 0 \tag{46}$$

which amounts to

$$\int_0^\pi x\log(\sin(x))\,dx = -\frac{\pi^2}{2}\log(2) \tag{47}$$

in agreement with both (3) and (43), the trailing sum of the latter being then vacuous. Precursor (33) is, *ipso facto*, likewise subsumed under (43) by the very manner of the latter's derivation.

And finally, even though it has now been sidestepped, recurrence (5) is duly assembled in [7, Section 2, Proposition 2.3*iv*], free from any reference to

[9]Eq. (44) is clearly accompanied by a symmetric counterpart around $u = 1/2$ for the even-indexed polynomials. These symmetry/antisymmetry attributes are corroborated under a different guise by the Fourier series evolved in [7, Section 3, Proposition 3.1, parts *i* & *ii*]. Consistent too with (44) are the null mid-point evaluations $B_{2m-1}(1/2) = (4^{1-m} - 1)b_{2m-1} = 0\ \forall m \geq 1$ [7, Section 2, Proposition 2.3*ii*], with the special evaluation $B_1(1/2) = 0$ being already self-evident from $B_1(u) = u - 1/2$. For the even-indexed polynomials, the overarching, global constraint $\int_0^1 B_{2n}(u)\,du = 0\ \forall n \geq 1$ is maintained despite their exhibiting non-zero mid-point values $B_{2n}(1/2) = (2^{1-2n} - 1)b_{2n} \neq 0\ \forall n \geq 0$.

Riemann's ζ. The basis of deduction at that point is yet another Bernoulli polynomial identity

$$B_n(u) = \sum_{m=0}^{n} \binom{n}{m} b_{n-m} u^m \quad \forall n \geq 0,$$

reinforced by the observation, also derived there, that $B_n(1) = b_n \ \forall n \geq 2$. Verily, verily, on this arena all roads seem to converge upon the proverbial Rome.

5 Appendix: a Fourier series grace note

A somewhat more pedestrian derivation of (14) rests upon consideration of the power series

$$\mathrm{Log}(1-z) = -\sum_{l=1}^{\infty} \frac{z^l}{l} \tag{48}$$

along the unit circle $z = e^{i\vartheta}$. Separation into real and imaginary parts emerges as a pair of Fourier series

$$\log\left(2\left|\sin\left\{\frac{\vartheta}{2}\right\}\right|\right) = -\sum_{l=1}^{\infty} \frac{\cos(l\vartheta)}{l} \tag{49}$$

and

$$\left\{\frac{\vartheta-\pi}{2},\ \mathrm{mod}\ \pi\right\} = -\sum_{l=1}^{\infty} \frac{\sin(l\vartheta)}{l}, \tag{50}$$

of which the second is of no interest *vis-à-vis* our immediate objective. The logarithmic divergence on both left and right in (49) whenever $\vartheta \equiv 0 \pmod{2\pi}$ remains integrable and is thus taken henceforth in easy stride.

Repeated integration by parts *vis-à-vis* series (49), when first multiplied by the argument power ϑ^n, advances by $\cos \to \sin \to \cos$ couplets, with end-point contributions arising only on the second beat, and the argument powers falling in steps of two.[10] One assembles in this manner the general formula

$$\begin{aligned} \int_0^{\pi} \vartheta^n \log(\sin(\vartheta))\, d\vartheta \quad &= \quad -\frac{\pi^{n+1}}{n+1}\log(2) \\ &+ \frac{n!}{2^{n+1}} \sum_{k=1}^{\lfloor n/2 \rfloor} (-1)^k \frac{(2\pi)^{n-2k+1}}{(n-2k+1)!} \zeta(2k+1) \end{aligned} \tag{51}$$

[10] In particular, this quadrature cadence provides a motivation, alternative to that previously given, as to why it is that the floor function affects the upper index cutoff $\lfloor n/2 \rfloor$ in both (14) and (51), allowing for unit growth in that cutoff only when n *per se* advances by two.

holding good unrestrictedly for n even or odd, and agreeing in every respect with (14). The only wrinkle to notice, perhaps, is that the sequence of integrations by parts which underlies (51) terminates, at each summation index l in (49), with a term proportional to either

$$\int_0^\pi \cos(2l\vartheta)\, d\vartheta = 0 \tag{52}$$

in the event that n is even, or

$$\int_0^\pi \vartheta \cos(2l\vartheta)\, d\vartheta = 0 \tag{53}$$

otherwise. Equation (52) is of course obvious whereas (53), while equally true and welcome as such, is, at first blush, mildly surprising. All in all the derivation which underlies (14) is far smoother and less apt to inflict bookkeeping stress, even if it is (51) which seems to rest on a more elementary underpinning.

It would be truly disappointing were we not able to utilize (49) so as to give an essentially one-line, zinger-style proof of (1). This anticipation is readily met simply by squaring both sides of (49), with summation indices l and l' figuring now on its right, and noting that when, as here, both $l \geq 1$ and $l' \geq 1$,

$$\int_0^\pi \cos(2l\vartheta)\cos(2l'\vartheta)\, d\vartheta = \frac{\pi}{2}\delta^l_{l'}, \tag{54}$$

with $\delta^l_{l'}$ being the Kronecker delta, unity when its indices match, and zero otherwise. In a gesture which embodies the essence of Parseval's theorem, it follows immediately that

$$\frac{1}{2}\int_0^\pi \Big\{ \log(2\sin(\vartheta)) \Big\}^2 d\vartheta = \frac{\pi}{4}\sum_{l=1}^\infty \frac{1}{l^2} = \frac{\pi^3}{24}, \tag{55}$$

and we are done.

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