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ON SCOTTISH BOOK PROBLEM 157

Abstract

This paper descibes our hunt for the solver of Problem 157 in the
Scottish Book, a problem originally posed by A. J. (Gus) Ward in 1937.
We first make the observation that a theorem of Richard O'Malley from
1975 yields an immediate positive solution. A further look at O'Malley's
references revealed a 1970 paper by Donald Ornstein that we now believe
contains the first solution of *SB 157*. We isolate the common elements in
the machinery used by both Ornstein and O'Malley and discuss several
consequences. We also examine an example function given by Ornstein.
There are some difficulties with this function but we provide a fix, and
show moreover that functions of that kind are typical in the sense of the
Baire category theorem.

1   The solution in brief

On March 23rd, 1937 A.J. Ward asked the following problem which is recorded
as Problem 157 in the famous *Scottish Book*.[1]

Scottish Book Problem 157. *Suppose f is approximately continuous and
at each point x_0 the quantity*

$$\limsup_{h \to 0^+} \frac{f(x_0 + h) - f(x)}{h},$$

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[1]The prize for the solution to this problem is lunch at the "The Dorothy" in Cambridge
which the authors now offer to purchase for Richard O'Malley and Donald Ornstein; trans-
portation costs are another matter!

neglecting any set of h that have zero density at $h = 0$, is positive. Is $f(x)$ monotone increasing?

We became aware of this problem only recently when one of us[2] was discussing the upcoming new edition of the *Scottish Book* with its editor, Dan Mauldin. Problem 157 was one of the problems marked as unresolved. Upon returning to campus, the authors decided to have a look at **Scottish Book Problem 157** and a natural approach to its resolution soon brought them to Richard O'Malley's paper on approximate maxima, [4]. In that paper, O'Malley proves [4, Theorem 1] from which the following result is an immediate corollary.

O'Malley's Theorem 1*. *Let $f : [0,1] \to \mathbb{R}$ be approximately continuous but not strictly increasing. Then f attains an approximate maximum at some point $x_0 \in [0,1)$.*

A follow up conversation with O'Malley then led us to the following theorem by Donald Ornstein, [6].

Ornstein's Theorem. *Let $f(x)$ be a real–valued function of a real variable satisfying the following:*

(a) *$f(x)$ is approximately continuous,*

(b) *For each x_0, let E be the set of x, such that $f(x) - f(x_0) \geq 0$. Then*

$$\limsup_{h \to 0^+} \lambda \left(E \cap (x_0, x_0 + h) \right) / h \neq 0.$$

Then f is monotone increasing and continuous.

This is clearly the solution to **Scottish Book 157**, and as far as we see is the first. There is a certain commonality to the machinery used in the proofs of O'Malley and Ornstein, and we'll try to isolate that common thread in the next section. Our proofs (largely reformulating those of O'Malley and Ornstein) will provide the slightly stronger result that the function under consideration is in fact strictly increasing (rather than just monotonically increasing). We'll also list some elementary consequences and state all the relevant background.

In Section 4, we'll examine an example given in [6]. We will show that this example needs amending and supply that amendment. Finally, we will also show in that section that functions satisfying the properties of Ornstein's example are typical in the sense of the Baire category theorem.

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[2]Humke

But for those familiar with the definitions, here is the solution to **Scottish Book 157** which we have drawn from O'Malley's work and which is a trivial consequence of Theorem 1*.

Theorem 1. *Suppose that* $f : [0,1] \to \mathbb{R}$ *is approximately continuous and that, for each* $x_0 \in [0,1)$ *and each set of* h *values* $E \subset \mathbb{R}$ *having zero density at* 0*, the quantity*

$$\limsup_{h \to 0^+, h \notin E} \frac{f(x_0 + h) - f(x_0)}{h} > 0.$$

Then f *is strictly increasing.*

PROOF. Given any points $0 \le x_1 < x_2 \le 1$, if $f(x_1) \ge f(x_2)$, then applying O'Malley's Theorem to the function f restricted to $[x_1, x_2]$, we obtain that f attains an approximate maximum (relative to $[x_1, x_2]$) at some point $x_0 \in [x_1, x_2)$. That is, there is some set of h values $E_0 \subset \mathbb{R}$ having zero density at 0 such that, on some neighborhood of x_0 in $[x_1, x_2] \setminus E_0$, f attains its absolute maximum at x_0.

We conclude therefore that

$$\limsup_{h \to 0^+, h \notin E_0} \frac{f(x_0 + h) - f(x_0)}{h} \le 0.$$

This contradicts the assumption made on f. We conclude that $f(x_1) < f(x_2)$, and that therefore f is strictly increasing on $[0,1]$.

□

2   The rest of the story

All sets and functions considered here will be assumed to be measurable with respect to λ, Lebesgue measure on $\mathbb{R}$. Suppose $E \subset \mathbb{R}$ and $I \subset \mathbb{R}$ is an interval. Then the density of E in I is defined as $\Delta(E, I) = \lambda(E \cap I)/\lambda(I)$. Now, if $x \in \mathbb{R}$, then the upper density of E at x is defined as $\overline{\Delta}(E, x) = \limsup_{r \to 0} \Delta(E, (x - r, x + r))$. The lower density at x, $\underline{\Delta}(E, x)$ is defined similarly where $\liminf$ replaces $\limsup$ and if these two are equal at x, their common value is called the density of E at x and is denoted $\Delta(E, x)$.

