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ON THE MIXED DERIVATIVES OF A SEPARATELY TWICE DIFFERENTIABLE FUNCTION

Abstract

We prove that a function $f(x,y)$ of real variables defined on a rectangle, having partial derivatives $f''_{xx}, f''_{yy} \in L_2([0,1]^2)$, has almost everywhere mixed derivatives f''_{xy} and f''_{yx}.

1 Introduction

In the well known "Scottish Book" [6], S. Mazur posed the following question (VII.1935, Problem 66):

The real function $z = f(x,y)$ of real variables x, y possesses the 1st partial derivatives f'_x, f'_y and the pure second partial derivatives f''_{xx}, f''_{yy}. Do there exist then almost everywhere the mixed 2nd partial derivatives f''_{xy}, f''_{yx}? According to a remark by P. Schauder, this theorem is true with the following additional assumptions: The derivatives f'_x, f'_y are absolutely continuous in the sense of Tonelli, and the derivatives f''_{xx}, f''_{yy} are square integrable. An analogous question for n variables.

Given a function $f : [a,b] \times [c,d] \to \mathbb{R}$ and $x \in [a,b]$, we denote by $V_1(x)$ the variation of the function $f^x : [c,d] \to \mathbb{R}$, $f^x(y) := f(x,y)$, and given $y \in [c,d]$ we denote by $V_2(y)$ the variation of $f_y : [a,b] \to \mathbb{R}$, $f_y(x) := f(x,y)$. A function f is of *Tonelli bounded variation* [11, p. 169] if $\int_a^b V_1(x)dx < \infty$ and $\int_c^d V_2(y)dy < \infty$. All integrals we consider (here and further in the paper) are Lebesgue integrals. A function f is *absolutely continuous in the sense of*

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Tonelli if f is of Tonelli bounded variation: for almost all $x \in [a,b]$ and for almost all $y \in [c,d]$ the functions f_y and f^x are absolutely continuous.

The existence and measurability of (mixed) partial derivatives were investigated by many mathematicians (see [1, Th. 79-81], [4], [3], [14], [13], [8], [2], [12], [5], [7] and the literature given there). Mainly, these results give some sufficient conditions for existence (and equality) almost everywhere of mixed second partial derivatives, but they do not give any answer to the Mazur problem. In particular, G. Tolstov in [13] proved the following result (see also [7, Lemma 4]).

Proposition 1.1. *Let $h \in L_1([0,1]^2)$ and*

$$f(x,y) = \int_0^x dx \int_0^y h(u,v)dudv.$$

Then there exists a measurable set $A \subseteq [0,1]$ with $\mu(A) = 1$ such that

$$f'_x(x_0,y) = \int_0^y h(u,v)dv$$

for every $x_0 \in A$ and $y \in [0,1]$.

Using this statement G. Tolstov proved that if a separately differentiable function $f : [0,1]^2 \to \mathbb{R}$ has $f'_x \in C([0,1]^2)$ and there exists f''_{xy} on a set $D \subseteq [0,1]^2$ of the second category, then there exists a rectangle $P \subseteq [0,1]^2$ such that $f''_{xy} = f''_{yx}$ almost everywhere on P. This result was developed in [7, Theorem 7]. Moreover, in [14, Theorem 7] G. Tolstov proved the following theorem.

Theorem 1.2. *Let a measurable function $f : (0,1)^2 \to \mathbb{R}$ and a measurable set $E \subseteq (0,1)^2$ of positive measure satisfy the following conditions:*

(i) for every $(x_0,y_0) \in E$ there exists $\delta > 0$ such that f has the partial derivatives $f'_x(p)$ and $f'_y(p)$ at every point $p \in ((x_0-\delta, x_0+\delta) \times \{y_0\}) \cup (\{x_0\} \times (y_0-\delta, y_0+\delta))$;

(ii) for every $p_0 \in E$ all derivative numbers of the functions f'_x and f'_y with respect to each variable at the point p_0 are finite.

Then a.e. on E there exist and are equal the mixed derivatives f''_{xy} and f''_{yx}.

A real $\lambda \in \mathbb{R}$ or a symbol $\lambda \in \{-\infty, +\infty\}$ is called *a derivative number of a function $f : (a,b) \to \mathbb{R}$ at a point $x_0 \in (a,b)$* if there exists a sequence $(\alpha_n)_{n=1}^{\infty}$ of reals $\alpha_n \neq 0$ such that $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\lim\limits_{n\to\infty} \frac{f(x_0+\alpha_n)-f(x_0)}{\alpha_n} = \lambda$.

