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ESSENTIAL CLOSURES

Abstract

Based on the Zermelo-Fraenkel system of axioms ZF, we introduce a theory of essential closures. It is a generalization of the concept of topological closures in which a set may not be contained in its essential closure. A typical essential closure collects all points which are essential with respect to a submeasure; hence it is called a submeasure closure. One of our main results states that a "nice" essential closure must be a submeasure closure. Many examples of known and new submeasure closures are discussed and their applications are demonstrated, especially in the study of the supports of measures.

1 Introduction

It was suggested in [1, Proposition 1] and [17, Lemma 10] that the probability mass of a complete dependence copula $C = C_{U,f(U)}$ is concentrated on the graph of f in the sense that $V_C(\operatorname{gr} f) = 1$. Here, the random variable U is uniformly distributed on $[0,1]$, $f\colon [0,1] \to [0,1]$ is measure-preserving and V_C denotes the Borel probability measure on $[0,1]^2$ induced by C. However, to the best of our knowledge, due probably to the lack of a suitable tool, no one

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had determined an explicit formula of the support of V_C in terms of the graph of f. Recently in [16], a formula of $\operatorname{supp} V_C$ in terms of $\operatorname{gr} f$ was obtained via a "new" tool called an *essential closure*. In $\mathbb{R}^2$, the essential closure $\widetilde{A}$ of a set A is the set of points $x \in \mathbb{R}^2$ for which the projection of each open neighborhood of x in A onto a coordinate axis has a positive Lebesgue outer measure. It was derived in [16, Theorem 3.3.3] that

$$\operatorname{supp} V_C = \widetilde{\operatorname{gr} f}$$

given that f is *essentially refined*, of which some examples are piecewise linear functions.

The adjective "essential" is quite ubiquitous in mathematical analysis and is often used to indicate that a defining condition holds outside of a negligible set. As such, taking essential closure should mean taking closure by ignoring "small" sets. For instance, in the above essential closure on $\mathbb{R}^2$, small sets are precisely those sets whose coordinate projections each have Lebesgue outer measure zero. As a tool in their study of absolutely continuous spectra for some linear operators, Gesztesy et al. [9] defined an essential closure on $\mathbb{R}$, with respect to which small sets are sets of Lebesgue measure zero. Both essential closures are our prototypes of general essential closures and share many satisfying properties. Thus far, there seems to be no systematic treatment of essential closures.

In this manuscript, our aim is to propose a set of postulates for general essential closures and to develop a theory of essential closures. Among many results, the concept of non-essential or "small" sets is (re)introduced and proved to be closely related to essential closures. Submeasure closures, defined as the essential closures whose non-essential sets are sets of submeasure zero, and their examples are investigated. They are shown to be useful in the study of the supports of measures. An interesting result is that the class of submeasure closures is large enough to contain all "nice" essential closures.

In section 2, we develop a theory of essential closures starting with a set of four postulates. In section 3, we present a motivation behind the set of postulates of essential closures from a topological point of view. In section 4, we construct concrete examples of essential closures via submeasures and demonstrate their applications. Finally, we discuss some existing concepts related to essential closures in the last section.

2 Essential closures

In sections 2 and 3, we denote both essential closures and essential closure operators by $\mathcal{E}$. Likewise, we use the notations $A \mapsto \overline{A}$ and cl to denote both topological closures and topological closure operators. In addition, for a given topological space X, $\mathfrak{N}(x)$ denotes the collection of open neighborhoods of $x \in X$ and $\mathcal{P}(X)$ denotes the collection of subsets of X.

2.1 Postulates for essential closures

After experimenting with various potential sets of postulates for essential closures, we have come to a conclusion that the following set of postulates seems the most natural.

Postulate 1. Let (X, τ) be a topological space equipped with an algebra Ω over X. We say that a unary operation $\mathcal{E}\colon \Omega \to \Omega$ is an *essential closure* if for every $A, B \in \Omega$, the following hold:

1. $\mathcal{E}(A)$ is a closed set;
2. $\mathcal{E}(A) \subseteq \overline{A}$;
3. $\mathcal{E}(A \cup B) = \mathcal{E}(A) \cup \mathcal{E}(B)$; and
4. $\mathcal{E}$ is idempotent; i.e., $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$.

Remark. It follows directly from 2 and 3 of Postulate 1 that $\mathcal{E}(\emptyset) = \emptyset$ and that $\mathcal{E}$ is monotonic with respect to the set inclusion, respectively.

Definition 2. A unary operation $\mathcal{E}$ on a topological space (X, τ) equipped with an algebra Ω over X is said to be

1. *strong* if $\mathcal{E}(A \setminus \mathcal{E}(A)) = \emptyset$ for every $A \in \Omega$; and
2. *weakly strong* if for each $A \in \Omega$ and $x \notin \mathcal{E}(A)$, there is $G \in \mathfrak{N}(x) \cap \Omega$ such that $\mathcal{E}(A \cap G) = \emptyset$.

Remark.

1. Let $\mathcal{E}$ be a unary operation on an algebra Ω satisfying 1, 2 and 3 of Postulate 1. If $\mathcal{E}(A \setminus \mathcal{E}(A)) = \emptyset$ for each $A \in \Omega$, then $\mathcal{E}$ is idempotent as

$$\mathcal{E}(A) = \mathcal{E}(A \setminus \mathcal{E}(A)) \cup \mathcal{E}(A \cap \mathcal{E}(A)) \subseteq \mathcal{E}(\mathcal{E}(A)) \subseteq \overline{\mathcal{E}(A)} = \mathcal{E}(A).$$

   Thus, if $\mathcal{E}$ satisfies 1, 2 and 3 and is strong then it is an essential closure. However, a weakly strong unary operation satisfying 1, 2 and 3 need not be an essential closure. See Example 8.

2. A strong unary operation satisfying 1 of Postulate 1 is also weakly strong.

Example 1. Let $X = \{0, 1\}$, $\tau = \{\emptyset, \{0\}, X\}$ and $\Omega = \mathcal{P}(X)$. Define a unary operation $\mathcal{E} \colon \Omega \to \Omega$ by $\mathcal{E}(\emptyset) = \emptyset$ and $\mathcal{E}(A) = \{1\}$ if A is not empty. It is easy to check that $\mathcal{E}$ is an essential closure. Moreover,

$$\mathcal{E}(X \setminus \mathcal{E}(X)) = \mathcal{E}(\{1\}^c) = \{1\} \neq \emptyset.$$

Hence $\mathcal{E}$ is not strong. Note that $\mathcal{E}(X) \neq X$.

Proposition 1. *Let $\mathcal{E}$ be an essential closure on an algebra Ω over a topological space X and suppose that $\mathcal{E}(X) = X$. Then the following hold:*

1. $\mathcal{E}(A)^c \subseteq \mathcal{E}(A^c)$ *for each* $A \in \Omega$*;*
2. $\mathcal{E}(G) = \overline{G}$ *for every open set* $G \in \Omega$*; and*
3. $\overline{\operatorname{int} A} \subseteq \mathcal{E}(A)$ *for each* $A \in \Omega$ *such that* $\operatorname{int} A \in \Omega$.

Proof. Recall the properties of essential closures in Postulate 1.

1. Observe that $X = \mathcal{E}(X) = \mathcal{E}(A \cup A^c) = \mathcal{E}(A) \cup \mathcal{E}(A^c)$. Hence we have $\mathcal{E}(A)^c \subseteq \mathcal{E}(A^c)$.
2. If $G \in \Omega$ is open, then $\mathcal{E}(G)^c \subseteq \mathcal{E}(G^c) \subseteq \overline{G^c} = G^c$. Thus $G \subseteq \mathcal{E}(G) \subseteq \overline{G}$. Since $\mathcal{E}(G)$ is closed, $\mathcal{E}(G) = \overline{G}$.
3. Since $\operatorname{int} A \in \Omega$ is open, $\overline{\operatorname{int} A} = \mathcal{E}(\operatorname{int} A) \subseteq \mathcal{E}(A)$. □

2.2 Non-essential sets

In this section, we introduce one of the most important concepts related to essential closures, namely the concept of essential and non-essential sets. This concept is at the core of the theory of essential closures. Non-essential sets can be viewed as small sets with respect to an essential closure.

Definition 3. Let $\mathcal{E}$ be an essential closure on Ω. Then a set $A \in \Omega$ is said to be *non-essential* if $\mathcal{E}(A) = \emptyset$; otherwise, A is said to be *essential*. The collection of non-essential sets is denoted by $\mathcal{N}_\Omega(\mathcal{E})$.

