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TURNING AUTOMATIC CONTINUITY AROUND: AUTOMATIC HOMOMORPHISMS

Abstract

Let G and H be Polish groups and let $\pi\colon G \to H$ be a function. The automatic continuity problem is the following: assuming π is a group homomorphism, find conditions on G, H, or π which imply that π is continuous. In this note, we initiate a study of a reverse problem: supposing π is continuous, find conditions on G, H, or π which imply that π is a homomorphism. Herein, we treat the case $G = H = \mathbb{R}$.

1 Introduction

A classic question posed by Augustin Cauchy is the following:

Question 1. *Suppose that* $\pi\colon \mathbb{R} \to \mathbb{R}$ *is additive, that is,* $\pi(x+y) = \pi(x)+\pi(y)$ *for all* $x, y \in \mathbb{R}$. *Must* π *be continuous?*

It is well-known that the answer is *no* in ZFC (It is consistent with ZF that every additive homomorphism $\pi\colon \mathbb{R} \to \mathbb{R}$ is continuous; see Theorem 6 of Mehdi [5] and Shelah [8].) To see this, fix a basis β for $\mathbb{R}$ as a vector space over $\mathbb{Q}$. Then $|\beta| = 2^{\aleph_0}$. Hence the permutation group $S_\beta := \{f : f\colon \beta \to \beta$ is bijective$\}$ has cardinality $2^{2^{\aleph_0}}$. Note that each $f \in S_\beta$ may be uniquely extended to an invertible $\mathbb{Q}$-linear transformation on $\mathbb{R}$. It follows that there are

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exactly $2^{2^{\aleph_0}}$ automorphisms of $(\mathbb{R},+)$. However, there are but $2^{\aleph_0}$ continuous functions on $\mathbb{R}$. Thus, "most" automorphisms of $(\mathbb{R},+)$ are not continuous on $\mathbb{R}$.

The negative answer to Question 1 spawned a mass of literature which grew far beyond the real line. Solutions to the following problem comprise what is known, contemporarily, as *automatic continuity theory*:

Problem 1 (The Automatic Continuity Problem)**.** *Let G and H be Polish groups (separable topological groups with topology given by a complete metric), and suppose $\pi\colon G \to H$ is a group homomorphism. Find conditions on G, H, or π which imply that π is continuous.*

A model case of this problem is provided when $G = H = \mathbb{R}$. An additive function that is continuous at a single point is automatically continuous everywhere. In fact, we have the following well-known lemma.

Lemma 1. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be additive. If f is continuous at a point, then f is continuous on $\mathbb{R}$. In this case, $f(x) = ax$ for some $a \in \mathbb{R}$.*

Proof. Suppose that f is continuous at some real number a, and let $b \in \mathbb{R}$ be arbitrary. Note that $f(a) = \lim_{x\to a} f(x) = \lim_{x\to b} f(x-b+a) = \lim_{x\to b} f(x) - f(b) + f(a)$. We deduce that $\lim_{x\to b} f(x) = f(b)$, and f is continuous on $\mathbb{R}$.

To prove that f is linear, a straightforward induction establishes that $f(m) = f(1)\cdot m$ for all $m \in \mathbb{Z}$. We then extend to $\mathbb{Q}$ as follows: for any integer $q \neq 0$, $f(1) = f(\frac{q}{q}) = q \cdot f(\frac{1}{q})$. Thus $f(\frac{1}{q}) = f(1)\cdot\frac{1}{q}$. Hence for any integer p, we obtain $f(\frac{p}{q}) = p\cdot f(\frac{1}{q}) = f(1)\cdot\frac{p}{q}$, which shows that $f(r) = f(1)\cdot r$ for every rational number r. The continuity of f now implies that $f(x) = f(1)\cdot x$ for all $x \in \mathbb{R}$. $\square$

It is not our purpose to exposit the important results of automatic continuity theory; we refer the reader instead to the bibliography for some excellent surveys on the topic. We mention one classical result with which the reader may be familiar: any Baire-measurable homomorphism between Polish groups is continuous (see Pettis [6]). Our motivation is to initiate a study of the following reverse problem:

Problem 2 (The Automatic Homomorphism Problem)**.** *Let G and H be Polish groups and suppose $\pi\colon G \to H$ is continuous. Find conditions on G, H, or π which imply that π is a homomorphism.*

Herein, we treat the special case $G = H = \mathbb{R}$. As noted above, the class of additive, continuous functions $f\colon \mathbb{R} \to \mathbb{R}$ is not very interesting in and of itself as every such function is linear. Of interest to us is the *implication* of additivity, using automatic continuity as a template. For example, in Lemma 1, there is an algebraic assumption (additivity) and an analytic assumption, weaker than continuity everywhere, that suffice to prove continuity. To examine Problem 2, we assume that π is continuous and various algebraic properties, each of which is weaker than additivity, to conclude that π is *automatically* a homomorphism.

The main results of the paper are summarized in Theorem 15. Our hope is that they will spur an investigation of the general problem. As the topic at hand intersects analysis, group theory, and lattice theory, we sacrifice some brevity for clarity by giving detailed definitions and proofs throughout the paper.

2 Preliminaries

We begin by stating some definitions and fixing notation which will be used throughout. If G and H are groups with H a subgroup of G, then we denote this by $H \leq G$. Further, $\mathcal{L}(G)$ will denote the lattice of subgroups of a group G, partially ordered by $\subseteq$. If $f\colon G \to H$ is a homomorphism and $G = H$, then f is said to be an *endomorphism* of G. A bijective endomorphism is an *automorphism*. A group $G \leq \mathbb{R}$ is *discrete* if and only if G is discrete as a topological subspace of $\mathbb{R}$ (with the usual topology). It is well-known that G is discrete if and only if G is cyclic. Moreover, if G is not cyclic, then G is dense in $\mathbb{R}$. For any real number r, $\mathbb{Z}r := \{mr : m \in \mathbb{Z}\}$ denotes the cyclic subgroup of $\mathbb{R}$ generated by r.