On the other hand, there are some interesting examples among the above mentioned results. G. Tolstov constructed in [13] a function $f : [0,1]^2 \to \mathbb{R}$ having first partial derivatives $f'_x, f'_y \in C([0,1]^2)$ and second partial derivatives f''_{xx} and f''_{yy} which are different on a set of positive measure. Moreover, J. Serrin constructed in [12] a measurable function $f : [0,1]^2 \to \mathbb{R}$ such that for almost all $y \in [0,1]$ the functions $f_y : [0,1] \to \mathbb{R}$, $f_y(x) = f(x,y)$, are differentiable and the set $\{p \in [0,1]^2 : \exists\ f'_x(p)\}$ is non-measurable.

The Mazur problem was solved in the negative in [9]. A separately twice differentiable function $f : [0,1]^2 \to \mathbb{R}$ was constructed in [9], whose partial derivative f'_x is discontinuous with respect to y on a set $A \times B \subseteq [0,1]^2$ with $\mu(A) = 1$ and $\mu(B) > 0$. This example shows that for a separately twice differentiable function $f : [0,1]^2 \to \mathbb{R}$ the continuity of f'_x with respect to y plays an important role for the existence of f''_{xy}.

Note that the second partial derivatives f''_{xx}, f''_{yy} of the function f from [9] are not integrable. Thus the following question naturally arises in the connection with Schauder's remark to the Mazur problem and the example from [9].

Problem 1.3. *Let $f : [0,1]^2 \to \mathbb{R}$ be a separately twice differentiable function and $f''_{xx}, f''_{yy} \in L_2([0,1]^2)$.*

(i) Does there exist a set $A \subseteq [0,1]$ with $\mu(A) = 1$ such that f'_x is continuous with respect to y at each point of $A \times [0,1]$?

(ii) Do there exist almost everywhere mixed derivatives f''_{xy} and f''_{yx}?

In this paper we give the positive answer to Problem 1.3. In Section 2 (Corollary 2.3) we give some sufficient conditions on a function $f(x,y)$ for $f'_x \in C([0,1]^2)$. In Section 3 we prove an auxiliary statement (Proposition 3.3) on the consistency of the Fourier series of a function f and its partial derivative f'_x, which we use in Section 4 for the proof of the main result of the paper (Theorem 4.1). Finally, in Section 5 we give two examples which show the essentiality of some assumptions in Corollary 2.3 and Theorem 4.1 and formulate open questions.

2 Jointly continuity of the first partial derivative

Lemma 2.1. *Let $Y \subseteq \mathbb{R}$, let a function $f : [0,1] \times Y \to \mathbb{R}$ be continuous with respect to y and f'_x be (uniformly) continuous with respect to x, uniformly on y. Then $f'_x : [0,1] \times Y \to \mathbb{R}$ is continuous.*

Proof. Fix $x_0 \in [0,1]$, $y_0 \in Y$ and $\varepsilon > 0$. Choose a neighborhood $U = [x_1, x_2]$ of x_0 such that for all $x, x' \in U$ and all y,

$$|f'_x(x,y) - f'_x(x',y)| < \frac{\varepsilon}{4}.$$

Using the continuity of f with respect to y, choose a neighborhood V of y_0 such that for all $y \in V$,

$$\left|\frac{f(x_2,y_0) - f(x_1,y_0)}{x_2 - x_1} - \frac{f(x_2,y) - f(x_1,y)}{x_2 - x_1}\right| < \frac{\varepsilon}{2}.$$

By the Lagrange theorem, for every $y \in V$ there is $x_y \in U$ such that

$$f'_x(x_y, y) = \frac{f(x_2,y) - f(x_1,y)}{x_2 - x_1}.$$

Hence, for each $y \in V$,

$$|f'_x(x_{y_0}, y_0) - f'_x(x_y, y)| < \frac{\varepsilon}{2}.$$

Therefore, for an arbitrary $(x,y) \in U \times V$,

$$\begin{aligned}|f'_x(x_0,y_0) - f'_x(x,y)| &\le |f'_x(x_0,y_0) - f'_x(x_{y_0},y_0)| + |f'_x(x_{y_0},y_0) - f'_x(x_y,y)| \\ &\quad + |f'_x(x_y,y) - f'_x(x,y)| \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