Theorem 2. *Let $\mathcal{E}$ be an essential closure on Ω. Then $\mathcal{E}$ is weakly strong if and only if, for each $A \in \Omega$, $\mathcal{E}(A)$ is the intersection of the closed sets $F \in \Omega$ such that $A \setminus F$ is non-essential.*

Proof. Assume that $\mathcal{E}$ is weakly strong. For each $A \in \Omega$, if $x \notin \mathcal{E}(A)$, then there exists $G \in \mathfrak{N}(x) \cap \Omega$ such that $\mathcal{E}(A \cap G) = \emptyset$. Therefore,

$$\mathcal{E}(A)^c \subseteq \bigcup\{G \in \Omega \colon G \text{ is open and } \mathcal{E}(A \cap G) = \emptyset\}.$$

So $\mathcal{E}(A) \supseteq \bigcap\{F \in \Omega \colon F \text{ is closed and } \mathcal{E}(A \setminus F) = \emptyset\}$. For the other inclusion, it suffices to show that any closed set $F \in \Omega$ with $\mathcal{E}(A \setminus F) = \emptyset$ necessarily contains $\mathcal{E}(A)$. Observe that for such a set F,

$$\mathcal{E}(A) = \mathcal{E}(A \cap F) \cup \mathcal{E}(A \setminus F) = \mathcal{E}(A \cap F) \subseteq \mathcal{E}(F) \subseteq \overline{F} = F.$$

To prove the converse, let $A \in \Omega$ and suppose $x \notin \mathcal{E}(A)$. Then, by the assumption, $x \in G$ for some open set $G \in \Omega$ such that $\mathcal{E}(A \cap G) = \emptyset$. In other words, $\mathcal{E}$ is weakly strong. □

According to Theorem 2, one can see that the collection of non-essential sets acts as a generator of its corresponding weakly strong essential closure. To study weakly strong essential closures, it suffices to study their non-essential sets.

Corollary 3. *Suppose $\mathcal{E}_1$ and $\mathcal{E}_2$ are weakly strong essential closures on Ω such that $\mathcal{N}_\Omega(\mathcal{E}_1) = \mathcal{N}_\Omega(\mathcal{E}_2)$. Then the two essential closures coincide.*

Definition 4. Let $\mathcal{E}$ be an essential closure on an algebra Ω over a topological space X. Then a set $A \in \Omega$ is said to be *locally essential* if $\mathcal{E}(G \cap A) \neq \emptyset$ for every open set $G \in \Omega$ such that $G \cap A \neq \emptyset$.

Proposition 4. *Let $\mathcal{E}$ be an essential closure on an algebra Ω over X and suppose $\mathcal{E}(X) = X$. Then every open set $O \in \Omega$ is locally essential.*

Proof. Let $G \in \Omega$ be an open set such that $G \cap O \neq \emptyset$. By Proposition 1(2), $\mathcal{E}(G \cap O) = \overline{G \cap O} \supseteq G \cap O \neq \emptyset$. □

Definition 5. An essential closure on Ω is said to be *σ-non-essential* if Ω is a σ-algebra and the union of every countable collection of non-essential sets is non-essential.

Lemma 5. *Let $\mathcal{E}$ be an essential closure on an algebra Ω and $x \in \mathcal{E}(A)$. Then for any $G \in \mathfrak{N}(x) \cap \Omega$, $G \cap A$ is essential.*

Proof. Suppose there exists $G \in \mathfrak{N}(x) \cap \Omega$ with $\mathcal{E}(G \cap A) = \emptyset$. Then

$$\mathcal{E}(A) = \mathcal{E}(A \cap G^c) \subseteq \mathcal{E}(A) \cap \mathcal{E}(G^c) \subseteq \mathcal{E}(A) \cap \overline{G^c} = \mathcal{E}(A) \setminus G,$$

which contradicts the fact that $\mathcal{E}(A) \setminus G$ is a proper subset of $\mathcal{E}(A)$. □

Definition 6. A *topological measurable space* is a triple (X, τ, Ω) where (X, τ) is a topological space and Ω is a σ-algebra over X containing the topology τ.

The following result requires a technical assumption that for each point $x \in X$ and each $G \in \mathfrak{N}(x)$, there is $O \in \mathfrak{N}(x)$ with $\overline{O} \subseteq G$. A topological space with such a property is called *regular*. More information on regular spaces can be found in Munkres' book [13]. A *regular measurable space* is a topological measurable space where the topology is regular.

Theorem 6. *Let $\mathcal{E}$ be a σ-non-essential essential closure on a regular measurable space. Then for every sequence of sets A_i in Ω,*

$$\mathcal{E}\left(\bigcup_{i=1}^{\infty} A_i\right) = \overline{\bigcup_{i=1}^{\infty} \mathcal{E}(A_i)}.$$

Proof. If $x \in \mathcal{E}\left(\bigcup_{i=1}^{\infty} A_i\right)$ and $G \in \mathfrak{N}(x)$, then there exists $O \in \mathfrak{N}(x)$ such that $\overline{O} \subseteq G$. By Lemma 5, $\mathcal{E}\left(\bigcup_{i=1}^{\infty}(O \cap A_i)\right) = \mathcal{E}\left(O \cap \bigcup_{i=1}^{\infty} A_i\right) \neq \emptyset$. Since the essential closure is σ-non-essential, there exists A_j with $\mathcal{E}(O \cap A_j) \neq \emptyset$. Hence

$$\emptyset \neq \mathcal{E}(O \cap A_j) \subseteq \mathcal{E}(O) \cap \mathcal{E}(A_j) \subseteq \overline{O} \cap \mathcal{E}(A_j) \subseteq G \cap \bigcup_{i=1}^{\infty} \mathcal{E}(A_i).$$

This implies that $x \in \overline{\bigcup_{i=1}^{\infty} \mathcal{E}(A_i)}$. The other inclusion follows trivially from the fact that the essential closure of a set is closed. $\square$

In what follows, we will consider various essential closures defined in the same manner. Given an algebra Ω over X, $\mathcal{I} \subseteq \Omega$ is an *ideal* if (1) $\emptyset \in \mathcal{I}$; (2) for every $A \in \mathcal{I}$, if $B \in \Omega$ is such that $B \subseteq A$, then $B \in \mathcal{I}$; and (3) if $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. Given an ideal $\mathcal{I}$ in Ω and a set $F \in \Omega \setminus \mathcal{I}$, it is straightforward to verify that the unary operation $\mathcal{E} = \mathcal{E}_{\mathcal{I},F}$ on Ω defined by

$$\mathcal{E}(A) = \begin{cases} \emptyset & \text{if } A \in \mathcal{I}, \\ F & \text{otherwise} \end{cases} \tag{1}$$

is an essential closure with respect to the topology $\tau = \{\emptyset, F^c, X\}$.

The following example shows that an essential closure on a σ-algebra is not necessarily σ-non-essential.

Example 2. Let $X = \mathbb{N}$, $\tau = \{\emptyset, \{1\}^c, X\}$ and $\Omega = \mathcal{P}(X)$. Consider an essential closure $\mathcal{E} = \mathcal{E}_{\mathcal{I},\{1\}}$ where $\mathcal{I} = \{A \in \Omega \colon A \text{ is finite and } 1 \notin A\}$. Observe that $\mathcal{E}(X \setminus \mathcal{E}(X)) = \{1\} \neq \emptyset = \bigcup_{x \neq 1} \mathcal{E}(\{x\})$. Hence there exists an essential closure on a σ-algebra that is neither strong nor σ-non-essential.

The following two examples show that the concepts of strong essential closures and σ-non-essential essential closures are not related in an obvious way, that is one does not imply the other.

Example 3. Let $X = \mathbb{N}$, $\tau = \{\emptyset, X\}$, $\Omega = \mathcal{P}(X)$ and $\mathcal{E} = \mathcal{E}_{\mathcal{I},X}$ where $\mathcal{I} = \{A \in \Omega \colon A \text{ is finite}\}$. It is easy to check that the essential closure $\mathcal{E}$ is strong. However, $\mathcal{E}(X) = X \neq \emptyset = \bigcup_{x \in X} \mathcal{E}(\{x\})$. Hence, there is a strong essential closure on a σ-algebra that is not σ-non-essential.

Example 4. Let $X = \mathbb{N}$, $\tau = \{\emptyset, \{1\}^c, X\}$, $\Omega = \mathcal{P}(X)$ and $\mathcal{E} = \mathcal{E}_{\{\emptyset\},\{1\}}$. Clearly, $\mathcal{E}$ is σ-non-essential. However, observe that

$$\mathcal{E}(X \setminus \mathcal{E}(X)) = \mathcal{E}(\{1\}^c) = \{1\} \neq \emptyset.$$

Hence, there is a σ-non-essential essential closure that is not strong.