Now suppose a function $f : \mathbb{R} \to \mathbb{R}$ is given. Then f is approximately continuous at x_0 if there is a set E with zero density at x_0, so that the limit of f as $x \to x_0$ on $\mathbb{R} \setminus E$ is $f(x_0)$. A function f has an approximate maximum at x_0 if $\Delta(H_{f(x_0)}, x_0) = 0$ where we define $H_y \equiv H_y(f) = \{x : f(x) > y\}$.

For the purpose of exposition, we isolate the following remarks concerning density and approximately continuous functions.

Remark 2. Although the definitions above of upper and lower density at a point x are given in terms of intervals which are symmetric around x only, it is easy to show that if E and F are sets having density zero and one at x respectively, then for any $\epsilon > 0$ there is a $\delta > 0$ small enough so that if I is any interval containing x in its closure with $\lambda(I) < \delta$, then $\Delta(E, I) < \epsilon$ and $\Delta(F, I) > 1 - \epsilon$.

Remark 3. Suppose $y \in \mathbb{R}$, f is approximately continuous at x and $\{I_n\}$ is a nested sequence of closed intervals so that that $\{x\} = \bigcap_n I_n$ and $\Delta(H_y, I_n) > \eta > 0$ for all $n \in \mathbb{N}$. Then $\overline{\Delta}(H_y, x) > \eta/2$.

Remark 4. If f is approximately continuous at x, $z < y$ and $\overline{\Delta}(H_y, x) > 0$ then $f(x) \geqslant y$ and $\Delta(H_z, x) = 1$.

An important first step is to see that there is an approximately continuous function with no relative extrema. This is, perhaps, not particularly surprising, but the fact that this function can be a derivative is a good introduction into the real nature of derivatives. This example can be found in Andy Bruckner's classic introduction, [1].

Bruckner's Theorem 3.1. *There is a bounded approximately continuous derivative which achieves no local maximum and no local minimum.*

To visit O'Malley's machinery, consider a measurable set H and an interval I so that $\lambda(H \cap I) > 0$ and let $\varepsilon > 0$ be given. We define

$$\mathcal{J}(H, I, \varepsilon) = \{J : J \subset I \text{ is an open interval with } \Delta(H, J) > \varepsilon\}$$

and we let $G_\varepsilon(H, I) = \bigcup \mathcal{J}(H, I, \varepsilon)$. O'Malley proves the following lemma, [4, Lemma 1] concerning components of $G_\varepsilon(H, I)$.

Lemma 5. *Suppose $H \subset [0, 1]$ is measurable and fix $(a_0, b_0) \subset (0, 1)$ with $\lambda(H \cap (a_0, b_0)) > 0$ and let $\varepsilon > 0$. Let (a_1, b_1) be a component of $G_\varepsilon(H, (a_0, b_0))$. Then*

1. $\Delta(H, (a_1, b_1)) \geqslant \varepsilon/2$, and

2. If $I \subset (a_0, b_0)$ is an open interval with $I \cap \{a_1, b_1\} \neq \emptyset$ then $\Delta(H, I) \leqslant \varepsilon$.

In particular, $\lambda(G_\varepsilon(H, (a_0, b_0))) \leqslant 2\lambda(H \cap (a_0, b_0))/\varepsilon$.

The first item holds since each component is comprised of intervals with density at least ε. The second part of the lemma follows from the definition of $G_\varepsilon(H, (a_0, b_0))$. If $I \subset (a_0, b_0)$ is an interval that either contains or overlaps

a component of $G_\varepsilon(H, (a_0, b_0))$ then, by definition, $I \notin J(H, (a_0, b_0), \varepsilon)$ and so $\Delta(H, I) \leqslant \varepsilon$.

The next lemma has been extracted from the proof of Theorem 1^* in [4] and is the main workhorse of the proof our main result. We postpone the proof to Section 3.

O'Malley's Lemma. *Suppose $f : [a, b] \to \mathbb{R}$ is approximately continuous satisfying $\lambda(f^{-1}(y)) = 0$ for all $y \in \mathbb{R}$. Let $[a_0, b_0] \subset [a, b]$ be such that $s_0 := \sup f[a_0, b_0] > \max\{f(a_0), f(b_0)\}$. Then for each $\varepsilon > 0$ and $y_0 < s_0$ there is a $y_1 \in (y_0, s_0)$ and a component (a_1, b_1) of $G_\varepsilon(H_{y_1}, (a_0, b_0))$ satisfying:*

1. $[a_1, b_1] \subset (a_0, b_0)$

2. $(b_1 - a_1) < 1/2(b_0 - a_0)$

3. $\max\{f(a_0), f(b_0), y_0\} < y_1$

4. $\max\{f(a_1), f(b_1)\} \leqslant y_1$

5. $\Delta(H_{y_0}, (a_1, b_1)) > 1/2$

6. *If $I \subset (a_0, b_0)$ is an open interval with $I \cap \{a_1, b_1\} \neq \emptyset$, then $\Delta(H_{y_1}, I) \leqslant \varepsilon$.*

Using the above we give the proof of O'Malley's Theorem 1^*.