If G and K are groups and $f\colon G \to K$ is a homomorphism, then f maps subgroups of G to subgroups of K. However, this property of group homomorphisms is weaker than being a homomorphism itself. We isolate it because it plays a central role in our exposition.

Property 1. *Let G and K be groups. The function $f\colon G \to K$ maps subgroups of G to subgroups of K.*

Property 1 is a weak assumption for our purposes, because as the next proposition shows, there are many functions $f\colon \mathbb{R} \to \mathbb{R}$ which possess an even stronger version of it, but are poorly behaved both algebraically and analytically. The functions fail to be additive and do not have a limit at nonzero points.

Proposition 2. *There exists a family $\mathcal{F}$ of $2^{2^{\aleph_0}}$ bijections $f\colon \mathbb{R} \to \mathbb{R}$ with the following properties:*

(a) $f(G) = G$ for every subgroup G of $\mathbb{R}$,

(b) for every nonzero r, there exists s such that $f(r+s) \neq f(r) + f(s)$, and

(c) f is continuous at 0, but fails to have a limit at every other point.

PROOF. We start by letting C denote the usual ternary Cantor set in $[0,1]$, and consider the collection $C \cap \mathbb{Q}^c$ of irrational members of C. Since $|C| = 2^{\aleph_0}$, it follows that $C \cap \mathbb{Q}^c$ also has the cardinality of the continuum. For any set $X \subseteq C \cap \mathbb{Q}^c$, define a function $f_X : \mathbb{R} \to \mathbb{R}$ as follows:

$$f_X(r) := \begin{cases} r & \text{if } r \in (X \cup \mathbb{Q}) \cap [0, \infty), \\ -r & \text{if } r \in (X \cup \mathbb{Q})^c \cap [0, \infty), \\ -f_X(-r) & \text{if } r \leq 0. \end{cases} \tag{1}$$

We now set $\mathcal{F} := \{f_X : X \subseteq C \cap \mathbb{Q}^c\}$. It is easy to see that if X and Y are distinct subsets of $C \cap \mathbb{Q}^c$, then $f_X \neq f_Y$. To see this, assume that there exists $x \in X - Y$. Then $x \neq 0$ and $f_X(x) = x$, yet $f_Y(x) = -x$. It follows that $|\mathcal{F}| = 2^{2^{\aleph_0}}$.

Fix an arbitrary set $X \subseteq C \cap \mathbb{Q}^c$. It is clear from the definition of f_X that

$$f_X \text{ is odd.} \tag{2}$$

It is also obvious from the definition of f_X that $f_X(r) = \pm r$ for every $r \in \mathbb{R}$. Moreover, since f_X is odd, it follows that for all $r \in \mathbb{R}$:

$$\text{if } f_X(r) = r, \text{ then } f_X(-r) = -r. \text{ If } f_X(r) = -r, \text{ then } f_X(-r) = r. \tag{3}$$

We conclude that f_X is bijective. Moreover, (a) follows at once from the above observations.

We now prove (b). Let $r > 0$ be arbitrary. Suppose first that $r \in X \cup \mathbb{Q}$, and pick $s > 0$ such that $r + s \in (X \cup \mathbb{Q})^c$. Then $f_X(r+s) = -(r+s) < r - s \leq r + f_X(s) = f_X(r) + f_X(s)$. Now assume that $r \in (X \cup \mathbb{Q})^c$. Pick a rational $s > 0$ such that $r+s \in (X \cup \mathbb{Q})^c$. Then $f_X(r+s) = -(r+s) < -r+s = f_X(r) + f_X(s)$. Suppose now that $r < 0$. Then $-r > 0$. By what we proved above, there exists $-s > 0$ such that $f_X(-r + -s) \neq f_X(-r) + f_X(-s)$. Since f_X is odd, we get $-f_X(r+s) \neq -f_X(r) + -f_X(s)$. Therefore, $f_X(r+s) \neq f_X(r) + f_X(s)$.

Finally, we establish (c). Since $f_X(r) = \pm r$ for all $r \in \mathbb{R}$, it is clear that f_X is continuous at 0. As f_X is odd, it suffices to show that $\lim_{x \to r} f_X(x)$ does not exist for any positive real number r. Thus let $r > 0$ be arbitrary. Since the Cantor set is closed and nowhere dense, it follows that $(C \cup \mathbb{Q})^c \cap [0,1]$ is dense in $[0,1]$. Therefore, $(C \cup \mathbb{Q})^c$ is dense in $[0, \infty)$. Noting that $(C \cup \mathbb{Q})^c \subseteq (X \cup \mathbb{Q})^c$, clearly $(X \cup \mathbb{Q})^c$ is dense in $[0, \infty)$ as well. Thus way may choose a sequence (s_n) in $(X \cup \mathbb{Q}) - \{r\}$ converging to r and a sequence (t_n) in $(X \cup \mathbb{Q})^c - \{r\}$

converging to r. Applying f_X to both sequences, it follows that $\lim_{x\to r} f_X(x)$ does not exist. $\square$

We further explicate the weakness of Property 1 by the following. Even the additional assumptions of continuity and surjectivity are not enough to guarantee additivity. More precisely, there exist continuous, surjective $f\colon \mathbb{R} \to \mathbb{R}$, mapping subgroups of $\mathbb{R}$ to subgroups of $\mathbb{R}$, that fail to be additive.