Clearly, the non-essential sets of the essential closure $\mathcal{E}_{\mathcal{I},F}$ defined by (1) are exactly the sets in the ideal $\mathcal{I}$. Observe also that the non-essential sets of any given essential closure form an ideal. Conversely, if one has in mind which sets should be considered small, then there always exists an essential closure with respect to which the pre-assigned small sets are non-essential. We will be more interested in σ-non-essential essential closures.

Definition 7. Let $\emptyset \neq S \subseteq \Omega$, where Ω is a σ-algebra over X. Define $\mathcal{N}_\Omega(S)$ to be the smallest collection which satisfies the following conditions for all $B \in \Omega$ and $A, A_1, A_2, \dots \in \mathcal{N}_\Omega(S)$:

1. $S \subseteq \mathcal{N}_\Omega(S) \subseteq \Omega$;
2. $B \subseteq A$ implies $B \in \mathcal{N}_\Omega(S)$; and
3. $\bigcup_{n=1}^{\infty} A_n \in \mathcal{N}_\Omega(S)$.

Remark. Notice that $\mathcal{N}_\Omega(S)$ is the smallest σ-ideal of Ω containing S; see page 13 in Bauer's book [3].

In the sequel, we often require that every subset of the space is Lindelöf. Such a topological space is called *hereditarily Lindelöf.*

Theorem 7. *Let (X, τ, Ω) be a Lindelöf measurable space and S be a non-empty subcollection of Ω. Then there exists a unique σ-non-essential weakly strong essential closure whose collection of non-essential sets is exactly the collection $\mathcal{N}_\Omega(S)$. In fact, it is defined by*

$$\mathcal{E}(A) = \bigcap \{F \in \Omega \colon F \text{ is closed and } A \setminus F \in \mathcal{N}_\Omega(S)\} \quad \text{for } A \in \Omega. \tag{2}$$

Proof. It is straightforward to verify that the unary operation $\mathcal{E}$ defined by (2) is an essential closure on Ω. By the definition of $\mathcal{E}$ in (2), $A \in \mathcal{N}_\Omega(S)$ implies $\mathcal{E}(A) = \emptyset$. Conversely, suppose $\mathcal{E}(A) = \emptyset$. Then for each $x \in X$, there exists $G \in \mathfrak{N}(x) \cap \Omega$ such that $A \cap G \in \mathcal{N}_\Omega(S)$. Let $\mathcal{G}$ be the collection of open sets $G \in \Omega$ such that $A \cap G \in \mathcal{N}_\Omega(S)$. Hence $\mathcal{G}$ covers X, which is Lindelöf. Let $\{G_n\}_{n\in\mathbb{N}}$ be a countable subcover of $\mathcal{G}$. Since $A \cap G_n \in \mathcal{N}_\Omega(S)$ for all $n \in \mathbb{N}$, $A = \bigcup_{n=1}^\infty (A \cap G_n) \in \mathcal{N}_\Omega(S)$ by property 3 in Definition 7. Hence the collections $\mathcal{N}_\Omega(\mathcal{E})$ and $\mathcal{N}_\Omega(S)$ coincide.

Let $A \in \Omega$ and $x \notin \mathcal{E}(A)$. Then $x \in G$ for some open set $G \in \Omega$ such that $A \cap G \in \mathcal{N}_\Omega(S)$, which implies that $\mathcal{E}(A \cap G) = \emptyset$. Thus $\mathcal{E}$ is weakly strong. Moreover, since the collection $\mathcal{N}_\Omega(S)$ is closed under countable union, the induced essential closure is σ-non-essential. The uniqueness part follows from Corollary 3. □

In the previous theorem, a similar result also holds if we replace Lindelöf and weakly strong with hereditarily Lindelöf and strong, respectively.

Theorem 8. *Let (X, τ, Ω) be a hereditarily Lindelöf measurable space and S be a non-empty subcollection of Ω. Then there exists a unique σ-non-essential strong essential closure, defined by* (2)*, whose collection of non-essential sets is exactly the collection $\mathcal{N}_\Omega(S)$.*

Proof. In view of Theorem 7, $\mathcal{E}$ is an essential closure and it suffices to show that $\mathcal{E}$ is strong. Let $A \in \Omega$. Since $\mathcal{E}(A)^c$ is Lindelöf,

$$\begin{aligned} A \setminus \mathcal{E}(A) &= A \setminus \bigcap \{F \in \Omega \colon F \text{ is closed and } A \setminus F \in \mathcal{N}_\Omega(S)\} \\ &= A \setminus \bigcap_{n=1}^{\infty} \{F_n \in \Omega \colon F_n \text{ is closed and } A \setminus F_n \in \mathcal{N}_\Omega(S)\} \\ &= \bigcup_{n=1}^{\infty} \{A \setminus F_n \colon F_n \in \Omega \text{ is closed and } A \setminus F_n \in \mathcal{N}_\Omega(S)\} \end{aligned}$$

for some countable subcollection $\{F_n\}_{n\in\mathbb{N}}$ of Ω. Hence, $A \setminus \mathcal{E}(A) \in \mathcal{N}_\Omega(S)$ by property 3 in Definition 7. In consequence, $\mathcal{E}$ is a strong essential closure. The uniqueness of $\mathcal{E}$ follows clearly from Theorem 7. □

In Theorems 7 and 8, since (X, τ, Ω) is a topological measurable space, Ω is assumed to contain the topology τ. If the σ-algebra does not contain the topology, the theorems may fail to hold. This is demonstrated in the following example.

Example 5. Choose pairwise distinct elements a, b and c. Put $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $S = \{\emptyset\}$ and $\Omega = \{\emptyset, \{a\}, \{b, c\}, X\}$. Notice that Ω does not contain τ and $\mathcal{N}_\Omega(S) = \{\emptyset\}$. Using the same construction as in Theorems 7 and 8, we have $\mathcal{E}(\{a\}) = X$ while $\overline{\{a\}} = \{a, c\}$. Hence the induced mapping is not an essential closure since it violates the second property of essential closures in Postulate 1.

2.3 Essential closedness

In this section, we introduce another important concept related to essential closures, namely the concept of essential closedness.

Definition 8. Let $\mathcal{E}$ be an essential closure on Ω. A set $F \in \Omega$ is said to be *essentially closed* if and only if $\mathcal{E}(F) = F$. We denote the collection of essentially closed sets by $\mathcal{C}_\Omega(\mathcal{E})$.

Proposition 9. *Let $\mathcal{E}$ be a strong essential closure on Ω. Then for any $A \in \Omega$, $\mathcal{E}(A) = \bigcap\{F \in \mathcal{C}_\Omega(\mathcal{E})\colon A \setminus F \in \mathcal{N}_\Omega(\mathcal{E})\}$.*

Proof. Since essentially closed sets are closed, it follows from Theorem 2 that for any $A \in \Omega$, $\mathcal{E}(A) \subseteq \bigcap\{F \in \mathcal{C}_\Omega(\mathcal{E})\colon A \setminus F \in \mathcal{N}_\Omega(\mathcal{E})\}$. The other inclusion follows from the fact that $\mathcal{E}(A)$ is essentially closed and $\mathcal{E}$ is strong. $\square$

Example 6. The above result does not generally hold for weakly strong essential closures. For example, let

$$X = \{0\} \cup \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\}$$

be equipped with the subspace topology inherited from the standard topology of $\mathbb{R}$ and let $\Omega = \mathcal{P}(X)$. For each $A \in \Omega$, define $\mathcal{E}(A) = \emptyset$ if A is finite and $0 \notin A$; otherwise, $\mathcal{E}(A) = \{0\}$.

It is straightforward to verify that $\mathcal{E}$ is a weakly strong essential closure and $\mathcal{C}_\Omega(\mathcal{E}) = \{\emptyset, \{0\}\}$. One can see that $\mathcal{E}(\{0\}^c) = \{0\}$, but on the other hand, there is no essentially closed set F such that $\mathcal{E}(\{0\}^c \setminus F) = \emptyset$. Therefore, $X = \bigcap\{F \in \mathcal{C}_\Omega(\mathcal{E})\colon \mathcal{E}(\{0\}^c \setminus F) = \emptyset\}$ as it is the empty intersection. Thus $\mathcal{E}(\{0\}^c) \neq \bigcap\{F \in \mathcal{C}_\Omega(\mathcal{E})\colon \{0\}^c \setminus F \in \mathcal{N}_\Omega(\mathcal{E})\}$.