$\square$

Lemma 2.2. *Let $g : [0,1] \to \mathbb{R}$ and $\int_0^1 (g')^2 dx \le c$. Then the function g is Hölder of the order $\frac{1}{2}$ and constant $\sqrt{c}$.*

Proof. For all $0 \le x_0 < x_1 \le 1$

$$|g(x_1) - g(x_0)| = \left|\int_{x_0}^{x_1} g' dx\right| \le \left((x_1 - x_0)\int_{x_0}^{x_1} (g')^2 dx\right)^{\frac{1}{2}} \le \sqrt{c(x_1 - x_0)}.$$

$\square$

The next theorem is a simple combination of the previous two lemmas.

Corollary 2.3. *Let $Y \subseteq \mathbb{R}$, let a function $f : [0,1] \times Y \to \mathbb{R}$ be continuous with respect to y, and suppose f''_{xx} exists and*

$$\sup_{y \in Y} \int_0^1 (f''_{xx}(x,y))^2 dx < \infty.$$

Then $f'_x : [0,1] \times Y \to \mathbb{R}$ is continuous.

3 Square integrable partial derivatives

Lemma 3.1. *Let* $\int_0^{2\pi} g^2(x)dx < \infty$ *,* $\int_0^{2\pi} g(x)dx = 0$ *and* $f(x) = a + \int_0^x g(t)d\mu$ *(in particular let* f *be a differentiable function such that* $f(0) = f(2\pi)$ *and the derivative* $g = f' \in L_1([0, 2\pi])$*). Then for every* $n \in \mathbb{N}$*,*

$$\int_0^{2\pi} f(x)\cos nx dx = -\frac{1}{n}\int_0^{2\pi} g(x)\sin nx dx$$

and

$$\int_0^{2\pi} f(x)\sin nx dx = \frac{1}{n}\int_0^{2\pi} g(x)\cos nx dx.$$

Proof. By [10, p. 251], for a differentiable function f we have $f(x) = f(0) + \int_0^x f'(t)d\mu$ for every $x \in [0, 2\pi]$. It remains to use the integration by parts [10, Ch. IX, §8, Th. 5]. □

For a function $f : [a, b] \to \mathbb{C}$ and $p \in \{1, 2\}$ the expression $f \in L_p([a, b])$ will denote that $\int_a^b |f(x)|^p\, dx < \infty$. The same for a function $f : [a, b] \times [c, d] \to \mathbb{C}$.

For a function $f : [0, 2\pi]^2 \to \mathbb{C}$, the expression

$$f \sim \sum\nolimits_{n,m\in\mathbb{Z}} a_{nm}e^{inx}e^{imy}$$

will denote that $f \in L_2([0, 2\pi]^2)$ and $\sum_{n,m\in\mathbb{Z}} a_{nm}e^{inx}e^{imy}$ is the Fourier series of f which converges to f in the L_2-norm. The same for a function $f : [0, 2\pi] \to \mathbb{C}$.

Lemma 3.2. *Let* $f \sim \sum_{n,m\in\mathbb{Z}} a_{nm}e^{inx}e^{imy}$ *,* $\alpha_n \sim \sum_{m\in\mathbb{Z}} a_{nm}e^{imy}$*,* $n \in \mathbb{Z}$ *and* $f_y(x) := f(x, y)$*.*

Then there exists a subset $B \subseteq [0, 2\pi]$ *with* $\mu(B) = 2\pi$ *such that* $\forall\, y \in B$ *the function* f_y *is square integrable and*

$$f_y \sim \sum\nolimits_{n\in\mathbb{Z}} \alpha_n(y)e^{inx}. \tag{1}$$

Proof. For every $n \in \mathbb{Z}$ we consider the linear continuous operator $T_{1,n} : L_2([0, 2\pi]^2) \to L_2([0, 2\pi])$, which any function $g \sim \sum_{n,m\in\mathbb{Z}} b_{nm}e^{inx}e^{imy}$ sends to the function $T_{1,n}g \sim \sum_{m\in\mathbb{Z}} b_{nm}e^{imy}$. Note that $T_{1,n}f = \alpha_n$ for every $n \in \mathbb{Z}$. Moreover, we consider the linear operator $T_{2,n} : L_2([0, 2\pi]^2) \to L_2([0, 2\pi])$,