Before constructing such an f, we first define some notation which will help to streamline the argument. Suppose that $I := [i_1, i_2]$ and $J := [j_1, j_2]$ are closed bounded intervals on the real line. Define $I < J$ to mean that $i_2 < j_1$. Further, assuming $I < J$, let $d(I, J) := j_i - i_2$. Lastly, if k is a real number, then we set $kI := \{kx : x \in I\}$.

Proposition 3. *There exists a continuous, surjective map $f\colon \mathbb{R} \to \mathbb{R}$ which is not additive, yet $f(G)$ is a subgroup of $\mathbb{R}$ for every subgroup G of $\mathbb{R}$.*

PROOF. For each $n \in \mathbb{Z}^+$, let $I_n := [\frac{1}{n}, n]$. Next observe that $\mathbb{Z}^+ \times \mathbb{Q} \times \mathbb{Q}$ is countable; let $\{(l_i, r_i, s_i) : i \in \mathbb{Z}^+\}$ be an enumeration. We now recursively define a sequence $(k_n : n \in \mathbb{Z}^+)$ of positive integers as follows: set $k_1 := 1$. Suppose that $k_1, \ldots, k_n$ have been defined. We then choose k_{n+1} to be any positive integer satisfying

(i) $k_n I_n < k_{n+1} I_{n+1}$, and

(ii) $d(k_n I_n, k_{n+1} I_{n+1}) \geq 1$.

Set $J_{(l_n, r_n, s_n)} := k_n I_n := [a_n, b_n]$ for every $n \in \mathbb{Z}^+$. Note that since I_n has rational endpoints, the same is true of $J_{(l_n, r_n, s_n)}$. We now define a piecewise-linear map $f\colon \mathbb{R} \to \mathbb{R}$ via the following steps:

Step 1. For each $n \in \mathbb{Z}^+$, define f on $[a_n, b_n]$ by $f(x) := (\frac{r_n}{k_n})x + s_n$. Since r_n, k_n, s_n, a_n, and b_n are rational, we see that $f(a_n)$ and $f(b_n)$ are also rational. Moreover, f is linear with rational slope and rational y-intercept.

Step 2. Let $n \in \mathbb{Z}^+$ be arbitrary. We now define f on $[b_n, a_{n+1}]$ as follows: let $m_n := \frac{f(a_{n+1}) - f(b_n)}{a_{n+1} - b_n}$. For $x \in [b_n, a_{n+1}]$, put $f(x) := m_n x + (f(b_n) - m_n b_n)$. Again, f is linear on $[b_n, a_{n+1}]$ with rational slope and rational y-intercept.

Step 3. To extend f to $\mathbb{R}$, we need only define f on $(-\infty, a_1]$. To do this, we set $f(x) := f(1) \cdot x$ for $x \leq a_1$. Once more, we see that f is linear on $(-\infty, a_1]$ with rational slope and rational y-intercept.

It is easy to check that the definitions given above for f agree at every a_n, b_n and that f is piecewise-linear. It follows that f is continuous on $\mathbb{R}$. We now prove that $f(G)$ is a subgroup of $\mathbb{R}$ for every subgroup G of $\mathbb{R}$.

Toward this end, we let G be an arbitrary subgroup of $\mathbb{R}$. If $G = \{0\}$, then by the definition in Step 3, we see that $f(G) = \{0\}$. Thus, assume that G is nontrivial and set $G^+ := \{g \in G : g > 0\}$. Then clearly,

$$\{rg : r \in \mathbb{Q}, g \in G^+\} := \mathbb{Q}G^+ = \mathbb{Q}G := \{rg : r \in \mathbb{Q}, g \in G\}. \tag{4}$$

The very definition of f yields

$$f(G) \subseteq \mathbb{Q}G + \mathbb{Q}. \tag{5}$$

Now let $g \in G^+$ and $x, y \in \mathbb{Q}$ be arbitrary. Note that $g \in I_n$ for all but finitely many n. It follows that $k_n g \in J_{(l_n, r_n, s_n)}$ for all but finitely many n. Now, there are infinitely many triples $(a, b, c) \in \mathbb{Z}^+ \times \mathbb{Q} \times \mathbb{Q}$ for which $b = x$ and $c = y$. Hence, there are infinitely many n such that $r_n = x$ and $s_n = y$. We deduce that for some such n, we have $k_n g \in J_{(l_n, r_n, s_n)}$. But now, by definition of f, $f(k_n g) = (\frac{r_n}{k_n})(k_n g) + s_n = r_n g + s_n = xg + y$. Since $g \in G^+$ and $x, y \in \mathbb{Q}$ were arbitrary, we have shown that

$$\mathbb{Q}G^+ + \mathbb{Q} \subseteq f(G^+). \tag{6}$$

We now complete the proof. We conclude from (4) - (6) above that $f(G) \subseteq \mathbb{Q}G + \mathbb{Q} = \mathbb{Q}G^+ + \mathbb{Q} \subseteq f(G^+) \subseteq f(G)$, whence equality holds throughout. It is apparent that $\mathbb{Q}G + \mathbb{Q}$ is a subgroup of $\mathbb{R}$. Moreover, $f(\mathbb{R}) = \mathbb{Q}\mathbb{R} + \mathbb{Q} = \mathbb{R}$; therefore f is surjective. Finally, we must justify that f is not additive, but this is easy. Suppose by way of contradiction that f is additive. Then f is linear by Lemma 1. However, it is clear from the definition of f that f is neither increasing nor decreasing on $\mathbb{R}$, and thus f cannot be linear. This contradiction completes the argument. $\square$

Mapping subgroups to subgroups is not a strong assumption, even in the presence of continuity. Therefore, we will make use of variants of Property 1 to produce automatic homomorphisms. For example, nonzero, continuous functions that map *discrete* subgroups of $\mathbb{R}$ to *discrete* subgroups of $\mathbb{R}$ are automatically automorphisms of $(\mathbb{R}, +)$.