PROOF OF O'MALLEY'S THEOREM 1^*. We first note that if the condition $\lambda(f^{-1}(y)) = 0$ found in the statement of O'Malley's Lemma fails for some $y \in \mathbb{R}$, then the set $f^{-1}(y)$ will have a density point $x_0 \in [0, 1)$. It is easy to see that f will achieve an approximate maximum at x_0, and we are done. Therefore we assume that $\lambda(f^{-1}(y)) = 0$ for all $y \in \mathbb{R}$. Note also that as f is not strictly increasing, there is a $b \in (0, 1]$ so that $f(b) \neq \sup f[0, b]$. Therefore without loss of generality we assume that $f(1) \neq \sup f[0, 1]$. If $f(0) = \sup f[0, 1]$ we are done, setting $x_0 = 0$. Therefore we assume further that $\sup f[0, 1] > \max\{f(0), f(1)\}$. Define $[a_0, b_0] = [0, 1]$, $s_0 = \sup f[0, 1]$, and let $y_0 < s_0$ be chosen arbitrarily. We apply O'Malley's Lemma iteratively with $\epsilon = 1/k$ at the kth stage to obtain a strictly increasing sequence of real numbers $\{y_k\}$ and a strictly nested sequence of intervals $\{(a_k, b_k)\}$ such that for each $k \in \mathbb{N}$, (a_{k+1}, b_{k+1}) is a component of $G_\varepsilon(H_{y_{k+1}}, (a_k, b_k))$ and the following items hold.

(i) $[a_{k+1}, b_{k+1}] \subset (a_k, b_k)$

(ii) $(b_{k+1} - a_{k+1}) < 1/2(b_k - a_k)$

(iii) $\max\{f(a_k), f(b_k), y_k\} < y_{k+1}$

(iv) $\max\{f(a_{k+1}), f(b_{k+1})\} \leqslant y_{k+1}$

(v) $\Delta(H_{y_k}, (a_{k+1}, b_{k+1})) > 1/2$

(vi) If $I \subset (a_k, b_k)$ is an open interval with $I \cap \{a_{k+1}, b_{k+1}\} \neq \emptyset$, then $\Delta(H_{y_{k+1}}, I) \leqslant 1/k$.

In order to justify this recursive construction (i.e. to ensure that at the kth stage the function f and the interval (a_k, b_k) satisfies the assumption made on f and (a_0, b_0) in the statement of the O'Malley's Lemma), observe that since (a_{k+1}, b_{k+1}) is a component of $G_\varepsilon(H_{y_{k+1}}, (a_k, b_k))$, $H_{y_{k+1}} \cap (a_k, b_k) \neq \emptyset$. Therefore setting $s_k = \sup f[a_k, b_k]$, we have $s_k > y_{k+1} > \max\{f(a_k), f(b_k)\}$.

Items (i) and (ii) above yield that the intersection of the intervals so obtained consists of a single point, $\{x_0\} = \bigcap_{k=0}^\infty (a_k, b_k)$. We have two claims: $f(x_0) \geqslant y_n$ for each $n \in \mathbb{N}$ and $\Delta(H_{f(x_0)}, x_0) = 0$. This final claim yields that f has an approximate maximum at $x_0 \in [0, 1)$, as desired.

Fix some positive integer n. Then for each positive integer k, $y_n < y_{n+k}$. Using this inequality and item (v) above

$$\Delta(H_{y_n}, (a_{n+k+1}, b_{n+k+1})) \geqslant \Delta(H_{y_{n+k}}, (a_{n+k+1}, b_{n+k+1})) > 1/2.$$

Using Remark 3, $\overline{\Delta}(H_{y_n}, x_0) > 1/4$ and so, by Remark 4, $f(x_0) \geqslant y_n$.

It remains to observe that $\Delta(H_{f(x_0)}, x_0) = 0$. Suppose by way of contradiction that $\overline{\Delta}(H_{f(x_0)}, x_0) = \eta > 0$. Choose some $m \in \mathbb{N}$ with $1/m < \eta/2$. Since $\overline{\Delta}(H_{f(x_0)}, x_0) = \eta$, we can find some $r > 0$ small enough so that the interval $I = (x_0 - r, x_0 + r)$ satisfies

$$I \subset (a_m, b_m), \text{ and } \Delta(H_{f(x_0)}, I) > \eta/2.$$

Choose $k > 0$ to be the smallest positive integer such that $I \not\subset (a_{m+k+1}, b_{m+k+1})$. Then

$$I \subset (a_{m+k}, b_{m+k}), \text{ and } I \cap \{a_{m+k+1}, b_{m+k+1}\} \neq \emptyset,$$

so we obtain the contradiction

$$\frac{\eta}{2} < \Delta(H_{f(x_0)}, I) \leqslant \Delta(H_{y_{m+k}}, I) < \frac{1}{m+k} \leqslant \frac{1}{m} < \frac{\eta}{2},$$

where the second inequality follows from the fact that $f(x_0) > y_{m+k}$, and the third inequality above follows from item (vi). This contradicts the choice of r. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Several old standards can be immediately generalized to approximately continuous functions using the theorem above. For example.

