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THE ESSENTIAL NORM OF MULTIPLICATION OPERATORS ON LORENTZ SEQUENCE SPACES

Abstract

We study some basic properties of Lorentz sequence spaces. Description of multiplication operators generated by a sequence is presented. We calculate the essential norm of multiplication operators acting on Lorentz sequence spaces.

1 Introduction

The multiplication operator, defined roughly speaking as the pointwise multiplication by a real-valued measurable function, is a well-studied transformation. This operator received considerable attention over the past several decades specially on Lebesgue and Bergman spaces and also played an important role in the study of operators on Hilbert spaces. For more detail on these operators we refer to [1], [5], [8] and [9]. Studies of multiplication operators on L_p spaces can be seen in [10] and [15], on Orlicz spaces in [12], on Lorentz spaces in [2], on Lorentz-Bochner spaces in [3], on weak L_p spaces in [6], and on Orlicz-Lorentz spaces in [7]. In the case of the Lorentz sequence space $l_{(p,q)}$, Arora, Datt and Verma [4] characterized the symbols inducing multiplication

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operators continuous, invertible, with closed range, compact and Fredholm. The multiplication operator M_u on the space $l_{(p,q)}$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ is induced by a sequence $u = \{u(n)\}_{n\in\mathbb{N}}$, where for a sequence $a = \{a(n)\}$, $M_u a$ is the sequence $u \cdot a = c = \{c(n)\}$ defined by

$$c(n) = u(n) \cdot a(n)$$

with $n \in \mathbb{N}$. Clearly, this operator is not 1-1 unless $u(n) \neq 0$ for all $n \in \mathbb{N}$. Furthermore, it follows immediately from [10], that the only compact multiplication operator on the non-atomic Lorentz space is the zero operator. However, in the case of the Lorentz sequence space, there exist compact non-zero multiplication operators on $l_{(p,q)}$, $1 < p \leq \infty$, $1 \leq q \leq \infty$. In fact, in [4], the authors showed the following result:

Theorem 1 ([4]). *The operator M_u is compact on $l_{(p,q)}$ if and only if $u(n) \to 0$ as $n \to \infty$.*

The aim of this note is to obtain an estimation of the essential norm of $M_u : l_{(p,q)} \to l_{(p,q)}$ which implies the above result. More precisely, in this note we will show the following result:

Main Theorem. *Let $u = \{u(n)\}$ be a bounded sequence. Then*

$$\|M_u\|_e = \limsup_{n\to\infty} |u(n)| . \tag{1}$$

We present the proof of the above result in Section 3, and in Section 2 we gather some properties of the Lorentz sequence space.

2 Some remarks on Lorentz sequence spaces

The Lorentz space is a two parameter family of functions $L_{(p,q)}$ which generalizes the Lebesgue space L_p. The $L_{(p,q)}$ spaces were introduced by Lorentz in [13] and [14]; a general treatment of Lorentz spaces is given in the article of Hunt [11]. In the case that the domain of the functions considered is $X = \mathbb{N}$ with the σ-algebra $A = 2^{\mathbb{N}}$, the power set of X, and the counting measure μ, we obtain the Lorentz sequence space $l_{(p,q)}$ with $1 < p \leq \infty$ and $1 \leq q \leq \infty$. More precisely, the Lorentz sequence space $l_{(p,q)}$, $1 < p \leq \infty$, $1 \leq q \leq \infty$, is the set of all complex sequences $a = \{a(n)\}$ such that $\|a\|^s_{(p,q)} < \infty$ where

$$\|a\|^s_{(p,q)} = \begin{cases} \left(\sum_{n=1}^{\infty} \left(n^{1/p} a^*(n)\right)^q \frac{1}{n}\right)^{1/q}, & 1 < p < \infty,\ 1 \leq q < \infty \\ \sup_{n\geq 1} n^{1/p} a^*(n), & 1 < p \leq \infty,\ q = \infty \end{cases}$$

where $a^*(n) = \inf\{\lambda > 0 : D_a(\lambda) \le n-1\}$ and the distribution function of any complex-valued function $a = \{a(n)\}_{n\ge 1}$ can be written as $D_a(\lambda) = \mu(\{n \in \mathbb{N} : |a(n)| > \lambda\})$ with $\lambda \ge 0$. The sequence $a^* = \{a^*(n)\}$ is obtained by permuting $\{|a(n)|\}_{n\in S}$ where $S = \{n : a(n) \ne 0\}$, in the decreasing order with $a^*(n) = 0$ for $n > \mu(S)$ if $\mu(S) < \infty$. The Lorentz sequence space $l_{(p,q)}$, $1 < p \le \infty$, $q = \infty$, is a linear space and $\|\cdot\|^s_{(p,q)}$ is a quasinorm. Moreover, $l_{(p,q)}$, $1 < p \le \infty$, $1 \le q \le \infty$, is complete with respect to the quasinorm $\|\cdot\|^s_{(p,q)}$. Observe that if the sequence $a = \{a(n)\} \in l_{(p,q)}$, then