3 Main results

We have seen that even in the presence of continuity, a surjective function $f\colon \mathbb{R} \to \mathbb{R}$ which maps subgroups to subgroups need not be additive. Our first proposition of the section demonstrates, however, that if we replace "surjective" with "injective," then the function f is much better-behaved.

Proposition 4. *Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous injective map which satisfies*

(a) $f(0) = 0$, and

(b) in every neighborhood of 0, there is a non-zero x such that $f(\mathbb{Z}x)$ is a subgroup of $\mathbb{R}$.

Then f is an automorphism of $(\mathbb{R}, +)$.

PROOF. Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous, injective map satisfying (a) and (b) above. We will show that f is an additive automorphism of $\mathbb{R}$. Replacing f by $-f$ if necessary, we may assume that f is strictly increasing on $\mathbb{R}$. Let $r \in \mathbb{R}$ be arbitrary. We claim:

$$\text{if } f(\mathbb{Z}r) \text{ is a subgroup of } \mathbb{R}, \text{ then } f(mr) = mf(r) \text{ for all } m \in \mathbb{Z}. \tag{7}$$

To verify (7), assume that $f(\mathbb{Z}r)$ is a subgroup of $\mathbb{R}$. Since f is strictly increasing, f is an order isomorphism between $\mathbb{Z}r$ and $f(\mathbb{Z}r)$. As mentioned in Section 2, every subgroup of $\mathbb{R}$ is either cyclic or dense in $\mathbb{R}$. Because $\mathbb{Z}r$ is a discrete subset of $\mathbb{R}$ and $f\colon \mathbb{Z}r \to f(\mathbb{Z}r)$ is an order isomorphism, we conclude that $f(\mathbb{Z}r)$ is a cyclic subgroup of $\mathbb{R}$. Since in addition $f(0) = 0$, (7) is clear.

Now suppose that $r \in \mathbb{R}$ and $f(mr) = mf(r)$ for every $m \in \mathbb{Z}$. Let $m_0 \in \mathbb{Z}$ be arbitrary. Then

$$\text{for any } m \in \mathbb{Z},\ f(m(m_0 r)) = mm_0 f(r) = m(m_0 f(r)) = mf(m_0 r). \tag{8}$$

Next, set $X := \{r \in \mathbb{R}\colon f(mr) = mf(r) \text{ for all } m \in \mathbb{Z}\}$. We deduce from assumption (b), (7), and (8) that X is a dense subset of $\mathbb{R}$. The continuity of f implies that $X = \mathbb{R}$. It then follows from the proof of Lemma 1 that $f(r) = f(1)r$ for all $r \in \mathbb{R}$. As f is injective, $f(1) \neq 0$, and thus f is an automorphism of $(\mathbb{R}, +)$. □

Remark 1. *The assumption that $f(0) = 0$ in the previous proposition cannot be eliminated. To see why, let $f(x) := x + 1$. Then f is injective, and for any $n > 0$, we have $f(\mathbb{Z}\frac{1}{n}) = (\mathbb{Z} + n)\frac{1}{n} = \mathbb{Z}\frac{1}{n}$. But since $f(0) \neq 0$, f is not an automorphism of $(\mathbb{R}, +)$.*

Corollary 5. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous injective map. If f maps discrete subgroups to discrete subgroups, then f is an automorphism of $(\mathbb{R}, +)$.*

If we remove the injectivity hypothesis, but replace it with nonincreasing or nondecreasing, then we may prove the following more general result using Corollary 5.

Proposition 6. *If $f: \mathbb{R} \to \mathbb{R}$ is a continuous, monotonic function which maps discrete subgroups to discrete subgroups, then f is additive.*

Proof. We may assume that f is not identically zero. Then either f is not identically 0 on $(-\infty, 0]$ or f is not identically 0 on $[0, \infty)$. We may assume without loss of generality, replacing $f(x)$ with $f(-x)$ if necessary, that

$$f \text{ is not identically } 0 \text{ on } [0, \infty). \tag{9}$$

Further, replacing f with $-f$ if necessary, we may suppose

$$f \text{ is increasing on } [0, \infty). \tag{10}$$

As f maps the subgroup $\{0\}$ to a subgroup of $\mathbb{R}$, we see that

$$f(0) = 0. \tag{11}$$

Since $f(0) = 0$, f is increasing, and f is not identically zero on $[0, \infty)$, clearly $f^{-1}(0)$ is bounded above; let $x^* :=\sup(f^{-1}(0))$. Then $x^* \geq 0$; we claim that $x^* = 0$. Suppose not, and choose x_1 such that $x^* < x_1 < 2x^*$ and $x_2 \in (x^*, x_1)$ such that $f(x_2) = \frac{2f(x_1)}{3}$. Finally, choose $x_3 \in (x^*, x_2)$ such that $f(x_3) = \frac{f(x_1)}{3}$. Next, consider the group $G := f(\mathbb{Z}x_3)$. Since f is increasing, $f(0) = 0$, and $f(x_3) > 0$, it follows that $f(x_3)$ is the least positive element of G. As G is a group, $2f(x_3) = f(x_2) \in G$. By definition of G, $f(x_2) = f(nx_3)$ for some integer $n > 1$. But $x^* < x_3$, and thus $x_1 < 2x^* \leq nx^* < nx_3$. Since f is increasing, we deduce that $f(x_1) \leq f(nx_3) = f(x_2)$, and this is a contradiction. Therefore, $x^* = 0$, and