Proposition 10. *Let $\mathcal{E}$ be an essential closure on Ω and $F \in \Omega$. If F is essentially closed, then F is closed and locally essential.*

Proof. Assume F is essentially closed. Then F is closed. Hence for any open set $G \in \Omega$ such that $G \cap F \neq \emptyset$, $\mathcal{E}(F \setminus G) \subseteq \overline{F \setminus G} = F \setminus G \subsetneq F$. Moreover, $F = \mathcal{E}(F) = \mathcal{E}(F \setminus G) \cup \mathcal{E}(F \cap G)$. Thus $\mathcal{E}(F \cap G) \neq \emptyset$. $\square$

Proposition 11. *Let $\mathcal{E}$ be a weakly strong essential closure on Ω and $F \in \Omega$. If F is closed and locally essential, then F is essentially closed.*

Proof. Since F is closed, $\mathcal{E}(F) \subseteq \overline{F} = F$. Suppose that $F \setminus \mathcal{E}(F) \neq \emptyset$ and let $x \in F \setminus \mathcal{E}(F)$. Then there exists $G \in \mathfrak{N}(x) \cap \Omega$ such that $\mathcal{E}(G \cap F) = \emptyset$. Since F is locally essential and $\mathcal{E}(G \cap F) = \emptyset$, $G \cap F = \emptyset$. This contradicts the fact that $x \in G \cap F$. Therefore, F is essentially closed. □

Together, Propositions 10 and 11 give a characterization of essential closedness for weakly strong essential closures.

Corollary 12. *Let $\mathcal{E}$ be a weakly strong essential closure on an algebra Ω and $F \in \Omega$. Then F is essentially closed if and only if F is closed and locally essential.*

3 Essential closure operators

In this section, we provide an alternative approach to postulating the concept of essential closures. An advantage of this approach is that we need not assume any a priori topological structure. Recall that a *topological closure operator* on a set X is defined as a unary operation $\mathrm{cl}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ satisfying the following properties for all $A, B \subseteq X$:

1. $\mathrm{cl}(\emptyset) = \emptyset$;
2. $A \subseteq \mathrm{cl}(A)$;
3. $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$; and
4. cl is idempotent.

It is well known that there is a one-to-one correspondence between the collection of topological closure operators and the collection of topological closures (equivalently, the collection of topologies) on a common space. If we want to add the prefix "essential," then the property that $A \subseteq \mathrm{cl}(A)$ should be excluded. We propose a set of postulates for essential closure operators accordingly.

Postulate 9. Let X be a non-empty set and Ω an algebra over X. An *essential closure operator* on (X, Ω) is a unary operation $\mathcal{E}\colon \Omega \to \Omega$ which satisfies the following properties for all sets $A, B \in \Omega$:

1. $\mathcal{E}(\emptyset) = \emptyset$;

2. $\mathcal{E}(A \cup B) = \mathcal{E}(A) \cup \mathcal{E}(B)$; and

3. $\mathcal{E}$ is idempotent.

Remark. A topological closure operator restricted to any algebra is an essential closure operator. Moreover, it is the unique essential closure operator with the property that $A \subseteq \mathcal{E}(A)$ for each A in the algebra.

Next, we demonstrate a relationship between essential closures and essential closure operators. First, we need the following two technical lemmas.

Lemma 13. *Let X be a non-empty set and Ω an algebra over X. Assume that $A \mapsto \overline{A}\colon \Omega \to \Omega$ satisfies the following properties for all $A, B \in \Omega$:*

1. $\overline{\emptyset} = \emptyset$;

2. $A \subseteq \overline{A}$;

3. $\overline{A \cup B} = \overline{A} \cup \overline{B}$; and

4. $A \mapsto \overline{A}$ is idempotent.

Then $A \mapsto \overline{A}$ can be extended to a topological closure operator on X.

PROOF. Define $\mathrm{cl}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$\mathrm{cl}(A) = \bigcap_{\overline{C} \supseteq A} \overline{C},$$

where C ranges over all sets in Ω. First, we verify that cl is indeed an extension. Suppose $A \in \Omega$. Then we have

$$\mathrm{cl}(A) = \bigcap_{\overline{C} \supseteq A} \overline{C} \subseteq \overline{A} \subseteq \bigcap_{\overline{C} \supseteq \overline{A}} \overline{C} \subseteq \bigcap_{\overline{C} \supseteq A} \overline{C} = \mathrm{cl}(A),$$

where the first inclusion follows from property 2 and the last inclusion follows from the fact that $A \subseteq \overline{C}$ implies $\overline{A} \subseteq \overline{C}$. Hence the unary operation cl is an extension of $A \mapsto \overline{A}$. Moreover, observe the following properties of cl.

- $\mathrm{cl}(\emptyset) = \overline{\emptyset} = \emptyset$ since $\emptyset \in \Omega$.

- $A \subseteq \bigcap_{\overline{C} \supseteq A} \overline{C} = \mathrm{cl}(A)$.

- Observe that

$$\bigcap_{\overline{C}\supseteq A\cup B}\overline{C}\subseteq\bigcap_{\overline{D}\supseteq A,\overline{E}\supseteq B}\overline{D}\cup\overline{E}=\Big(\bigcap_{\overline{D}\supseteq A}\overline{D}\Big)\cup\Big(\bigcap_{\overline{E}\supseteq B}\overline{E}\Big).$$

Hence $\mathrm{cl}(A\cup B)\subseteq\mathrm{cl}(A)\cup\mathrm{cl}(B)$. Moreover, the other inclusion follows from the fact that cl is monotonic with respect to the set inclusion.

- If $A\subseteq\overline{C}$, then $\mathrm{cl}(A)\subseteq\mathrm{cl}(\overline{C})=\overline{C}$ since $\overline{C}\in\Omega$. Hence

$$\mathrm{cl}(\mathrm{cl}(A))=\bigcap_{\overline{C}\supseteq\mathrm{cl}(A)}\overline{C}\subseteq\bigcap_{\overline{C}\supseteq A}\overline{C}=\mathrm{cl}(A).$$

The opposite inclusion holds.

Therefore, cl is a topological closure operator on X. □

Lemma 14. *Let $\mathcal{E}$ be an essential closure operator on an algebra Ω over X. Then there exists a topology τ on X such that $\mathcal{E}(A)$ is closed and $\mathcal{E}(A)\subseteq\mathrm{cl}(A)$ for every $A\in\Omega$.*

PROOF. Define $\overline{A}=A\cup\mathcal{E}(A)$ for each $A\in\Omega$. It is straightforward to check that $A\mapsto\overline{A}\colon\Omega\to\Omega$ satisfies the properties in Lemma 13. Let cl be a topological closure operator extended from $A\mapsto\overline{A}\colon\Omega\to\Omega$ and let τ be the topology induced by cl. Observe that $\mathcal{E}(A)\subseteq A\cup\mathcal{E}(A)=\mathrm{cl}(A)$ for each $A\in\Omega$. Moreover, $\mathrm{cl}(\mathcal{E}(A))=\mathcal{E}(A)\cup\mathcal{E}(\mathcal{E}(A))=\mathcal{E}(A)$. Hence $\mathcal{E}(A)$ is closed with respect to the topology τ for each $A\in\Omega$. □

Given an essential closure, if we take out its underlying topological structure, what we obtain is an essential closure operator. The following result, which is one of our main results, shows that there is a natural way to induce an underlying topology for a given essential closure operator. However, it is not guaranteed that the induced topology coincides with the given topology.

Theorem 15. *Let $\mathcal{E}$ be an essential closure operator on Ω. Define $\tau_{\mathcal{E}}=\bigcap\tau_\alpha$, where the non-empty intersection is taken over all topologies τ_α on X satisfying the properties in Lemma 14, and let $\mathrm{cl}_{\mathcal{E}}$ be the topological closure relative to $\tau_{\mathcal{E}}$. Then $\mathcal{E}\colon\Omega\to\Omega$ satisfies the following properties for all $A\in\Omega$:*

1. *$\mathcal{E}(A)$ is closed in $(X,\tau_{\mathcal{E}})$; and*
2. *$\mathcal{E}(A)\subseteq\mathrm{cl}_{\mathcal{E}}(A)$.*

In other words, $\mathcal{E}$ is an essential closure on $(X,\tau_{\mathcal{E}},\Omega)$. Furthermore, $\tau_{\mathcal{E}}$ is generated by the collection $\{\mathcal{E}(A)^c\}_{A\in\Omega}$.

Proof. Let $A \in \Omega$. Observe that $\mathcal{E}(A)$ is closed in $(X, \tau_{\mathcal{E}})$ because $\mathcal{E}(A)^c \in \tau_\alpha$ for all α. Moreover, $\mathcal{E}(A) \subseteq \mathrm{cl}_\alpha(A) \subseteq \mathrm{cl}_{\mathcal{E}}(A)$ because $\tau_{\mathcal{E}} \subseteq \tau_\alpha$. Therefore, $\mathcal{E}$ is an essential closure on $(X, \tau_{\mathcal{E}}, \Omega)$.