Rolle's Theorem AP. *Let* $f : [0,1] \to \mathbb{R}$ *be approximately continuous and approximately differentiable on* $(0,1)$ *with* $f(0) = f(1)$. *Then there is a point* $x_0 \in (0,1)$ *at which* $f'_{ap}(x_0) = 0$.

And hence, also its immediate consequence:

The Mean Value Theorem AP. *Let* $f : [a,b] \to \mathbb{R}$ *be approximately continuous and approximately differentiable on* (a,b). *Then there is a point* $x_0 \in (0,1)$ *at which* $f'_{ap}(x_0) = \frac{f(b)-f(a)}{b-a}$.

Mean Value Theorems for the approximate derivative are well known and in much greater generality; in fact, O'Malley showed in [5] that x_0 can be chosen so that $f'(x_0) = \frac{f(b)-f(a)}{b-a}$.

3   Proof of O'Malley's Lemma

We need the following easy remark.

Remark 6. Suppose $f : [a,b] \to \mathbb{R}$ is approximately continuous, so that for all $y \in \mathbb{R}$, $\lambda(f^{-1}(y)) = 0$. Then, setting $s = \sup(f[a,b])$, $\lim_{y \to s} \lambda(H_y) = 0$. Therefore by Lemma 5, for such a function f and any $\varepsilon > 0$, $\lambda(G_\varepsilon(H_y, (a,b))) \to 0$ as $y \to s$.

We restate O'Malley's Lemma for reference.

O'Malley's Lemma. *Suppose* $f : [a,b] \to \mathbb{R}$ *is approximately continuous satisfying* $\lambda(f^{-1}(y)) = 0$ *for all* $y \in \mathbb{R}$. *Let* $[a_0, b_0] \subset [a,b]$ *be such that* $s_0 := \sup f[a_0, b_0] > \max\{f(a_0), f(b_0)\}$. *Then for each* $\varepsilon > 0$ *and* $y_0 < s_0$ *there is a* $y_1 \in (y_0, s_0)$ *and a component* (a_1, b_1) *of* $G_\varepsilon(H_{y_1}, (a_0, b_0))$ *satisfying:*

1. $[a_1, b_1] \subset (a_0, b_0)$

2. $(b_1 - a_1) < 1/2(b_0 - a_0)$

3. $\max\{f(a_0), f(b_0), y_0\} < y_1$

4. $\max\{f(a_1), f(b_1)\} \leqslant y_1$

5. $\Delta(H_{y_0}, (a_1, b_1)) > 1/2$

6. *If* $I \subset (a_0, b_0)$ *is an open interval such that* $I \cap \{a_1, b_1\} \neq \emptyset$, *then* $\Delta(H_{y_1}, I) \leqslant \varepsilon$.

The reader will note a marked similarity between the proof given of O'Malley's Theorem 1* and the following proof of O'Malley's Lemma, both depending on a recursive construction of an increasing sequence of real numbers and a nested sequence of intervals. The thing to notice is that, in the following proof of O'Malley's Lemma, the two sequences are chosen so that the corresponding members satisfy the first four items in the statement of O'Malley's Lemma, and then one real number and interval is chosen which also satisfies items (5) and (6) as well. By contrast, each of the real numbers and corresponding intervals found in the proof of O'Malley's Theorem 1* satisfy all six items from the statement of O'Malley's Lemma. It was then shown that the intersection of all the intervals consists of a single point, which turns out to be the point we were looking for (at which the function achieves a local approximate maximum).

PROOF. Fix $f, (a, b), (a_0, b_0), s_0, y_0$ and $\varepsilon > 0$ as in the hypotheses. Find $\alpha > 0$ so that $\max\{f(a_0), f(b_0), y_0\} < \alpha < s_0$. Since f is approximately continuous there is a $\delta > 0$ so that if I is an interval in $[a, b]$ with either a_0 or b_0 as an endpoint, and with $\lambda(I) < \delta$,

$$\Delta(H_\alpha, I) < \varepsilon/2. \tag{1}$$

In order to choose our number y_1 and interval (a_1, b_1), we will first construct a strictly increasing sequence of real numbers $\{r_k\}$ and nested intervals $\{(c_k, d_k)\}$. To initialize this construction, we set $r_0 = y_0$ and $(c_0, d_0) = (a_0, b_0)$.