$$f(x) > 0 \text{ for all } x > 0. \tag{12}$$

We now show that f is injective on $[0, \infty)$. Suppose not. Then $f(a) = f(b) := y$ for some a, b with $0 \leq a < b$. Since $f(0) = 0$ and $f(x) > 0$ for every $x > 0$, we deduce $0 < a < b$ and $y > 0$. Without loss of generality, we may assume that a is the least positive real number for which $f(a) = y$. Using (10)–(12) above along with the continuity of f, we obtain $x_1 \in \mathbb{R}$ with the following properties:

(i) $0 < f(x_1) < y$ (hence $0 < x_1 < a$),

(ii) $f(x_1) \neq \frac{y}{n}$ for any positive integer n, and

(iii) $x_1 < b - a$.

Again, note that $f(x_1)$ is the positive generator of the group $f(\mathbb{Z}x_1)$. Now let k be the largest positive integer for which $kf(x_1) < y$. Then $kf(x_1) \in f(\mathbb{Z}x_1)$, whence $kf(x_1) = f(mx_1)$ for some positive integer m. We may assume that m is the largest integer for which $f(mx_1) = kf(x_1)$. But then we have:

(i) $kf(x_1) < y < (k+1)f(x_1)$ (the second inequality follows from (ii)), and

(ii) $f((m+1)x_1) = (k+1)f(x_1)$ (since f is increasing).

However, note from above that $f(mx_1) = kf(x_1) < y$ and $f((m+1)x_1) > y$. Thus $mx_1 < a$ and $(m+1)x_1 > b$. This contradicts (iii) above. Thus f is injective on $[0,\infty)$. Since f is injective on $[0,\infty)$, it follows that $f(\mathbb{Z})$ is a nonzero subgroup of $\mathbb{R}$. Therefore, f is surjective. A symmetric argument shows that f is injective on $(-\infty,0]$. As f is surjective, we see that f is strictly increasing on $(-\infty,0]$, hence is strictly increasing on the entire line. Corollary 5, then, yields the desired conclusion. $\square$

Recall from Proposition 3 that if $f\colon \mathbb{R} \to \mathbb{R}$ is continuous and $f(G) \leq \mathbb{R}$ for every $G \leq \mathbb{R}$, then it does not follow that f is additive, even when f is surjective. If we assume instead that f is one-to-one, then Corollary 5 shows that f is an automorphism of $(\mathbb{R},+)$. The next proposition examines the natural question that arises when we replace injectivity with the preservation of discrete subgroups.

Proposition 7. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. If f maps discrete subgroups to discrete subgroups, then f is additive.*

Proof. First, we claim that

$$f \text{ is monotonic on } [0,\infty). \tag{13}$$

Suppose not. Then there exist real numbers $0 \leq a < c$ such that $f(a) = f(c)$ and f is not constant on $[a,c]$. Let $\max(f)$ denote the absolute max of f on $[a,c]$. By replacing f with $-f$ if necessary, we may assume that $\max(f) > f(a)$. Let $b \in (a,c)$ be such that $f(b) = \max(f)$, and choose some rational number $q \in (1, \frac{c}{b})$.

Case 1. $\max(f) > 0$. Without loss of generality, we may assume

(i) $f(a) = f(c) > 0$,

(ii) $f(x) > f(a)$ for all $x \in (a,b)$, and

(iii) $f(c) < f(x) < f(b)$ for all $x \in (b, c)$.

Define $G\colon [a,b] \to \mathbb{R}$ by $G(x) := \frac{f(x)}{f(qx)}$. Then G is well-defined and continuous on $[a,b]$. Furthermore, $G(a) = \frac{f(a)}{f(qa)} \leq 1$ and $G(b) = \frac{f(b)}{f(qb)} > 1$. Since G is continuous and nonconstant on $[a,b]$, we conclude that the range of G must contain an irrational number; let $x \in [a,b]$ be such that $\frac{f(x)}{f(qx)}$ is irrational. Then $\{f(x), f(qx)\}$ is linearly independent over $\mathbb{Q}$. Since q is a positive rational number, $q = \frac{m}{n}$ for some positive integers m and n. Now consider the group $H := \mathbb{Z}\frac{x}{n}$, and note that $\{x, qx\} \subseteq H$. But then $\{f(x), f(qx)\} \subseteq f(H)$, and $f(H)$ is not discrete, a contradiction.

Case 2. $\max(f) < 0$. This is analogous.

Case 3. $\max(f) = 0$. Let $b := \sup([a,c] \cap f^{-1}(0))$. Recall from above that $q \in (1, \frac{c}{b}) \cap \mathbb{Q}$. Choose $\alpha \in \mathbb{R}$ such that $qb < q\alpha < c$, and define G on $[b, \alpha]$ as above by $G(x) := \frac{f(x)}{f(qx)}$. Again, G is well-defined. Moreover, $G(b) = 0$ and $G(\alpha) > 0$. The remainder of the argument proceeds just as in Case 1 and is omitted.