Let τ be the topology generated by the collection $\{\mathcal{E}(A)^c\}_{A\in\Omega}$. Since $\mathcal{E}(A)$ is closed in $(X, \tau_{\mathcal{E}})$ for all $A \in \Omega$, $\tau \subseteq \tau_{\mathcal{E}}$. Consequently, $\mathrm{cl}_{\mathcal{E}}(A) \subseteq \mathrm{cl}_\tau(A)$ for all $A \in \Omega$. Since $\mathcal{E}$ is an essential closure, $\mathcal{E}(A) \subseteq \mathrm{cl}_{\mathcal{E}}(A) \subseteq \mathrm{cl}_\tau(A)$ for all $A \in \Omega$. Moreover, $\mathcal{E}(A)$ is closed in (X, τ) for all $A \in \Omega$ since τ is generated by $\{\mathcal{E}(A)^c\}_{A\in\Omega}$. Hence τ is a topology satisfying the properties in Lemma 14, which implies that $\tau_{\mathcal{E}} \subseteq \tau$. Thus the two topologies coincide. $\square$

Remark. Let $A \mapsto \overline{A}$ be a topological closure operator, hence an essential closure operator. One can see that the topology induced by a topological closure operator $A \mapsto \overline{A}$, as an essential closure operator, coincides with the topology induced by $A \mapsto \overline{A}$ as a topological closure operator.

Given an essential closure operator $\mathcal{E}$ on (X, Ω), any topology τ containing $\tau_{\mathcal{E}}$ with the property that $\mathcal{E}(A) \subseteq \mathrm{cl}_\tau(A)$ for all $A \in \Omega$ is said to be *compatible* with $\mathcal{E}$. Notice that if τ is a compatible topology, then $(X, \tau, \Omega, \mathcal{E})$ is an essential closure space.

On a given essential closure operator space, there can be several compatible topologies, among which the topology $\tau_{\mathcal{E}}$ is the smallest. The induced topology $\tau_{\mathcal{E}}$ is called the *canonical topology.* The following result gives a characterization of the canonical topologies.

Theorem 16. *Let $\mathcal{E}$ be an essential closure on (X, τ, Ω). Then τ is the canonical topology $\tau_{\mathcal{E}}$ if and only if there exists a subbase of τ whose elements are of the form $\mathcal{E}(A)^c$ where $A \in \Omega$.*

Proof. If τ is canonical, then it is generated by the collection $\{\mathcal{E}(A)^c\}_{A\in\Omega}$. On the other hand, assume that τ is generated by a subcollection of $\{\mathcal{E}(A)^c\}_{A\in\Omega}$. Then $\tau \subseteq \tau_{\mathcal{E}}$. Moreover, for each $A \in \Omega$, $\mathcal{E}(A)$ is closed with respect to τ since $\mathcal{E}$ is an essential closure. Thus $\tau_{\mathcal{E}} \subseteq \tau$. $\square$

4 Submeasure closures

In this section, we construct concrete examples of essential closures and demonstrate some of their applications, especially in the study of the supports of measures. Let us remark that, even though we can avoid the argument of the Axiom of Choice in all of our proofs, many (if not most) existing concepts and results used below are so relevant to the axiom that it cannot be completely disregarded. An example is the countability of the union of a countable collection of countable sets, which is required in defining Lebesgue measures. As such, in the sequel, we additionally assume the Axiom of Choice, hence ZFC (the Zermelo-Fraenkel system of axioms with the Axiom of Choice).

4.1 Definition and properties

Definition 10. Let Ω be a σ-algebra over X. A *submeasure* on (X, Ω) is a set function $\mu\colon \Omega \to [0, \infty]$ satisfying

1. $\mu(\emptyset) = 0$;
2. $\mu(A) \leq \mu(B)$ for any $A, B \in \Omega$ such that $A \subseteq B$; and
3. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \mu(A_i)$ for any $A_1, A_2, \ldots \in \Omega$.

Remark. Let us note a few facts about our submeasures.

1. Our submeasures are defined on σ-algebras and are countably subadditive, unlike classical submeasures which are defined on algebras and are finitely subadditive.
2. Every submeasure on a σ-algebra can be extended, perhaps not uniquely, to an outer measure. In other words, every submeasure is a restriction of some outer measure. The reason we do not simply call it an outer measure or a restriction of an outer measure is for convenience in stating our results.

Definition 11. A topological submeasure space is a quadruple (X, τ, Ω, μ) where (X, τ, Ω) is a topological measurable space and μ is a submeasure on (X, Ω).

Definition 12. Let (X, τ, Ω, μ) be a topological submeasure space. For any measurable set $A \in \Omega$, we say that $x \in \overline{A}^{\mu}$ if $\mu(G \cap A) > 0$ for every $G \in \mathfrak{N}(x)$. The set $\overline{A}^{\mu}$ is called the *μ-closure* of A.

Remark. The following are immediate results from the definition.

1. If $\mu(A) = 0$, then $\overline{A}^{\mu} = \emptyset$.
2. Every submeasure closure is weakly strong.
3. If μ and ν are submeasures on a common measurable space such that $\mu \ll \nu$, then $\overline{A}^{\mu} \subseteq \overline{A}^{\nu}$ for every measurable set A.

Example 7. Recall the definition of the submeasure closure $(A \mapsto \overline{A}^{e})$ on the real line defined in [9]. It is an essential closure (with respect to the standard topology τ_s) on the Lebesgue σ-algebra $\mathfrak{L}(\mathbb{R})$. Now, we temporarily remove the topology and view the essential closure as an essential closure operator on $\mathfrak{L}(\mathbb{R})$. We will show that the canonical topology (i.e., the induced topology $\tau_{A\mapsto\overline{A}^{e}}$ in Theorem 15) is, in fact, the given standard topology.

Recall that the collection of non-empty open intervals forms a subbase for the standard topology. Moreover, notice that each non-empty open interval (a, b) can be written as $\left(\overline{A}^{e}\right)^{c}$ where $A = (-\infty, a] \cup [b, \infty) \in \mathfrak{L}(\mathbb{R})$. Hence by Theorem 16, the canonical topology is the standard topology.

Example 8. Let $X = (-\infty, 0]$ and τ be the topology on X generated by the collection of singletons $\{x\}$ where $x \in (-\infty, 0)$. Notice that $\{x\}$ is a neighborhood of x for every point $x \in (-\infty, 0)$. However, the only neighborhood of 0 is X.

Define a measure μ on $\mathcal{P}(X)$ by setting $\mu(A) = 0$ if A is countable and $\mu(A) = \infty$ otherwise. Observe that $\overline{X}^{\mu} = \{0\}$ while $\overline{\overline{X}^{\mu}}^{\mu} = \emptyset$. Thus the μ-closure is not idempotent. Hence it is not an essential closure.

According to Example 8, a submeasure closure need not be idempotent. Nevertheless, it is easy to verify that every submeasure closure satisfies the other three properties in Postulate 1. As a result,

$$\overline{\overline{A}^{\mu}}^{\mu} \subseteq \overline{\overline{A}^{\mu}} = \overline{A}^{\mu}$$

for every μ-measurable set A.

Two sufficient conditions for a submeasure closure to be an essential closure are given in the following result.

Theorem 17. *Assume that (X, τ, Ω, μ) is either a hereditarily Lindelöf submeasure space or an inner regular measure space. Then $A \mapsto \overline{A}^{\mu}$ is a strong essential closure.*

Proof. Let A be a measurable set and $\mathcal{G}$ be the collection of the open sets G such that $\mu(G \cap (A \setminus \overline{A}^{\mu})) = 0$. Observe that $x \in A \setminus \overline{A}^{\mu}$ implies $x \notin \overline{A \setminus \overline{A}^{\mu}}^{\mu}$.

If (X, τ, Ω, μ) is a hereditarily Lindelöf submeasure space, then for each $x \in A \setminus \overline{A}^{\mu}$, there is $G \in \mathfrak{N}(x)$ such that $\mu(G \cap (A \setminus \overline{A}^{\mu})) = 0$. Thus $\mathcal{G}$ covers $A \setminus \overline{A}^{\mu}$. Hence there exists a countable subcover $\{G_1, G_2, \dots\}$ of $\mathcal{G}$. Consequently,

$$\mu(A \setminus \overline{A}^{\mu}) \leq \sum_{i=1}^{\infty} \mu\left(G_i \cap (A \setminus \overline{A}^{\mu})\right) = 0.$$

Therefore, $\overline{A \setminus \overline{A}^{\mu}}^{\mu} = \emptyset$.