$$a(n) \to 0 \quad \text{as} \quad n \to \infty. \tag{2}$$

Indeed, if (2) is false then we can find a $\delta > 0$ and a subsequence $\{a(n_k)\}$ such that $|a(n_k)| \ge \delta$ for all $k \in \mathbb{N}$. Hence $D_a(\lambda) = +\infty$ for all $\lambda \in (0, \delta]$, and $\{\lambda > 0 : D_a(\lambda) \le n-1\} \subset (\delta, +\infty)$ for all $n \in \mathbb{N}$. We conclude that $a^*(n) \ge \delta$ for all $n \in \mathbb{N}$ which implies that $\|a\|^s_{(p,q)} = +\infty$, giving us a contradiction with the fact that $a \in l_{(p,q)}$.

The Lorentz sequence space $l_{(p,q)}$ is a normed linear space if and only if $1 \le q \le p < \infty$; see [11]. Moreover, $l_{(p,q)}$ is normable when $1 < p < q \le \infty$; that is, there exists a norm equivalent to $\|\cdot\|^s_{(p,q)}$. For the remaining cases $l_{(p,q)}$ cannot be equipped with an equivalent norm. The normable case for $p < q$ comes up in the following way:

$$\|a\|^*_{(p,q)} = \begin{cases} \left(\sum_{n=1}^{\infty} (a^{**}(n))^q \, n^{q/p-1}\right)^{1/q}, & q < \infty \\ \sup_{n\ge 1} \left\{ n^{1/p} a^{**}(n) \right\}, & q = \infty \end{cases}$$

where $a^{**} = \{a^{**}(n)\}$ is called the maximal sequence of $a^* = \{a^*(n)\}$ and it is defined as

$$a^{**}(n) = \frac{1}{n} \sum_{k=1}^{n} a^*(k).$$

It is known that for $1 < p \le q < \infty$ the following relation holds:

$$\|a\|^s_{(p,q)} \le \|a\|^*_{(p,q)} \le \left(\frac{p}{p-1}\right)^q \|a\|^s_{(p,q)}.$$

Remark 2. By definition of the rearrangements, if $a = \{a(n)\}$ and $b = \{b(n)\}$ are complex sequences, $b \in l_{(p,q)}$ with $1 < p \le \infty$, $1 \le q \le \infty$ and $|a(n)| \le |b(n)|$ for all $n \in \mathbb{N}$, then $a^*(n) \le b^*(n)$ for all $n \in \mathbb{N}$. Hence $a \in l_{(p,q)}$ and $\|a\|^*_{(p,q)} \le \|b\|^*_{(p,q)}$.

The following result is due to Hardy-Littlewood.

Theorem 3. *If $a = \{a(n)\}$ and $b = \{b(n)\}$ are complex sequences, then*

$$\sum_{n=1}^{\infty} |a(n)b(n)| \leq \sum_{n=1}^{\infty} a^*(n)b^*(n). \tag{3}$$

As a consequence of the above result, we have that if $a = \{a(n)\} \in l_{(p,t)}$ and $b = \{b(n)\} \in l_{(q,r)}$, where $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{t} + \frac{1}{r} = 1$, then

$$\sum_{n=1}^{\infty} |a(n)b(n)| \leq \|a\|_{(p,t)}^{s} \|b\|_{(q,r)}^{s}.$$

This last inequality tells us that each $a \in l_{(p,q)}$ defines a bounded linear functional h on $l_{(r,t)}$, where $\frac{1}{p} + \frac{1}{r} = 1$ and $\frac{1}{q} + \frac{1}{t} = 1$, which is given by

$$h(b) = \sum_{n=1}^{\infty} a(n) \cdot b(n). \tag{4}$$

Conversely (see [11]), each bounded linear functional h on $l_{(r,t)}$ is of the form (4) for some $a \in l_{(p,q)}$ and $\|h\| \simeq \|a\|_{(p,q)}$. We refer this fact as Riesz's theorem for Lorentz sequence spaces.

3 The essential norm of multiplication operators on Lorentz sequence spaces

Recall that if X is a Banach space and $T : X \to X$ a continuos operator, then the *essential norm* of T, denoted by $\|T\|_e$, is the distance of T to the class of the compact operators on X; that is,

$$\|T\|_e = \inf \{\|T - K\| : K : X \to X \text{ is compact}\},$$

where $\|T\|$ denotes the operator norm of T, which is defined by $\|T\| = \sup \{\|Tf\|_X : \|f\|_X = 1\}$. Observe that $T : X \to X$ is compact if and only if $\|T\|_e = 0$. With this notation, now we are ready to show our main result:

Proof of Main Theorem

For each $N \in \mathbb{N}$, we set $u_N = (u(1), u(2), ..., u(N), 0, 0, ...)$. Then by Theorem 1, the multiplication operator M_{u_N} is compact on $l_{(p,q)}$. Hence

$$\|M_u\|_e \leq \|M_u - M_{u_N}\| = \|M_{u-u_N}\|.$$

But if $a = \{a(n)\} \in l_{(p,q)}$ is such that $\|a\|^*_{(p,q)} = 1$, then clearly

$$|(u(n) - u_N(n)) \cdot a(n)| \leq S_N\, |a(n)|$$

for all $n \in \mathbb{N}$, where $S_N = \sup\{|u(k)| : k \geq N\}$. Thus, by Remark 2, we conclude that

$$\|M_{u-u_N}(a)\|^*_{(p,q)} = \|(u - u_N) \cdot a\|^*_{(p,q)} \leq S_N\ \|a\|^*_{(p,q)} = S_N,$$

and therefore $\|M_u\|_e \leq S_N$ for all $N \in \mathbb{N}$. That is,

$$\|M_u\|_e \leq \limsup_{n\to\infty} |u(n)|\,.$$

On the other hand, let $K : l_{(p,q)} \to l_{(p,q)}$ be any compact operator and consider the sequence $\{e_k\} \subset l_{(p,q)}$ given by

$$e_k(n) = \begin{cases} 1, & n = k \\ 0, & \text{otherwise} \end{cases}.$$

Then $\|e_k\|^s_{(p,q)} = 1$ for all $k \in \mathbb{N}$, and $\{e_k\}$ is a bounded sequence in $l_{(p,q)}$. We claim that

$$\lim_{k\to\infty} \|K(e_k)\|^*_{(p,q)} = 0. \tag{5}$$

Indeed, if (5) is false, then we can find a $\delta > 0$ and a subsequence $\{e_{k_m}\}$ such that

$$\|K(e_{k_m})\|^*_{(p,q)} \geq \delta \tag{6}$$

for all $m \in \mathbb{N}$. Since $K : l_{(p,q)} \to l_{(p,q)}$ is compact and $\{e_{k_m}\}$ is bounded in $l_{(p,q)}$, by passing to a subsequence if is necessary, we can suppose that $\{K(e_{k_m})\}$ converges in $l_{(p,q)}$. That is, there exists a $b \in l_{(p,q)}$ such that

$$\lim_{m\to\infty} \|K(e_{k_m}) - b\|^*_{(p,q)} = 0.$$

Note that, if $b = 0$, then this fact leads us to contradict (6). By Hahn-Banach's theorem, it is enough to show that $h(b) = 0$ for all bounded linear functionals h on $l_{(p,q)}$. Let h be any bounded linear functional on $l_{(p,q)}$. Then the composition hK is also a bounded linear functional on $l_{(p,q)}$, so that, by the Riesz representation theorem for $l_{(p,q)}$ spaces, there exists $a = \{a(n)\} \in l_{(r,t)}$ with $\frac{1}{p} + \frac{1}{r} = 1$ and $\frac{1}{q} + \frac{1}{t} = 1$ such that

$$hK(c) = \sum_{n=1}^{\infty} a(n) \cdot c(n)$$

for all $c = \{c(n)\} \in l_{(p,q)}$. In particular, by (2) and evaluating at e_{k_m}, we have $hK(e_{k_m}) = a(k_m) \to 0$ as $m \to \infty$. Hence

$$\begin{aligned} |h(b)| &\leq |h(b) - hK(e_{k_m})| + |a(k_m)| \\ &\leq \|h\| \, \|b - K(e_{k_m})\|^*_{(p,q)} + |a(k_m)| \to 0 \end{aligned}$$

as $m \to \infty$ and $h(b) = 0$. This proves the claim.

Next we can conclude the proof of our result. Observe that $e_k^{**} = e_1^{**} = \left(1, \frac{1}{2}, ..., \frac{1}{n}, ...\right)$ for all $k \in \mathbb{N}$. Hence there exists a constant $D_{(p,q)} > 0$, depending only on p and q, such that $D_{(p,q)} = \|e_k\|^*_{(p,q)}$ for all $k \in \mathbb{N}$. Thus, for each $k \in \mathbb{N}$, the vector

$$f_k = \frac{e_k}{\|e_k\|^*_{(p,q)}}$$

is unitary in $l_{(p,q)}$, and we can write

$$\begin{aligned} \|M_u - K\| &\geq \|M_u(f_k) - K(f_k)\|^*_{(p,q)} \\ &\geq \frac{1}{D_{(p,q)}} \|M_u(e_k)\|^*_{(p,q)} - \frac{1}{D_{(p,q)}} \|K(e_k)\|^*_{(p,q)} \\ &= |u(k)| - \frac{1}{D_{(p,q)}} \|K(e_k)\|^*_{(p,q)} . \end{aligned}$$

Thus taking limit when $k \to \infty$, we conclude that

$$\|M_u - K\| \geq \limsup_{k \to \infty} |u(k)| ,$$

and therefore $\|M_u\|_e \geq \limsup_{k \to \infty} |u(k)|$ since the compact operator K on $l_{(p,q)}$ was arbitrary. This finishes the proof. ■

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