$$T_{2,n}g(y) = \frac{1}{2\pi}\int_0^{2\pi} g(x, y)e^{-inx}dx$$

for all $y \in B(g) = \{v \in [0, 2\pi] : g_v \in L_2([0, 2\pi])\}$, where $g_v(x) := g(x, v)$. Note that $\mu(B(g)) = 2\pi$. Since

$$\left(\int_0^{2\pi} h(x)dx\right)^2 \le 2\pi \int_0^{2\pi} |h(x)|^2 dx$$

for every measurable on $[0, 2\pi]$ function h, by the Fubini theorem we have

$$\begin{aligned}\int_0^{2\pi} |T_{2,n}g(y)|^2 \, dy &\le \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} |g(x,y)e^{-inx}|^2 dxdy \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^{2\pi} |g(x,y)|^2 dxdy.\end{aligned}$$

Thus $T_{2,n}$ is continuous. Since $T_{1,n}(e^{ikx}e^{imy}) = T_{2,n}(e^{ikx}e^{imy})$ for every $k, m \in \mathbb{Z}$, $T_{1,n} = T_{2,n}$; in particular, $T_{2,n}f = \alpha_n$. Now we choose a set $B \subseteq B(f)$ with $\mu(B) = 2\pi$ such that $\alpha_n(y) = \frac{1}{2\pi}\int_0^{2\pi} f(x,y)e^{-inx}dx$ for every $n \in \mathbb{Z}$ and $y \in B$. This gives (1). □

Proposition 3.3. *Let a function $f : [0, 2\pi]^2 \to \mathbb{R}$ be differentiable with respect to x, $f'_x \in L_2([0, 2\pi]^2)$ and, moreover, $f(0, y) = f(2\pi, y) = \alpha(y)$ for every $y \in [0, 2\pi]$ with $\alpha \in L_2([0, 2\pi])$. Then $f \in L_2([0, 2\pi]^2)$. Moreover, if*

$$f \sim \sum\nolimits_{n,m\in\mathbb{Z}} a_{nm}e^{inx}e^{imy}, \qquad \textit{then} \qquad f'_x \sim \sum\nolimits_{n,m\in\mathbb{Z}} ina_{nm}e^{inx}e^{imy}.$$

PROOF. Let

$$f'_x \sim \sum\nolimits_{n,m\in\mathbb{Z}} b_{nm}e^{inx}e^{imy}.$$

By the Fubini theorem there exists a set $B \subseteq [0, 2\pi]$ with $\mu(B) = 2\pi$ such that for every $y \in B$ we have $g_y \in L_2([0, 2\pi])$, where $g_y(x) := f'_x(x, y)$, and in particular $g_y \in L_1([0, 2\pi])$. Note that for an arbitrary $y \in B$ we have

$$\int_0^{2\pi} f'_x(x,y)dx = f(2\pi, y) - f(0, y) = 0.$$

So, by the Fubini theorem,

$$b_{0m} = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f'_x(x,y)e^{-imy}dxdy = 0$$

for all $m \in \mathbb{Z}$.

Consider the function $h(x,y) = f(x,y) - \alpha(y)$. For every $y \in B$ and $x \in [0,2\pi]$ we have

$$h^2(x,y) = \left(\int_0^x g_y(t)dt\right)^2 \leq x\int_0^x g_y^2(t)dt \leq 2\pi \int_0^{2\pi} g_y^2(t)dt.$$

Thus

$$\int_0^{2\pi}\int_0^{2\pi} h^2(x,y)dxdy \leq 4\pi^2 \int_0^{2\pi}\int_0^{2\pi} (f'_x(x,y))^2 dxdy,$$

and $h \in L_2([0,2\pi]^2)$. So $f \in L_2([0,2\pi]^2)$, too. Let

$$f \sim \sum\nolimits_{n,m\in\mathbb{Z}} a_{nm} e^{inx} e^{imy}.$$

Now using Lemma 3.1 and the Fubini theorem for every $m, n \in \mathbb{Z}$, $n \neq 0$, we have

$$\begin{aligned} a_{nm} &= \frac{1}{4\pi^2}\int_B e^{-imy}dy \int_0^{2\pi} f(x,y)e^{-inx}dx \\ &= \frac{1}{in}\frac{1}{4\pi^2}\int_B e^{-imy}dy \int_0^{2\pi} f'_x(x,y)e^{-inx}dx \\ &= \frac{b_{nm}}{in}. \end{aligned}$$

□

4 Main result

The following theorem gives a positive answer to the Mazur problem for functions with square integrable pure partial derivatives (Problem 1.3).