We have shown that f is monotonic on $[0, \infty)$. Similarly, it follows that

$$f \text{ is monotonic on } (-\infty, 0]. \tag{14}$$

If f is identically zero on $\mathbb{R}$, then f is additive and we are done. Assume now that f is not identically zero; say $f(r) \neq 0$. Then $f(\mathbb{Z}r)$ is a nonzero subgroup of $\mathbb{R}$, and it follows that f is surjective. This fact along with (13) and (14) above implies that f is monotonic on $\mathbb{R}$. We apply Proposition 6 to conclude that f is additive. $\square$

Corollary 8. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. Then $f(D) = D$ for every discrete subgroup D of $\mathbb{R}$ if and only if $f(x) = x$ or $f(x) = -x$.*

Proof. If $f(x) = x$ or $f(x) = -x$, the result is clear. Suppose now that $f(D) = D$ for every discrete $D \leq \mathbb{R}$. The previous proposition implies that f is an automorphism of $(\mathbb{R}, +)$, and therefore $f(x) = ax$ for some non-zero real number a. But then $\mathbb{Z}a = \mathbb{Z}$. In particular, $a = 1 \cdot a \in \mathbb{Z}$. Therefore, a generates $\mathbb{Z}$, and so $a = \pm 1$. $\square$

Having considered continuous functions f which map subgroups to subgroups, we now turn our attention to lattice endomorphisms. Recall that a *lattice* is a poset $(X, \leq)$ with the property that every pair $\{x, y\}$ of elements of X has both a greatest lower bound $x \wedge y$ (the *meet* of x and y) and a smallest upper bound $x \vee y$ (the *join* of x and y) with respect to $\leq$. A lattice is *bounded* provided X has a smallest element 0 and a largest element 1. A *bounded lattice endomorphism* of X is a function $\varphi\colon X \to X$ with the property that $\varphi(0) = 0$, $\varphi(1) = 1$, $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$, and $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$ for all $x, y \in X$. A bijective bounded lattice endomorphism is called a *bounded lattice automorphism*. It is easy to see that for any group G, the collection $\mathcal{L}(G)$ of subgroups of G is a bounded lattice with respect to set inclusion. Specifically, $0 = \{e\}$, $1 = G$, $H \wedge K = H \cap K$, and $H \vee K = \langle H, K\rangle$ for any $H, K \in \mathcal{L}(G)$. Note $\langle H, K\rangle$ is the subgroup of G generated by H and K.

Suppose now that G is a group and φ is a bounded lattice endomorphism of $\mathcal{L}(G)$. Then for all subgroups H and K of G,

$$\text{if } H \cap K = \{e\}, \text{ then } \varphi(H) \cap \varphi(K) = \{e\}. \tag{15}$$

We will shortly investigate a class of continuous functions on the real line which satisfy a more general version of (15). We will make use of the following lemma.

Lemma 9. *Let $[a, c]$ be an interval of positive measure for which 0 is not interior, and suppose that $f\colon [a, c] \to \mathbb{R}$ is a continuous function with $f(a) = f(c)$. Then there exist distinct $\alpha, \beta \in [a, c]$ such that $f(\alpha) = f(\beta)$ and $\{\alpha, \beta\}$ is linearly independent over $\mathbb{Q}$.*

PROOF. We prove the lemma only for the case $0 \leq a$, as the case $c \leq 0$ is analogous. Clearly, we may assume that f is not constant on $[a, c]$. Thus there is some number $b \in (a, c)$ such that $f(b) \neq f(a)$. We assume that $f(b) > f(a)$ (a similar argument applies if $f(b) < f(a)$). Now define a function $f^* : [f(a), f(b)] \to \mathbb{R}$ by

$$f^*(y) := \frac{\inf([a, b] \cap f^{-1}(y))}{\sup([b, c] \cap f^{-1}(y))}. \tag{16}$$

As y increases, the Intermediate Value Theorem implies that the numerator of the above function strictly increases while the denominator strictly decreases. Therefore, f is injective. We deduce that $f^*(y)$ is irrational for some $y \in [f(a), f(b)]$. Set $x_1 := \inf([a, b] \cap f^{-1}(y))$ and $x_2 := \sup([b, c] \cap f^{-1}(y))$. Since $[a, b] \cap f^{-1}(y)$ is closed, we see that $x_1 \in [a, b] \cap f^{-1}(y)$, and thus $f(x_1) = y$. Similarly, $f(x_2) = y$. As $\frac{x_1}{x_2} \notin \mathbb{Q}$, it follows that x_1 and x_2 are linearly independent over $\mathbb{Q}$, completing the proof. $\square$

Remark 2. *In the statement of Lemma 9, we cannot dispense with the assumption that 0 is not in the interior of the interval $[a, b]$. A counterexample is provided by $f(x) = |x|$ on $[-1, 1]$. Then, $f(\alpha) = f(\beta)$ only for $\alpha = \pm\beta$.*

Recall from Proposition 3 that the assumption that $f\colon \mathbb{R} \to \mathbb{R}$ maps discrete subgroups of $\mathbb{R}$ to subgroups of $\mathbb{R}$ is not sufficient to imply that f is additive. However, if we assume in addition that f satisfies (15) above for all discrete subgroups H and K of $\mathbb{R}$, then this is sufficient.

Proposition 10. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying the following two conditions:*

(a) f maps discrete subgroups of $\mathbb{R}$ to subgroups of $\mathbb{R}$ (not assumed to be discrete), and

(b) for any discrete subgroups H and K of $\mathbb{R}$ for which $H \cap K = \{0\}$, we have $f(H) \cap f(K) = \{0\}$.

Then f is additive.