By Remark 6, we may find some r_1 with $\alpha < r_1 < s_0$ so that $\lambda(G_\varepsilon(H_{r_1}, (a_0, b_0))) < \min\{\delta, 1/2(d_0 - c_0)\}$. Let (c_1, d_1) be any component of $G_\varepsilon(H_{r_1}, (a_0, b_0))$. Assuming that (c_1, d_1) shares an endpoint with (c_0, d_0) yields the contradiction

$$\varepsilon/2 \leqslant \Delta(H_{r_1}, (c_1, d_1)) \leqslant \Delta(H_\alpha, (c_1, d_1)) < \varepsilon/2.$$

The first inequality comes from Lemma 5(1), the second from $r_1 > \alpha$, and the third from (1). Thus $[c_1, d_1] \subset (c_0, d_0)$.

So far (1), (2) and (3) are satisfied for (c_1, d_1). To see (4), suppose, by way of contradiction, that $r_1 < \max\{f(c_1), f(d_1)\}$, and without loss of generality that $r_1 < f(c_1)$. The approximate continuity of f at c_1 then yields that $\Delta(H_{r_1}, c_1) = 1$. This, in turn, implies that there is an open interval I contained in (c_0, d_0) and containing c_1 with $\Delta(H_{r_1}, I) > \varepsilon$. This contradicts Lemma 5(2).

At this point we have that the interval (c_1, d_1) and the value r_1 satisfy items (1)-(4) in the statement of the lemma (replacing (a_1, b_1) with (c_1, d_1) and y_1 with r_1). The next step is to iterate this construction, obtaining a strictly nested sequence of intervals (c_k, d_k) and a strictly increasing sequence of numbers $r_1 < r_2 < \cdots < s_0$ such that, for each $k \geqslant 1$, (c_{k+1}, d_{k+1}) is a component of $G_\varepsilon(H_{r_{k+1}}, (c_k, d_k))$ satisfying

(i) $[c_{k+1}, d_{k+1}] \subset (c_k, d_k)$

(ii) $(d_{k+1} - c_{k+1}) < 1/2(d_k - c_k)$.

(iii) $\max\{f(c_k), f(d_k), r_k\} < r_{k+1}$

(iv) $\max\{f(c_{k+1}), f(d_{k+1})\} \leqslant r_{k+1}$

In order to justify this recursive construction (i.e. to ensure that at the kth stage the function f and the interval (c_k, d_k) satisfies the assumption made on f and (a_0, b_0) in the statement of the lemma), observe that since (c_{k+1}, d_{k+1}) is a component of $G_\varepsilon(H_{r_{k+1}}, (c_k, d_k))$, $H_{r_{k+1}} \cap (c_k, d_k) \neq \emptyset$. Therefore setting $s_k = \sup f[c_k, d_k]$, we have $s_k > r_{k+1} > \max\{f(c_k), f(d_k)\}$.

Items (i) and (iv) above guarantee that the intersection of the intervals $[c_k, d_k]$ consists of a single point, $\{x_0\} = \bigcap_k [c_k, d_k]$.

For each $k \geqslant 1$, $r_1 < r_{k+1}$, so $\Delta(H_{r_1}, (c_k, d_k)) \geqslant \Delta(H_{r_{k+1}}, (c_k, d_k)) \geqslant \varepsilon/2$ (by Lemma 5(1)). Therefore using Remark 3 we obtain $\overline{\Delta}(H_{r_1}, x_0) \geqslant \varepsilon/4$. Remark 4 now yields that $f(x_0) \geqslant r_1$ and since $r_1 > y_0$, $\Delta(H_{y_0}, x_0) = 1$. Therefore using Remark 2, we can find some n so that $\Delta(H_{y_0}, (c_n, d_n)) > 1/2$. Define $y_1 = r_n$ and $(a_1, b_1) = (c_n, d_n)$.

For this choice of (a_1, b_1) and y_1 it is easy to verify items (1)–(4) from the statement of the lemma. Item (5) follows immediately from the choice of n.

It remains to establish item (6). To this end, let $(c, d) \subset (a_0, b_0)$ be any open interval with $(c, d) \cap \{a_1, b_1\} \neq \emptyset$. Now, $(c, d) \subset (c_0, d_0) = (a_0, b_0)$, and $(c, d) \not\subset (c_n, d_n) = (a_1, b_1)$, so we may choose the least $m \in \{1, \dots, n\}$ such that $(c, d) \not\subset (c_m, d_m)$ (and thus $(c, d) \subset (c_{m-1}, d_{m-1})$). Moreover, (c, d) intersects (a_1, b_1), which is in turn contained in (c_m, d_m), so we have that $(c, d) \cap \{c_m, d_m\} \neq \emptyset$. Therefore, since (c_m, d_m) is a component of $G_\varepsilon(H_{r_m}, (c_{m-1}, d_{m-1}))$, it follows from Lemma 5(2) that $\Delta(H_{r_m}, (c, d)) \leq \varepsilon$. Since $y_1 \geqslant r_m$, it follows that $\Delta(H_{y_1}, (c, d)) \leqslant \varepsilon$ as required.

$\square$

4 An example

In order to show that item (b) in Ornstein's Theorem may not be significantly weakened without losing the result of the theorem, in [6] a continuous function $f : [0, 1] \to \mathbb{R}$ is described which satisfies the following weaker assumption (b'), but which is not monotonic on $[0, 1]$.