Theorem 4.1. *Let $f : [0,1]^2 \to \mathbb{R}$ be a separately twice differentiable function and $f''_{xx}, f''_{yy} \in L_2([0,1]^2)$. Then*

(i) a.e. there are equal mixed derivatives $f''_{xy}, f''_{yx} \in L_2([0,1]^2)$;

(ii) there exists $A \subseteq [0,1]$ with $\mu(A) = 1$ such that f'_x is continuous with respect to y at every point of $A \times [0,1]$;

(iii) $f \in C([0,1]^2)$.

Proof. (i). We consider a function $f : [0,2\pi]^2 \to \mathbb{R}$. It is sufficient to prove this assertion for a sequence of products $f \cdot \varphi_n$ of f by twice differentiable functions φ_n with bounded derivatives and the conditions $\varphi_n(x,y) = 1$ for all $(x,y) \in [\frac{1}{n}, 2\pi - \frac{1}{n}]^2$ and $\varphi_n(x,y) = 0$ on an open set in $[0,2\pi]^2$ which contains the boundary of $[0,2\pi]^2$. Thus it is sufficient to consider a function f which satisfies the additional assumption

$$f(x,y) = f'_x(x,y) = f'_y(x,y) = f''_{xx}(x,y) = f''_{yy}(x,y) = 0$$

on an open set in $[0,2\pi]^2$ which contains the boundary of $[0,2\pi]^2$.

According to Proposition 3.3, we have $f, f'_x, f'_y \in L_2([0,2\pi]^2)$. Let

$$f \sim \sum\nolimits_{n,m\in\mathbb{Z}} a_{nm} e^{inx} e^{imy}.$$

Then, by Proposition 3.3,

$$f'_x \sim \sum\nolimits_{n,m\in\mathbb{Z}} ina_{nm} e^{inx} e^{imy}, \quad f'_y \sim \sum\nolimits_{n,m\in\mathbb{Z}} ima_{nm} e^{inx} e^{imy},$$

$$f''_{xx} \sim \sum\nolimits_{n,m\in\mathbb{Z}} -n^2 a_{nm} e^{inx} e^{imy} \quad \text{and} \quad f''_{yy} \sim \sum\nolimits_{n,m\in\mathbb{Z}} -m^2 a_{nm} e^{inx} e^{imy}.$$

Let

$$\alpha_m \sim \sum\nolimits_{n\in\mathbb{Z}} ina_{nm} e^{inx},\ m \in \mathbb{Z}.$$

Using Proposition 3.2, we choose a set $A_1 \subseteq [0,2\pi]$ so that $\mu(A_1) = 2\pi$ and for every $x \in A_1$ we have $g^x \in L_2([0,2\pi])$ and

$$g^x \sim \sum\nolimits_{m\in\mathbb{Z}} \alpha_m(x) e^{imy},$$

where $g^x(y) := f'_x(x,y)$. Since there exist open neighborhoods V_1 and V_2 of points 0 and 2π in $[0,2\pi]$ such that $g^x(y) = 0$ for every $x \in A_1$ and $y \in V_1 \cup V_2$, according to the well-known localization theorem of Riemann we have

$$\sum\nolimits_{m\in\mathbb{Z}} \alpha_m(x) = 0$$

for every $x \in A_1$. From the Fourier expansions of f''_{xx} and f''_{yy} we have the bound

$$\sum\nolimits_{n,m\in\mathbb{Z}} m^2 n^2 |a_{nm}|^2 \le \sum\nolimits_{n,m\in\mathbb{Z}} (m^4 + n^4)|a_{nm}|^2 < \infty.$$