Proof. Suppose that $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function on $\mathbb{R}$ satisfying (a) and (b). We may further assume that f is not identically zero. We now establish a sequence of claims. Since f is not identically 0, it follows that f is not identically 0 on $(-\infty, 0]$ or f is not identically 0 on $[0, \infty)$. Replacing $f(x)$ with $f(-x)$ if necessary, we may assume

$$f \text{ is not identically } 0 \text{ on } [0, \infty). \tag{17}$$

Since f maps discrete subgroups of $\mathbb{R}$ to subgroups of $\mathbb{R}$, it follows that

$$f(0) = 0. \tag{18}$$

We now prove:

$$\text{If } 0 \le a < c \text{ and } f(a) = f(c), \text{ then } f \text{ is identically } 0 \text{ on } [a, c]. \tag{19}$$

We first show that $f(a) = f(c) = 0$. Suppose not. Replacing f by $-f$ if necessary, we may assume that $f(a) = f(c) := y > 0$. Observe that f cannot be constant on $[a, c]$; if so, choose $\alpha, \beta \in [a, c]$ which are linearly independent over $\mathbb{Q}$. Then $\mathbb{Z}\alpha \cap \mathbb{Z}\beta = \{0\}$. By (b) above, we obtain $f(\mathbb{Z}\alpha) \cap f(\mathbb{Z}\beta) = \{0\}$. However, this is impossible since $0 \ne y \in f(\mathbb{Z}\alpha) \cap f(\mathbb{Z}\beta)$. Thus there exists $b \in (a, c)$ such that $f(b) > 0$ and $f(b) \ne f(a)$. Suppose first that $f(b) > f(a)$. Then we may assume without loss of generality that $f(b)$ is the global max of f on $[a, c]$. By continuity of f, there exists an interval $[a', c']$ contained in $[a, c]$ for which b is interior and $f(x) > f(a) = f(c)$ on $[a', c']$. By the Intermediate Value Theorem, we may assume that $f(a') = f(c')$. We now invoke Lemma 9 to obtain linearly independent elements $\alpha', \beta' \in [a, c]$ such that $f(\alpha') = f(\beta') := y' \ge \min(f(a), f(b)) > 0$. We obtain the same conclusion via an analogous argument in case $f(b) < f(a)$. But then $\mathbb{Z}\alpha' \cap \mathbb{Z}\beta' = \{0\}$, yet $0 < y' \in f(\mathbb{Z}\alpha') \cap f(\mathbb{Z}\beta')$, another contradiction. We conclude that $f(a) = f(c) = 0$. Suppose again by contradiction that f is not identically 0 on $[a, c]$. Without loss of generality, we may assume there exists $b \in (a, c)$ with $f(b) > 0$. By the Intermediate Value Theorem, there exists $x_1 \in (a, b)$ such that $f(x_1) = \frac{f(b)}{2}$ and there exists $x_2 \in (b, c)$ such that $f(x_2) = \frac{f(b)}{2}$. But then by what we just proved above, $f(x_1) = f(x_2) = 0$, a contradiction. Thus (19) is established.

It follows that f is monotonic on $[0, \infty)$. A symmetric argument can be used to show that f is monotonic on $(-\infty, 0]$. Since f is not identically zero and as f maps discrete subgroups to subgroups, we conclude as before that f is surjective. Therefore, f is monotonic on $\mathbb{R}$. Proposition 6 implies that f is additive. □

Corollary 11. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then the following are equivalent:*

(a) f is an automorphism of $(\mathbb{R}, +)$.

(b) The map $G \mapsto f(G)$ is a bounded lattice automorphism of $\mathcal{L}(\mathbb{R})$.

(c) The map $G \mapsto f(G)$ is a bounded lattice endomorphism of $\mathcal{L}(\mathbb{R})$.

Our final study will be of continuous functions $f : \mathbb{R} \to \mathbb{R}$ for which the preimage of a discrete subgroup of $\mathbb{R}$ is also a discrete subgroup of $\mathbb{R}$. Observe that not every such function is additive, as the following example shows.

Example 1. *Let $f(x) := |x|$ for all $x \in \mathbb{R}$. Then f is not additive, yet $f^{-1}(G) = G$ for every subgroup G of $\mathbb{R}$.*

We now prove a lemma which will be used in our final proposition.

Lemma 12. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and that $f^{-1}(D)$ is a discrete subgroup of $\mathbb{R}$ for every discrete subgroup D of $\mathbb{R}$. Then*

(a) $f(0) = 0$, and

(b) f is strictly monotonic on $(-\infty, 0]$ and strictly monotonic on $[0, \infty)$.

PROOF. Assume that f is as stated above. Then $f^{-1}(\{0\})$ is a (discrete) subgroup of $\mathbb{R}$. In particular, $0 \in f^{-1}(\{0\})$, and we have verified (a). As for (b), we only show that f is strictly monotonic on $[0, \infty)$, the other case being analogous. It suffices to show that f is one-to-one on $[0, \infty)$. Suppose not. Then there exist $a, b \in [0, \infty)$ such that $f(a) = f(b)$ and $a < b$. Invoking

Lemma 9, we obtain distinct elements $\alpha, \beta \in [a, b]$ which are linearly independent over $\mathbb{Q}$ such that $f(\alpha) = f(\beta)$. Therefore, $\{\alpha, \beta\} \subseteq f^{-1}(\mathbb{Z}(f(\alpha)))$. But $f^{-1}(\mathbb{Z}(f(\alpha)))$ is discrete. Since a discrete subgroup of $\mathbb{R}$ cannot contain two distinct elements which are linearly independent over $\mathbb{Q}$, we have reached a contradiction. □

We are now ready to prove our final proposition.