(b') For each point $x_0 \in [0, 1]$, the set $E = \left\{x : \frac{f(x) - f(x_0)}{x - x_0} \geq 0\right\}$ does not have zero density at x_0.

In this section we identify a problem with this example and provide a fix. In the last section we show that the typical continuous function satisfies condition (b') but is not monotonically increasing.

4.1 The Function and the Problem

Begin by choosing the eight points, $p_i = i/7$ for $i = 0, 1, \ldots, 7$ in $[0, 1]$ (the points $\{p_i\}$ have been explicitly chosen for the sake of concreteness, but with any other choice of these points the same problem would occur) and defining a function g at each of those points as follows:

$$g(p_0) = 1, \ \ g(p_1) = \frac{4}{3}, \ \ g(p_2) = \frac{1}{3}, \ \ g(p_3) = \frac{4}{3},$$
$$g(p_4) = \frac{-1}{3}, \ \ g(p_5) = \frac{2}{3}, \ \ g(p_6) = \frac{-1}{3}, \ \ g(p_7) = 0. \tag{2}$$

Extend g linearly on the intervening intervals and let

$$E_{x_0} = \left\{ x \in [0, 1] : \frac{g(x) - g(x_0)}{x - x_0} \geq 0 \right\}. \tag{3}$$

Then $\phi(x) = \Delta(E_x, [0, 1])$ is continuous and, by inspection, positive at each point $x \in [0, 1]$. Hence, there an $\alpha > 0$ such that $\phi(x) > \alpha$ for each $x \in [0, 1]$.

A sequence of functions, g_n is defined inductively by first setting $g_0 = g$.

> Assuming g_n has been defined, g_{n+1} is obtained by replacing each decreasing segment of g_n with a suitable affine copy of g in such a way that the resulting g_{n+1} is continuous. $\qquad (\star)$

We refer to the process described in $(\star)$ as the *insertion of g into g_n*. Specifically, this entails that if $[a, b]$ denotes a maximal interval on which g_n is decreasing, define $g_{n+1}(x) = S \circ g \circ T(x)$ for each $x \in [a, b]$ where

$$T(x) = \frac{x - a}{b - a} \ \text{ and } \ S(x) = xg(a) + (1 - x)g(b). \tag{4}$$

The claim is that the sequence g_n converges pointwise to a continuous function, and that this limit function f satisfies the condition (b').

Unfortunately, $\{g_n\}$ does not converge to a continuous function. To see this, we will show that there is a sequence of points $\{x_n\} \in [0, 1]$ such that for each n the sequence $\{g_k(x_n)\}$ is eventually constant, and these constants approach ∞ as $n \to \infty$. By the compactness of $[0, 1]$ this implies that the functions g_n do not converge to a continuous function.

To that end, we define a nested sequence $I_1 \supset I_2 \supset I_3 \cdots$ of sub-intervals of $[0, 1]$ inductively as follows. We set $I_1 = [3/7, 4/7]$. Suppose that I_k has been defined for all $k < n$, and $I_{n-1} = [a_{n-1}, b_{n-1}]$. Then we define

$$I_n = \left[a_{n-1} + \frac{3}{7^n}, a_{n-1} + \frac{4}{7^n}\right].$$

The construction of $\{g_n\}$ above immediately implies that for each n, I_n is a maximal interval of decrease for g_n. Let us also define a sequence $\{\Delta y_n\}$ by $\Delta y_n = g_n(b_n) - g_n(a_n)$, the net change in g_n on I_n. Since $g(1) - g(0) = 1$, and when g_n is defined on I_{n-1}, the vertical scaling factor used is $|\Delta y_{n-1}|$, an easy induction argument shows that

$$\Delta y_n = \Delta y_0 \cdot |\Delta y_{n-1}| = -|\Delta y_0|^{n+1} = -\left(\frac{5}{3}\right)^{n+1}.$$

Using the same reasoning, it may also be shown that

$$g_n(a_n) = 1 + \sum_{i=0}^{n} \frac{1}{3} \cdot \left(\frac{5}{3}\right)^i.$$

Of course this sequence $\{g_n(a_n)\} \to \infty$, and we note that since I_n is a maximal interval of decrease of g_n, for any $m > n$, $g_m(a_n) = g_n(a_n)$. Therefore putting $x_n = a_n$, the sequence has the properties described above. We conclude therefore that the sequence g_n does not converge to a continuous function.

In actuality and with a little more computation it's not hard to see that $\{g_n(\frac{1}{2})\} \to +\infty$. The purpose of this note, however, is simply to point out that the example needs repair and in the following subsection we show how this can be done in a rather straightforward manner.

4.2   A Fix

In this section we adapt Ornstein's function so that the change in y values (as in the Δy_n from our discussion of Ornstein's functions) on each decreasing interval is strictly less than 1. We will use h's here, rather than g's to avoid confusion.