Then there exists a function $h \in L_2([0,2\pi]^2)$ with

$$h \sim -\sum\nolimits_{n,m\in\mathbb{Z}} mna_{nm} e^{inx} e^{imy}.$$

Once more, using Proposition 3.2 we choose a set $A_2 \subseteq A_1$ so that $\mu(A_2) = 2\pi$ and for every $x \in A_2$, we have $h^x \in L_2([0,2\pi])$ and

$$h^x \sim \sum\nolimits_{m\in\mathbb{Z}} im\alpha_m(x) e^{imy},$$

where $h^x(y) := h(x,y)$. Put

$$F(x,y) = \int_0^y h(x,t)dt.$$

Note that $h^x \in L_1([0,2\pi])$ and $\sum_{m\in\mathbb{Z}} \alpha_m(x) = 0$ for every $x \in A_2$. Using Theorem 3 of [10, Ch. X, §4] on the termwise integration of Fourier series of integrable functions of one variable, we obtain that for every $x \in A_2$ the equality

$$F(x,y) = \sum\nolimits_{m\in\mathbb{Z}} \alpha_m(x)(e^{imy} - 1) = \sum\nolimits_{m\in\mathbb{Z}} \alpha_m(x)e^{imy}$$

is satisfied. Note that $F \in L_2([0,2\pi]^2)$ (it may be obtained analogously as the inclusion $h \in L_2([0,2\pi]^2)$ in the proof of the Proposition 3.3). Thus $F = f'_x$ in $L_2([0,2\pi]^2)$.

Hence, $F(x,y) = f'_x(x,y)$ almost everywhere on $[0,2\pi]^2$. Therefore, by the Fubini theorem, there exists a set $B \subseteq [0,2\pi]$ with $\mu(B) = 2\pi$ such that

$$\mu\left(\{x \in [0,2\pi] : f'_x(x,y) = F(x,y)\}\right) = 2\pi,$$

and the function $g_y \in L_1([0,2\pi])$ for every $y \in B$, where $g_y(x) := f'_x(x,y)$.

Consider the function

$$G(x,y) = \int_0^x du \int_0^y h(u,v)dv.$$

Now for every $x \in [0,2\pi]$ and $y \in B$ we have

$$f(x,y) = \int_0^x f'_x(u,y)du = \int_0^x F(u,y)du = \int_0^x du \int_0^y h(u,v)dv = G(x,y).$$

Since f is continuous with respect to y, $G \in C([0,2\pi]^2)$ and B is dense in $[0,2\pi]$, we have that $f(x,y) = G(x,y)$ for every $(x,y) \in [0,2\pi]^2$.

Note that the function f'_x is continuous with respect to x. According to [9, Proposition 2.1] the set E of all points $(x,y) \in [0,2\pi]^2$ at which f''_{xy} exists is a $F_{\sigma\delta}$-set. In particular, E is measurable. It follows from Proposition 1.1 that the set E has the measure $4\pi^2$ and a.e. $f''_{xy} = h$. Analogously a.e. there exist mixed derivatives f''_{yx} which are a.e. equal to h. Hence, $f''_{x,y}, f''_{yx} \in L_2([0,2\pi]^2)$.

(ii), (iii). It follows from the proof of (i) that for every $n \in \mathbb{N}$ there exists $h_n \in L_2([0,1]^2)$ such that

$$f(x,y) = \int_0^x du \int_0^y h_n(u,v)dv$$

for every $(x,y) \in [\frac{1}{n}, 1-\frac{1}{n}]^2$. Therefore f is continuous on $(0,1)^2$ and according to Proposition 1.1 there exists $A_0 \subseteq [0,1]$ with $\mu(A_0) = 1$ such that f'_x is continuous with respect to y at every point of $A \times (0,1)$. It remains to use this fact to some separately twice differentiable extension $\tilde{f} : [-1,2]^2 \to \mathbb{R}$ of f with $\tilde{f}''_{xx}, \tilde{f}''_{yy} \in L_2([-1,2]^2)$. □

Corollary 4.2. *Let $f(x,y)$ have on $[0,1]^2$ the second pure partial derivatives. Then there exists an open dense set $G \subseteq [0,1]^2$ on which there are equal mixed partial derivatives f''_{xy} and f''_{yx}.*

Proof. Note that by [14, p. 427] the functions f''_{xx} and f''_{yy} are of the first Baire class. Hence, there exists an open dense subset $G \subseteq [0,1]^2$ on which the pure derivations are locally bounded. It remains to use Theorem 4.1. □

5 Examples, questions

For a real valued function f, we denote $\mathrm{supp} f = \{x \in \mathbb{R} : f(x) \neq 0\}$.