If (X, τ, Ω, μ) is an inner regular measure space, consider any compact set $K \subseteq A \setminus \overline{A}^{\mu}$. Then $x \in K$ implies $x \notin \overline{A \setminus \overline{A}^{\mu}}^{\mu}$. Therefore, for each $x \in K$, there is $G \in \mathfrak{N}(x)$ such that $\mu(G \cap (A \setminus \overline{A}^{\mu})) = 0$. Thus $\mathcal{G}$ covers K. Hence there exists a finite subcover $\{G_1, \ldots, G_n\}$ of $\mathcal{G}$. Consequently,

$$\mu(K) \leq \sum_{i=1}^{n} \mu(G_i \cap K) \leq \sum_{i=1}^{n} \mu\left(G_i \cap (A \setminus \overline{A}^{\mu})\right) = 0.$$

Therefore, $\mu(A \setminus \overline{A}^{\mu}) = 0$ by inner regularity. Thus $\overline{A \setminus \overline{A}^{\mu}}^{\mu} = \emptyset$. □

Observe that the two parts of the proof of Theorem 17 are very similar. One proof uses a countable subcover while the other uses a finite subcover. So in the sequel, if there are twin results like these, we shall omit the proof for the case of inner regular measure spaces.

Theorem 18. *Assume that (X, τ, Ω, μ) is either a Lindelöf submeasure space or an inner regular measure space. Then $\overline{A}^{\mu} = \emptyset$ if and only if $\mu(A) = 0$.*

Proof. Assume that (X, τ, Ω, μ) is a Lindelöf submeasure space and $\overline{A}^{\mu} = \emptyset$. Let $\mathcal{G}$ be the collection of the open sets G such that $\mu(G \cap A) = 0$. Since $\overline{A}^{\mu} = \emptyset$, $\mathcal{G}$ covers X. Thus there is a countable subcover $\{G_1, G_2, \ldots\}$ of $\mathcal{G}$. Therefore,

$$\mu(A) \leq \sum_{i=1}^{\infty} \mu(G_i \cap A) = 0.$$

The converse is an immediate result from the definition of submeasure closures. The case of inner regular measure spaces can be proved similarly. □

Corollary 19. *Assume that (X, τ, Ω, μ) is either a Lindelöf submeasure space or an inner regular measure space.*

1. *If $A \mapsto \overline{A}^{\mu}$ is an essential closure, then it is σ-non-essential.*
2. *If $A \mapsto \overline{A}^{\mu}$ is a strong essential closure, then $\mu(\overline{A}^{\mu}) \geq \mu(A)$ for every measurable set A.*

Proof. 1. Let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of non-essential sets. By Theorem 18, $\mu(A_i) = 0$. Consequently, $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 0$. Again, by Theorem 18, $\bigcup_{i=1}^{\infty} A_i$ is a non-essential set.

2. Observe that

$$\begin{aligned}\mu(A) &\leq \mu(A \cap \overline{A}^{\mu}) + \mu(A \setminus \overline{A}^{\mu}) \\ &= \mu(A \cap \overline{A}^{\mu}) \\ &\leq \mu(\overline{A}^{\mu}).\end{aligned}$$

This completes the proof. □

The following result gives a characterization of the σ-non-essential strong essential closures on a hereditarily Lindelöf measurable space.

Theorem 20. *Assume that (X, τ, Ω) is a hereditarily Lindelöf measurable space. Then an essential closure $\mathcal{E}$ on Ω is strong and σ-non-essential if and only if it is a submeasure closure on (X, τ, Ω).*

Proof. Since the case $\mathcal{E}(X) = \emptyset$ is trivial, assume that $\mathcal{E}(X) \neq \emptyset$. Suppose $\mathcal{E}$ is a σ-non-essential strong essential closure on Ω. Define $\mu\colon \Omega \to [0, \infty]$ by $\mu(A) = 0$ if $\mathcal{E}(A) = \emptyset$; otherwise, $\mu(A) = 1$. We show that μ is a submeasure on (X, Ω). Let $\{A_i\}_{i=1}^{\infty}$ be a countable collection of measurable sets.

- Since $\emptyset$ is non-essential, $\mu(\emptyset) = 0$.
- Suppose $A_1 \subseteq A_2$. Then $\mathcal{E}(A_1) \subseteq \mathcal{E}(A_2)$. If A_2 is non-essential, then A_1 is also non-essential. Hence $\mu(A_1) = 0 = \mu(A_2)$. If A_2 is essential, then $\mu(A_1) \leq 1 = \mu(A_2)$.
- If there is an essential set A_j in $\{A_i\}$ then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \leq 1 \leq \sum_{i=1}^{\infty} \mu(A_i)$. If every A_i is non-essential, then $\bigcup_{i=1}^{\infty} A_i$ is also non-essential. So

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 0 = \sum_{i=1}^{\infty} \mu(A_i).$$

Therefore, μ is a submeasure on (X, Ω) and the μ-closure on (X, τ, Ω) is a σ-non-essential strong essential closure by Theorem 17 and Corollary 19(1). Moreover, the fact that $\mathcal{E}$ and the μ-closure coincide follows directly from Corollary 3 and Theorem 18. Finally, the converse follows from Theorem 17 and Corollary 19(1). □

A submeasure is said to be *trivial* if the space is of submeasure zero and is said to be *normalized* if the space is of submeasure one.

Remark. An essential closure induces a normalized submeasure if the space is essential. Otherwise, it induces the trivial submeasure.

Example 9. Consider a μ_1-closure and a μ_2-closure on a common topological measurable space. One can verify that the set function $\mathcal{E}$ defined as

$$\mathcal{E}(A) = \overline{A}^{\mu_1} \cup \overline{A}^{\mu_2}$$

is, in fact, the $(\mu_1 + \mu_2)$-closure. Moreover, if both the μ_1-closure and the μ_2-closure are essential closures, then so is the $(\mu_1 + \mu_2)$-closure.

Example 10. If a μ-closure is a strong essential closure, then $A \setminus \overline{A}^{\mu}$ is a non-essential set. However, $\overline{A}^{\mu} \setminus A$ can be an essential set. For example, take $\mu = \lambda_1$, the 1-dimensional Lebesgue measure on $[0,1]$, and $A = [0,1] \setminus C$, where C is a positive Lebesgue measure Cantor set on $[0,1]$. Then for each $x \in [0,1]$ and $G \in \mathfrak{N}(x)$, $G \cap A$ contains a non-empty open interval. Hence $\lambda_1(G \cap A) > 0$. Therefore, $\overline{A}^{\lambda_1} = [0,1]$. As a result,

$$\overline{A}^{\lambda_1} \setminus A = [0,1] \setminus A = C,$$

which is of positive Lebesgue measure.

4.2 Applications

In this section, we demonstrate some applications of submeasure closures, especially the study of the supports of measures. An essential closure can be viewed as a tool to eliminate the non-essential part of a set. In the case of an essential closure defined via a measure, one can expect that eliminating the non-essential part of the space should give the support of that measure.

4.2.1 The supports of measures

The support of a submeasure is defined analogously to the definition of the support of a measure.

Theorem 21. *Let μ be a submeasure on (X, τ, Ω). Then*

$$\operatorname{supp} \mu = \overline{A}^{\mu}$$

for any measurable set A such that $\mu(A^c) = 0$. In particular, if the μ-closure is an essential closure, then $\operatorname{supp} \mu$ *is μ-essentially closed.*

Proof. If $x \notin \operatorname{supp} \mu$, then there exists $G \in \mathfrak{N}(x)$ such that $\mu(G) = 0$. Thus $\mu(G \cap A) = 0$ for any measurable set A. Hence $x \notin \overline{A}^{\mu}$. Conversely, if $x \notin \overline{A}^{\mu}$ then there exists $G \in \mathfrak{N}(x)$ such that $\mu(G \cap A) = 0$. Since $\mu(A^c) = 0$, we have $\mu(G) \leq \mu(G \cap A) + \mu(G \cap A^c) = 0$. Therefore, $x \notin \operatorname{supp} \mu$. $\square$

Theorem 22. *Let (X, τ, Ω, μ) be a hereditarily Lindelöf measure space. Then a set $A \in \Omega$ is μ-essentially closed if and only if there is an absolutely continuous measure $\nu \ll \mu$ such that* $\operatorname{supp}\nu = A$.

PROOF. For each measurable set B, define $\nu(B) = \mu(A \cap B)$. Clearly, $\nu \ll \mu$. It is left to show that $\operatorname{supp}\nu = A$. First of all, observe that A is closed and $\nu(A^c) = 0$. Hence $\operatorname{supp}\nu \subseteq A$. We prove the opposite inclusion by first noting that

$$\mu((\operatorname{supp}\nu)^c \cap A) = \nu((\operatorname{supp}\nu)^c) = 0.$$

Suppose $(\operatorname{supp}\nu)^c \cap A \neq \emptyset$. Let $x \in (\operatorname{supp}\nu)^c \cap A$. Then $(\operatorname{supp}\nu)^c \in \mathfrak{N}(x)$ and $x \in A = \overline{A}^{\mu}$. Therefore, $\mu((\operatorname{supp}\nu)^c \cap A) > 0$, a contradiction. Hence $(\operatorname{supp}\nu)^c \cap A = \emptyset$. In other words, $A \subseteq \operatorname{supp}\nu$.