Define h to be the continuous function defined $[0, 13]$ with the following prescribed values at the integers, and linear in the intervening intervals.

$$
\begin{array}{llllll}
h(0) &=& 4/4 & h(1) &=& 6/4 & h(2) &=& 3/4 \\
h(3) &=& 5/4 & h(4) &=& 2/4 & h(5) &=& 4/4 \\
h(6) &=& 1/4 & h(7) &=& 3/4 & h(8) &=& 0/4 \\
h(9) &=& 2/4 & h(10) &=& \text{-}1/4 & h(11) &=& 1/4 \\
h(12) &=& \text{-}2/4 & h(13) &=& 0/4 &&&
\end{array}
$$

The function h has been chosen so that at each point x_0, h_0 takes a smaller value to the left of x_0 or a larger value to the right of x_0, thus again assuring that the difference quotient is positive on a density (in $[0, 13]$) α set (for some $\alpha > 0$ independent of x_0). If we choose $\alpha > 0$ a bit smaller, we can say a bit more, that for each $x_0 \in [0, 13]$, the set

$$
E_{x_0} = \left\{ x \in [0, 13] : \frac{h_0(x) - h_0(x_0)}{x - x_0} \geq 0 \right\},
$$

where we disregard all x-values on which h_0 is decreasing, has density α in $[0, 13]$. This will be of use to us when we show that our final function h_∞ satisfies the property (b').

Let h_n be the sequence of functions with domain $[0, 13]$ defined recursively in precisely the same manner as Ornstein's function was defined, but using $h = h_0$ as our "seed function", rather than Ornstein's g. That is,

    ... to get h_{n+1} we simply replace each line segment of the graph of g_n having negative slope with an affine copy of h.

We will now show that our sequence h_n does converge uniformly on $[0, 13]$. If h_0 is decreasing on an interval $[a, b] \subset [0, 13]$, then

$$
h_0(b) - h_0(a) \geq -1 \cdot \frac{3}{4}.
$$

Recursively, if h_n is decreasing on an interval $[a, b] \subset [0, 13]$, then

$$
h_n(b) - h_n(a) \geq -1 \cdot \left(\frac{3}{4} \right)^n.
$$

Therefore the difference between h_{n+1} and h_n on $[a, b]$ is at most $(3/4)^n$ times the difference between h_0 and the line $y = -1/13x + 1$ on the interval $[0, 13]$. That is, the sequence $\{h_n\}$ converges uniformly.

4.3   Show that h_∞ Satisfies (b')

If $x_0 \in [0, 13]$ is contained in one of the intervals on which some one of the h_n's is increasing, then the desired result holds immediately, because the function values of all later h_{n+k}'s will not change on that interval.

Suppose that x_0 is not in any such interval. Define $I_0 = [0, 13]$, and for each $n > 0$, let $I_n \subset [0, 13]$ denote the interval on which h_{n-1} is decreasing which contains x_0. (That is, I_n is the interval containing x_0 on which h_{n-1} is changed to form h_n.) It is easy to see that $m(I_n) = \left(\dfrac{1}{13}\right)^{n-1}$.

Moreover, if we set

$$E_n = \left\{ x \in I_n : \frac{h_\infty(x) - h_\infty(x_0)}{x - x_0} \geq 0 \right\},$$

(again including only the x values at which h_n is increasing) then we will show that $\Delta(E_n, I_n) \geq 1/26$. Since the intervals $I_n \to \{x_0\}$, this will immediately imply that (b') holds.

We first show that, if $I_1 = [1, 2]$, then

$$\Delta(E_0, I_0) = \Delta(E_0, [0, 13]) \geq 1/26.$$

The idea is that, regardless of the value of $h_\infty(x_0)$, the union $[0, .5] \cup [2.5, 3]$ (ie the left half of the interval of increase to the left of I_1 and the right half of the interval of increase to the right of I_1) contains at least mass $1/2$ of E_0. Put $y_0 = h_\infty(x_0)$. We proceed by cases.

Case 6.1. $y_0 \geq 1.25$.

By inspection, E_0 contains the interval $[0, 0.5]$. Thus $\Delta(E_0, [0, 13]) \geq 1/26$.

Case 6.2. $y_0 \leq 1$.

By inspection, E_0 contains the interval $[2.5, 3]$. Thus $\Delta(E_0, [0, 13]) \geq 1/26$.

Case 6.3. $1 \leq y_0 \leq 1.25$.

This is the interesting case. Consider the intervals of the graph with x-values $[0, .5]$ and $[2.5, 3]$. h_∞ has constant slope on these intervals, and

$$h_\infty(0) = h_\infty(2.5) = 1, \text{ and } h_\infty(.5) = h_\infty(3) = 1.25.$$

Therefore if we choose $\epsilon \in (0, 1/2)$ so that $h_\infty(\epsilon) = y_0$, then E_0 contains the union $[0, \epsilon] \cap [2.5 + \epsilon, 3]$.

But $m([0, \epsilon] \cap [2.5 + \epsilon, 3]) = .5$, so we conclude that $\Delta(E_0, [0, 13]) \geq 1/26$.

This argument extends immediately to show that $\Delta(E_0, [0, 13]) \geq 1/26$ for every other choice of I_1, always finding points in E_0 with mass at least $1/2$ in the intervals of increase of h_0 directly to the left and right of I_1.