Proposition 13. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function. Then $f^{-1}(D)$ is a discrete subgroup of $\mathbb{R}$ for every discrete subgroup D of $\mathbb{R}$ if and only if there exists a positive $a \in \mathbb{R}$ such that $|f(x)| = a|x|$ for all $x \in \mathbb{R}$.*

Proof. Suppose first that $|f(x)| = a|x|$ for some $a > 0$. Then it is easy to see that $f^{-1}(G) = \frac{G}{a}$ for any $G \leq \mathbb{R}$. Therefore, $f^{-1}(\mathbb{Z}r) = \mathbb{Z}(\frac{r}{a})$ for any $r \in \mathbb{R}$.

Conversely, assume that $f^{-1}(D)$ is a discrete subgroup of $\mathbb{R}$ for every discrete subgroup D of $\mathbb{R}$. Using Lemma 12 and replacing f with $-f$ if necessary, we may assume that f is strictly increasing on $[0, \infty)$. We claim

$$f \text{ is not bounded above on } [0, \infty). \tag{20}$$

Suppose not and let y be the sup of f on $[0, \infty)$. Since $f(0) = 0$ and f is strictly increasing on $[0, \infty)$, it follows that $y > 0$ and that $y \notin f([0, \infty))$. Let $x > 0$ be such that $f(x) = \frac{y}{2}$. Then x is the unique positive member of the group $f^{-1}(\mathbb{Z}\frac{y}{2})$, which is absurd, and (20) is established. Lemma 12 implies that f is either strictly increasing or strictly decreasing on $(-\infty, 0]$. We consider these two cases separately to finish the proof.

Case 1. f is strictly increasing on $(-\infty, 0]$. By a similar argument, we see that f is not bounded below on $(-\infty, 0]$. But then $f\colon \mathbb{R} \to \mathbb{R}$ is bijective, whence $f^{-1}\colon \mathbb{R} \to \mathbb{R}$ is a continuous injective function with the property that $f^{-1}(D) \leq \mathbb{R}$ for every discrete $D \leq \mathbb{R}$. Lemma 1 and Corollary 5 imply that $f^{-1}(x) = bx$ for some nonzero $b \in \mathbb{R}$. Setting $a := |\frac{1}{b}|$, we see that $|f(x)| = a|x|$ for all $x \in \mathbb{R}$, completing this case.

Case 2. f is strictly decreasing on $(-\infty, 0]$. Again, argument similar to the one used to prove (20) above shows that f is not bounded above on $(-\infty, 0]$. Define a new function $G\colon \mathbb{R} \to \mathbb{R}$ by

$$G(x) := \begin{cases} f(x) & \text{if } x \geq 0, \\ -f(x) & \text{if } x \leq 0. \end{cases} \tag{21}$$

Then $G\colon \mathbb{R} \to \mathbb{R}$ is bijective, and therefore so is G^{-1}. Now let K be an arbitrary subgroup of $\mathbb{R}$ and let $x \geq 0$. Then $x \in G^{-1}(K)$ iff $G(x) \in K$

iff $f(x) \in K$ iff $x \in f^{-1}(K)$. Now suppose $x < 0$. Then $x \in G^{-1}(K)$ iff $G(x) \in K$ iff $-f(x) \in K$ iff (since K is a group) $f(x) \in K$ iff $x \in f^{-1}(K)$. It follows that $G^{-1}(D)$ is a subgroup of $\mathbb{R}$ for every discrete subgroup D of $\mathbb{R}$. Again, we invoke Lemma 1 and Corollary 5 to conclude that $G^{-1}(x) = bx$ for some nonzero $b \in \mathbb{R}$. Setting $a := |\frac{1}{b}|$, we see that $|f(x)| = a|x|$ for all $x \in \mathbb{R}$, concluding the proof. $\square$

Corollary 14. *Let f be a countinuous function. If $f^{-1}(D)$ is a discrete subgroup of $\mathbb{R}$ for every discrete subgroup D of $\mathbb{R}$, then the map $G \mapsto f^{-1}(G)$ is a bounded lattice automorphism of $\mathcal{L}(\mathbb{R})$.*

Proof. By the previous proposition, $|f(x)| = a|x|$ for some $a > 0$. This implies that $f^{-1}(G) = \frac{G}{a}$ for every $G \leq \mathbb{R}$. It is now a simple task to verify that $G \mapsto \frac{G}{a}$ is a bounded lattice automorphism of $\mathcal{L}(G)$. $\square$

We conclude the paper with the following synopsis of our results.

Theorem 15. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function which is not identically 0. Then the following are equivalent:*

(a) f is an automorphism of $(\mathbb{R}, +)$.

(b) f maps discrete subgroups to discrete subgroups.

(c) $f(D)$ is a subgroup of $\mathbb{R}$ for every discrete subgroup D of $\mathbb{R}$. Moreover, if H and K are discrete subgroups of $\mathbb{R}$ such that $H \cap K = \{0\}$, then also $f(H) \cap f(K) = \{0\}$.

(d) The map $G \mapsto f(G)$ is a bounded lattice automorphism of $\mathcal{L}(\mathbb{R})$.

(e) The map $G \mapsto f(G)$ is a bounded lattice endomorphism of $\mathcal{L}(\mathbb{R})$.

(f) f is injective and $f^{-1}(D)$ is a discrete subgroup of $\mathbb{R}$ for all discrete subgroups D of $\mathbb{R}$.

(g) f is surjective and $f^{-1}(D)$ is a discrete subgroup of $\mathbb{R}$ for all discrete subgroups D of $\mathbb{R}$.

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