Conversely, it suffices to show that $\operatorname{supp}\nu \subseteq \overline{\operatorname{supp}\nu}^{\mu}$. If $x \notin \overline{\operatorname{supp}\nu}^{\mu}$, then there exists $G \in \mathfrak{N}(x)$, $\mu(G \cap \operatorname{supp}\nu) = 0$. By the absolute continuity, we have $\nu(G) = \nu(G \cap \operatorname{supp}\nu) = 0$. So $x \notin \operatorname{supp}\nu$. □

The following result can be proved similarly. Notice the difference in the inner regularity of the measure ν.

Corollary 23. *Let (X, τ, Ω, μ) be a topological inner regular measure space. Then a set $A \in \Omega$ is μ-essentially closed if and only if there is an inner regular measure $\nu \ll \mu$ such that* $\operatorname{supp}\nu = A$.

Theorem 24. *Let (X, τ, Ω, μ) be a hereditarily Lindelöf measure space where μ is σ-finite. For any σ-finite measure η on (X, τ, Ω) with Lebesgue decomposition $\eta = \eta_a + \eta_s$ with respect to μ, if $\mu(\operatorname{supp}\eta_s) = 0$, then* $\operatorname{supp}\eta_a = \overline{\operatorname{supp}\eta}^{\mu}$.

PROOF. It is straightforward to verify that $\operatorname{supp}\eta = \operatorname{supp}\eta_a \cup \operatorname{supp}\eta_s$. If $x \notin \operatorname{supp}\eta_a = \overline{\operatorname{supp}\eta_a}^{\mu}$ (since $\eta_a \ll \mu$, $\operatorname{supp}\eta_a$ is μ-essentially closed), then there exists $G \in \mathfrak{N}(x)$ such that $\mu(G \cap \operatorname{supp}\eta_a) = 0$. Thus

$$\mu(G \cap \operatorname{supp}\eta) \leq \mu(G \cap \operatorname{supp}\eta_a) + \mu(G \cap \operatorname{supp}\eta_s) = 0.$$

Hence $x \notin \overline{\operatorname{supp}\eta}^{\mu}$. Conversely, if $x \notin \overline{\operatorname{supp}\eta}^{\mu}$, then there exists $G \in \mathfrak{N}(x)$ such that $\mu(G \cap \operatorname{supp}\eta) = 0$. Therefore,

$$\mu(G \cap \operatorname{supp}\eta_a) \leq \mu(G \cap \operatorname{supp}\eta) = 0.$$

Hence $x \notin \overline{\operatorname{supp}\eta_a}^{\mu} = \operatorname{supp}\eta_a$. □

4.2.2 The essential supports of functions

In this section, we introduce the concept of the essential supports of functions, which is partly motivated by the study of the supports of Radon-Nikodym derivatives; see Chapter 23 in Fremlin's book [8]. We are particularly interested in the study of Radon-Nikodym derivatives via techniques from geometric measure theory.

For any pair of Radon measures (see Definition 1.5 and Corollary 1.11 in Mattila's book [12]) ν and μ on $\mathbb{R}^n$, equipped with a σ-algebra containing the Borel sets, such that $\nu \ll \mu$, it was shown in [12, Theorem 2.12] that the function

$$D_{\nu,\mu}(x) = \lim_{\epsilon \to 0^+} \frac{\nu(B(x,\epsilon))}{\mu(B(x,\epsilon))} \tag{3}$$

is defined μ-almost everywhere on $\mathbb{R}^n$ and coincides μ-almost everywhere with the Radon-Nikodym derivative of ν with respect to μ.

Similarly, for any locally finite measure ν defined on the Borel σ-algebra over $\mathbb{R}^n$ such that $\nu \ll \lambda_n$, it was shown in [2, Theorem 2.3.8] that the function D_{ν,λ_n} is defined Lebesgue almost everywhere on $\mathbb{R}^n$ and coincides Lebesgue almost everywhere with the Radon-Nikodym derivative of ν with respect to λ_n.

Definition 13. Let ν and μ be σ-finite measures on a metric measurable space (X, d, Ω). We say that ν is *differentiable* with respect to μ if $\nu \ll \mu$ and $D_{\nu,\mu}$ defined in (3) exists μ-almost everywhere and coincides μ-almost everywhere with the Radon-Nikodym derivative of ν with respect to μ.

Proposition 25. *Let ν and μ be σ-finite measures on a metric measurable space such that ν is differentiable with respect to μ. Then* $\operatorname{supp} D_{\nu,\mu} = \operatorname{supp} \nu$.

Proof. If $x \notin \operatorname{supp} \nu$, then there exists $\epsilon > 0$ such that $\nu(B(x,\epsilon)) = 0$. Hence $D_{\nu,\mu}(x) = 0$. So $\{x \colon D_{\nu,\mu}(x) \neq 0\} \subseteq \operatorname{supp} \nu$. Therefore, $\operatorname{supp} D_{\nu,\mu} \subseteq \operatorname{supp} \nu$. Conversely, if $x \notin \operatorname{supp} D_{\nu,\mu}$, then there is $G \in \mathfrak{N}(x)$ such that $D_{\nu,\mu} \equiv 0$ on G. Observe that $\nu(G) = \int_G D_{\nu,\mu}\, d\mu = 0$. Thus $x \notin \operatorname{supp} \nu$. Hence $\operatorname{supp} \nu \subseteq \operatorname{supp} D_{\nu,\mu}$. Therefore, $\operatorname{supp} D_{\nu,\mu} = \operatorname{supp} \nu$. $\square$

Radon-Nikodym derivatives are unique up to a set of measure zero. As a result, the concept of topological supports fails to detect the essential parts of such functions. We demonstrate an extreme case in the following example.

Example 11. Consider the trivial measure $\nu \equiv 0$ on the Lebesgue σ-algebra $\mathfrak{L}(\mathbb{R})$, which is absolutely continuous with respect to the Lebesgue measure. Observe that both $f \equiv 0$ and $g = \chi_{\mathbb{Q}}$ are the Radon-Nikodym derivatives of ν with respect to λ_1. However, $\operatorname{supp} f = \emptyset$ while $\operatorname{supp} g = \mathbb{R}$.

In the above example, even though $\mathbb{Q}$ is negligible in the sense that it has Lebesgue measure zero, it is dense in $\mathbb{R}$.

Definition 14. Let f be an extended real-valued measurable function on a topological submeasure space (X, τ, Ω, μ). Define the *essential support* of f with respect to μ by $\operatorname{ess\,supp}_\mu f = \overline{\{x \in X \colon f(x) \neq 0\}}^{\mu}$.

Remark. In Ondreját's work [15], the essential support of a function f on a set $D \subseteq \mathbb{R}^n$ is defined to be the intersection of all closed subsets F in D such that $f = 0$ Lebesgue almost everywhere on the complement of F. It is straightforward to verify that this existing concept agrees with Definition 14.

Similarly to the concept of almost everywhere for measures, for the case of submeasures, we say that a property holds *almost everywhere* if the set of elements for which the property does not hold is a subset of a submeasure zero set.

Proposition 26. *Let f and g be extended real-valued measurable functions on a topological submeasure space (X, τ, Ω, μ). If f and g are equal μ-almost everywhere, then* $\operatorname{ess\,supp}_\mu f = \operatorname{ess\,supp}_\mu g$.

Proof. Since $f = g$ μ-almost everywhere, we have

$$\mu(\{x \in X \colon g(x) \neq 0\}) = \mu(\{x \in X \colon f(x) \neq 0\}),$$

which implies that the essential supports of f and g coincide. $\square$

Theorem 27. *Assume that (X, τ, Ω, μ) is either a hereditarily Lindelöf submeasure space or an inner regular measure space. For each extended real-valued measurable function f, let $[f]_\mu$ denote the class of extended real-valued measurable functions on X which are equal to f μ-almost everywhere. Then there exists $f_0 \in [f]_\mu$ such that*

$$\operatorname{supp} f_0 = \operatorname{ess\,supp}_\mu f$$

which is μ-essentially closed.