Let us now consider the second interval I_1. We wish to show that $\Delta(E_1, I_1) \geq 1/26$. The first (ie from left to right) possibility for I_2 is the interval $[14/13, 15/13]$.

Here the argument is the same with the y values for the cases being: Case 1) $y_0 \geq 27/16$, Case 2) $y_0 \leq 1.5$, and Case 3) $1.5 \leq y_0 \leq 27/16$, and the intervals where we will be finding points in E_1 are $[26/26, 27/26]$ (the first half of the interval of increase of h_1 to the left of I_2) and $[31/26, 32/26]$ (the second half of the interval of increase of h_1 to the right of I_2). The conclusion is that $m(E_1)$ is greater than or equal to the length of one of these intervals, $m(E_1) \geq 1/26$, and $m(I_1) = 1$. Thus $\Delta(E_1, I_1) \geq 1/26$. This iterates nicely (we always pick up a factor of $1/13$ in both the numerator and the denominator of our density calculation, which cancel), thus $\Delta(E_n, I_n) \geq 1/26$, and thus the upper density of the set

$$E = \left\{ x \in [0, 13] : \frac{h_\infty(x) - h_\infty(x_0)}{x - x_0} \geq 0 \right\}$$

at x_0 is greater than or equal to $1/26 > 0$, proving that h_∞ satisfies (b').

5   Counterexamples are typical

The goal of this final section is to show that continuous functions with Property (b') are ubiquitous in the complete space $C[0, 1]$ of all continuous functions on [0,1] endowed with the sup metric. However, "ubiquitous" can be defined in several ways.

A property is *typical* in a complete metric space of functions, $C[0, 1]$ if the set of functions enjoying that property is residual (the complement of a set of the first Baire Category) in $C[0, 1]$. A well known method of establishing whether a given set A is residual or not is the so-called Banach-Mazur Game which we describe briefly here. See [7] for more details and generalizations.

This is a two player game and the players take turns selecting open balls from $C[0, 1]$. Suppose $A \subset C[0, 1]$ is fixed. Player P_1 selects a ball, B_1, then player two, P_2 selects a ball $B_2 \subset B_1$ and so on so that the game produces a nested sequence of balls, $\{B_n : n \in \mathbb{N}\}$. P_2 wins the game if $A \cap \bigcap_{n=1}^{\infty} B_n \neq \emptyset$ otherwise P_1 wins. And P_2 has a winning strategy provided P_2 can always win the game, meaning independently of the balls P_1 selects.

Banach-Mazur Theorem. *P_2 has a winning strategy iff A is residual.*

The Banach-Mazur Game is a convenient way to see why the next result is true. The proof uses the following notation. If $p_i = (x_i, y_i) \in \mathbb{R}^2$, $i = 1, 2$ we define

$$DQ(p_1, p_2) = \frac{y_2 - y_1}{x_2 - x_1}.$$

Theorem 1. *The typical continuous function has Property (b').*

PROOF. Suppose P_1 has chosen the ball $B_n \equiv B_\epsilon(f) \subset C[0,1]$ at the nth stage of play. We describe the strategy for P_2.

1. First partition $[0,1]$ into sufficiently small intervals such that the insertion (see ($\star$) on page 340) of h into any partition interval lies within the $\epsilon/2$ ball about f. Let $g : [0,1] \to \mathbb{R}$ be the conjunction of all such insertions and define

$$E_{x_0} = \left\{ x : \frac{g(x) - g(x_0)}{x - x_0} > 0 \right\}.$$

   Then for every partition interval J and every $x_0 \in J$, $\Delta(E_{x_0}, J) > \alpha$. This function g is the center of the ball P_2 will respond with.

2. To determine the response radius, first fix a partition interval J and an $x_0 \in J$. There exists $\eta(x_0) > 0$ such that $\Delta(E_{x_0} \backslash B_{\eta(x_0)}, J) > \alpha$. Hence by compactness there is a $\eta > 0$ such that for every $x \in [0,1]$ and every partition interval J containing x,

$$\Delta(E_x \backslash B_\eta(x), J) > \alpha.$$

   Hence, again by compactness, there is a $0 < \delta < \epsilon/2$ such that $\delta < \eta$ and if $x_0 \in [0,1]$ and $x_1 \in E_{x_0} \backslash B_\eta$ then whenever

$$p_0 \in B_\delta((x_0, g(x_0))) \text{ and } p_1 \in B_\delta((x_1, g(x_1))) \text{ then } DQ(p_0, p_1) > 0. \quad (5)$$

Player P_2 returns the ball $B_\delta(g)$ where δ is the radius just determined above.

Now, any sequence of plays converges uniformly in the sense that if f_n is any choice of a function in B_n, then the sequence of functions $\{f_n\}$ converges uniformly. Due to (5), at each $x \in [0,1]$ the density of the set of points for which the difference quotient is positive at x exceeds α at the scale of each play of P_2. That is, the limit function satisfies (b'); this then, completes the proof.   □

As it is well known that the set of monotone functions is nowhere dense in $C[0,1]$, the following is an immediate corollary.

Corollary 7. *The typical continuous function satisfies property (b') but is not monotonically increasing on any interval.*

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