The following example shows that the assumption

$$\sup_{y \in Y} \int_0^1 (f''_{xx}(x,y))^2 dx < \infty$$

in Corollary 2.3 cannot be replaced by

$$\sup_{y \in Y} \int_0^1 |f''_{xx}(x,y)| dx < \infty.$$

Theorem 5.1. *There exists a function $f : [0,1]^2 \to \mathbb{R}$ satisfying the following conditions:*

- (i) *f is separately infinitely differentiable;*
- (ii) $\sup_{y\in[0,1]} \int_0^1 |f''_{xx}(x,y)|dx = \sup_{x\in[0,1]} \int_0^1 |f''_{yy}(x,y)|dy < \infty$;
- (iii) *f'_x and f'_y are discontinuous at every point of some closed set E of positive measure.*

Proof. Let

$$C = [0,1] \setminus \bigsqcup_{n=1}^{\infty} \bigsqcup_{k=1}^{2^{n-1}} (a_{n,k}, b_{n,k})$$

be a Cantor type set of positive measure such that

(1) $0 < a_{n,k} < b_{n,k} < 1$ for every n and k;

(2) $a_{n,k} \neq b_{m,l}$ for every $n, m \in \mathbb{N}$, $k \leq 2^{n-1}$ and $l \leq 2^{m-1}$;
(3) $b_{n,k} - a_{n,k} = b_{n,l} - a_{n,l}$ for every $n \in \mathbb{N}$ and $k, l \leq 2^{n-1}$.

Let $\{a_{n,k}, b_{n,k} : n \in \mathbb{N}, 1 \leq k \leq 2^{n-1}\} = \{p_n : n \in \mathbb{N}\}$, $\varphi : \mathbb{N} \to \mathbb{N}^3$ be a bijection. Inductively for n we choose a sequence $(W_n)_{n=1}^{\infty}$ of rectangles $W_n = U_n \times V_n$ such that

(a) $U_n = (a_n, b_n), V_n = (c_n, d_n) \in \{(a_{m,k}, b_{m,k}) : m \in \mathbb{N}, 1 \leq k \leq 2^{m-1}\}$;
(b) $U_n \cap U_m = V_n \cap V_m = \emptyset$ for all distinct $n, m \in \mathbb{N}$;
(c) $b_n - a_n = d_n - c_n$ for all $n \in \mathbb{N}$;
(d) $W_n \subseteq \{(x, y) \in \mathbb{R}^2 : \max\{|x - c_k|, |y - c_m|\} \leq \frac{1}{l}\}$ for every $n \in \mathbb{N}$, where $(k, m, l) = \varphi(n)$.

Note that $E = C^2 \subseteq \overline{\{w_n : n \in \mathbb{N}\}}$ for every sequence $(w_n)_{n=1}^{\infty}$ of points $w_n \in W_n$.

Let $\psi : \mathbb{R} \to \mathbb{R}^+$ be an arbitrary infinitely differentiable function with $\operatorname{supp} \psi(y) = (0, 1)$ and $\max_{x \in [0,1]} |\psi(x)| = 1$. For every $n \in \mathbb{N}$ we put $\varepsilon_n = b_n - a_n = d_n - c_n$,

$$\varphi_n(x) = \psi\left(\frac{x - a_n}{\varepsilon_n}\right)$$

and

$$\psi_n(y) = \psi\left(\frac{y - c_n}{\varepsilon_n}\right).$$

Consider the function $f : [0, 1]^2 \to \mathbb{R}$,

$$f(x, y) = \sum\nolimits_{n=1}^{\infty} \varepsilon_n \varphi_n(x) \psi_n(y).$$

It follows from (b) that f is separately infinitely differentiable. Clearly,

$$\sup_{y \in [0,1]} \int_0^1 |f''_{xx}(x, y)| dx = \sup_{x \in [0,1]} \int_0^1 |f''_{yy}(x, y)| dy = \int_0^1 |\psi''(x)| dx.$$

Thus f satisfies the condition (ii).

We show that f satisfies the condition (iii). For every $n \in \mathbb{N}$ we choose $u_n \in U_n$ and $v_n \in V_n$ such that

$$\varphi_n(u_n) = A := \max_{x \in [0,1]} |\psi'(x)| \quad \text{and} \quad |\psi_n(v_n)| = 1.$$

Therefore $|f'_x(u_n, v_n)| = A$ for every $n \in \mathbb{N}$. Since $f'_x(z) = 0$ for every $z \in E$ and $E = \subseteq \overline{\{(u_n, v_n) : n \in \mathbb{N}\}}$, f'_x is discontinuous at every point of E.

Analogously f'_y is jointly discontinuous at every point of E. $\square$

The following modification of the example from [9, Theorem 3.2] shows that the assumptions of the existence of f''_{yy} and f''_{xx} everywhere on the rectangle $[0, 2\pi]^2$ in Theorem 4.1 cannot be weakened.

Theorem 5.2. *There exists a function $f : [0,1]^2 \to \mathbb{R}$ satisfying the following conditions:*

(i) *f has continuous partial derivative f''_{yy};*

(ii) *for every $y \in [0,1]$ there exists a finite set $A(y)$ such that $f''_{xx}(x,y) = 0$ for all $x \in [0,1] \setminus A(y)$;*

(iii) *the set $\bigcup_{y\in[0,1]} A(y)$ is countable;*

(iv) *f'_x is discontinuous with respect to y at every point of some set E of positive measure; in particular, f''_{xy} does not exist at all points of E.*

PROOF. We construct a function f similarly as in the proof of Theorem 3.2 from [9], modifying the functions φ_n only.

Let $B \subset [0,1]$ be a closed set without isolated points with $\mu(B) > 0$, whose complement $[0,1] \setminus B$ is dense in $[0,1]$ and

$$[0,1] \setminus B = \bigsqcup_{n=1}^{\infty} (a_n, b_n).$$

Let $\psi : \mathbb{R} \to \mathbb{R}^+$ be an arbitrary twice differentiable function with $\operatorname{supp} \psi(y) = (0,1)$,

$$\psi_n(y) = \psi\left(\frac{y - a_n}{b_n - a_n}\right), \quad n = 1, 2, \dots$$

$\varepsilon_n > 0$ so that $\lim\limits_{n\to\infty} \frac{\varepsilon_n}{(b_n - a_n)^2} = 0$.

We choose continuous functions $\varphi_n : [0,1] \to [0, \varepsilon_n]$ so that $|\varphi'_n(x)| = 1$ for all $x \in [0,1] \setminus A_n$, where A_n is some finite set.

The function $f : [0,1]^2 \to \mathbb{R}$,

$$f(x,y) = \sum\nolimits_{n=1}^{\infty} \varphi_n(x)\psi_n(y),$$

satisfies conditions $(i) - (iii)$ and condition (iv) for

$$E = \left([0,1] \setminus \bigcup_{n=1}^{\infty} A_n\right) \times B.$$

$\square$

In connection with this example and Theorem 4.1, the following question naturally arises.

Problem 5.3. *Let $f : [0,1]^2 \to \mathbb{R}$ be a separately twice differentiable function and $f''_{xx}, f''_{yy} \in L_1([0,1]^2)$.*

(*i*) *Do there exist a.e. the mixed derivatives f''_{xy} and f''_{yx}?*

(*ii*) *Does there exist a set $A \subseteq [0,1]$ with $\mu(A) = 1$ such that f'_x is continuous with respect to y in each point of $A \times [0,1]$?*

(*iii*) *Is the function f jointly continuous?*

It follows from the proof of Theorem 4.1 that the conditions $f''_{xx}, f''_{yy} \in L_2([0,1]^2)$ can be replaced by the condition $f''_{xx}, f''_{yy} \in L_1([0,1]^2)$ and by the existence of an function $h \in L_1([0,1]^2)$ with

$$h \sim -\sum\nolimits_{n,m\in\mathbb{Z}} mna_{nm}e^{inx}e^{imy}.$$

In this connection the following question naturally arises.

Problem 5.4. *Let $f : [0,2\pi]^2 \to \mathbb{R}$ be a separately twice differentiable function with $f(x,y) = 0$ on an open set in $[0,2\pi]^2$ which contains the boundary of $[0,2\pi]^2$ and $f''_{xx}, f''_{yy} \in L_1([0,1]^2)$. Let*

$$f \sim \sum\nolimits_{n,m\in\mathbb{Z}} a_{nm}e^{inx}e^{imy}.$$

Does there exist an function $h \in L_1([0,1]^2)$ with

$$h \sim -\sum\nolimits_{n,m\in\mathbb{Z}} mna_{nm}e^{inx}e^{imy}?$$

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