Proof. Define f_0 to be the function that coincides with f on $\operatorname{ess\,supp}_\mu f$ and vanishes elsewhere. Since the μ-closure is a strong essential closure,

$$\begin{aligned}\{x \in X \,:\, f(x) \neq f_0(x)\} &= \{x \in X \,:\, f(x) \neq 0\} \setminus \operatorname{ess\,supp}_\mu f \\ &= \{x \in X \,:\, f(x) \neq 0\} \setminus \overline{\{x \in X \,:\, f(x) \neq 0\}}^{\mu}\end{aligned}$$

is μ-non-essential. Hence $\mu(\{x \in X \,:\, f(x) \neq f_0(x)\}) = 0$ by Theorem 18. Thus f and f_0 are equal μ-almost everywhere; i.e., $f_0 \in [f]_\mu$. By Proposition 26, we have $\operatorname{ess\,supp}_\mu f = \operatorname{ess\,supp}_\mu f_0$.

If $f_0(x) \neq 0$, then $x \in \operatorname{ess\,supp}_\mu f$ by the construction. Therefore, we have that $\{x \in X\colon f_0(x) \neq 0\} \subseteq \operatorname{ess\,supp}_\mu f$. Hence

$$\operatorname{supp} f_0 \subseteq \operatorname{ess\,supp}_\mu f = \operatorname{ess\,supp}_\mu f_0 \subseteq \operatorname{supp} f_0.$$

Thus $\operatorname{supp} f_0 = \operatorname{ess\,supp}_\mu f_0 = \operatorname{ess\,supp}_\mu f$. As a consequence, $\operatorname{supp} f_0$ is μ-essentially closed. $\square$

Proposition 28. *Let ν and μ be σ-finite measures on (X, τ, Ω) with $\nu \ll \mu$, and let $\dfrac{d\nu}{d\mu}$ denote the Radon-Nikodym derivative. Then* $\operatorname{ess\,supp}_\mu \dfrac{d\nu}{d\mu} = \operatorname{supp} \nu$.

Proof. Let f denote $\frac{d\nu}{d\mu}$. If $x \notin \operatorname{supp}\nu$, then there exists $G \in \mathfrak{N}(x)$ such that $\nu(G) = 0$. Thus $f = 0$ μ-almost everywhere on G. Therefore, we have that $\mu(G \cap \{x \in X\colon f(x) \neq 0\}) = 0$. Hence $x \notin \operatorname{ess\,supp}_\mu f$. Conversely, if $x \notin \operatorname{ess\,supp}_\mu f$, then $x \notin \overline{\{x \in X\colon f(x) \neq 0\}}^{\mu}$. Therefore, there exists $G \in \mathfrak{N}(x)$ such that $\mu(G \cap \{x \in X\colon f(x) \neq 0\}) = 0$. Thus $f = 0$ μ-almost everywhere on G. Hence $\nu(G) = 0$. So $x \notin \operatorname{supp}\nu$. $\square$

Corollary 29. *Let ν and μ be σ-finite measures on a metric measurable space such that ν is differentiable with respect to μ. Then* $\operatorname{ess\,supp}_\mu D_{\nu,\mu} = \operatorname{supp} D_{\nu,\mu}$.

Proof. This follows directly from Propositions 25 and 28. $\square$

Example 12. There exists an absolutely continuous measure $\nu \ll \mu$ with full support such that μ is not absolutely continuous with respect to ν. To see this, let μ be the 1-dimensional Lebesgue measure on $[0,1]$ and let ν be a measure on $[0,1]$ defined, for each Lebesgue measurable set $B \subseteq [0,1]$, by $\nu(B) = \lambda_1(B \cap A^c)$, where A is a positive Lebesgue measure Cantor set on $[0,1]$. Obviously, $\nu \ll \lambda_1$ by construction. Moreover, by Proposition 28,

$$\operatorname{supp}\nu = \operatorname{ess\,supp}_{\lambda_1} \chi_{A^c} = \overline{A^c}^{\lambda_1} = [0,1].$$

Therefore, ν has full support. However, $\nu(A) = 0$ while $\lambda_1(A) > 0$.

Example 13. Let (X, τ, Ω, μ) be a hereditarily Lindelöf measure space and let f be an extended real-valued measurable function. We already know that

$$\int_X f\ d\mu = \int_{\operatorname{supp} f} f\ d\mu.$$

Let $\mathcal{G}$ be the collection of the open sets G such that $\mu(G\cap\{f\neq 0\})=0$. Since for each $x\notin \text{ess supp}_\mu f$, there is $G\in\mathfrak{N}(x)$ with $\mu(G\cap\{f\neq 0\})=0$, $\mathcal{G}$ covers $(\text{ess supp}_\mu f)^c$. Thus there is a countable subcover $\{G_1,G_2,\dots\}$ of $\mathcal{G}$.

By the countable additivity of measures, it is straightforward to show that $\mu((\text{ess supp}_\mu f)^c\cap\{f\neq 0\})=0$. Thus

$$\int_X f\ d\mu=\int_{\text{ess supp}_\mu f} f\ d\mu=\int_{\text{supp} f_0} f_0\ d\mu,$$

where f_0 is a representative of the class $[f]_\mu$ in Theorem 27. Also note that

$$\operatorname{supp} f_0=\text{ess supp}_\mu f\subseteq \operatorname{supp} f.$$

In this case, we see that f_0 is indeed a good representative of the class $[f]_\mu$.

4.2.3 Local Hausdorff dimension

In the sequel, let $\mathcal{H}^{\text{s}}$ denote the s-dimensional Hausdorff measure. More details on the Hausdorff measures and Hausdorff dimension $\dim_{\text{H}}$ can be found for example in Falconer's book [5] and Fremlin's book [8].

Definition 15. Let (X,d) be a metric space and τ_d denote the topology induced by the metric d. The *s-Hausdorff closure* is defined to be the submeasure closure on $(X,\tau_d,\mathcal{P}(X))$ induced by $\mathcal{H}^{\text{s}}$.

Lemma 30. *If a set A is s-Hausdorff essentially closed, then it has local Hausdorff dimension at least s.*

PROOF. Suppose there exist $x\in A$ and $G\in\mathfrak{N}(x)$ such that $\dim_{\text{H}}(G\cap A)<s$, where $\dim_{\text{H}}$ denotes the Hausdorff dimension. See [5]. Then $\mathcal{H}^{\text{s}}(G\cap A)=0$, contradicting the fact that $x\in A=\overline{A}^{\mathcal{H}^{\text{s}}}$. □

Theorem 31. *Let ν be an n-stochastic measure on $[0,1]^n$. Then $\operatorname{supp}\nu$ is 1-Hausdorff essentially closed. In particular, by Lemma 30, $\operatorname{supp}\nu$ has local Hausdorff dimension at least one.*

PROOF. It suffices to show that $\operatorname{supp}\nu\subseteq\overline{\operatorname{supp}\nu}^{\mathcal{H}^1}$. If $x\notin\overline{\operatorname{supp}\nu}^{\mathcal{H}^1}$, then there exists $G\in\mathfrak{N}(x)$, $\mathcal{H}^1(G\cap\operatorname{supp}\nu)=0$. Note that $\nu(G)=\nu(G\cap\operatorname{supp}\nu)$. Suppose $\nu(G\cap\operatorname{supp}\nu)>0$. Then

$$\mathcal{H}^1(\pi_1(G\cap\operatorname{supp}\nu))=\lambda_1(\pi_1(G\cap\operatorname{supp}\nu))>0,$$

where π_1 denotes the orthogonal projection onto the first variable. Thus $\mathcal{H}^1(G\cap\operatorname{supp}\nu)>0$, a contradiction. So $\nu(G)=\nu(G\cap\operatorname{supp}\nu)=0$, which implies $x\notin\operatorname{supp}\nu$. Therefore, $\operatorname{supp}\nu=\overline{\operatorname{supp}\nu}^{\mathcal{H}^1}$. Hence $\operatorname{supp}\nu$ is 1-Hausdorff essentially closed. □

It is well known that there is a one-to-one correspondence between the collection of n-stochastic measures and the collection of n-copulas. More information on n-copulas can be found in Nelsen's book [14].

Example 14. In [6, Theorem 1], Fredricks et al. show that for each $s \in (1, 2)$, there is a copula with a fractal support of Hausdorff dimension s. Also, there are copulas with supports of Hausdorff dimension 1 and 2, examples of which include the Fréchet-Hoeffding bounds and the independence copula, respectively. Moreover, Theorem 31 implies that the support of a copula is of Hausdorff dimension at least 1. Together with the result of Fredricks et al., it follows that the supports of copulas are of Hausdorff dimension at least 1 and for each possible value $s \in [1, 2]$, there is a copula whose support is of Hausdorff dimension s.

5 Existing and related concepts

In this section, we discuss various concepts that are related to the concept of essential closures. Most of them are related to measures and submeasures as expected.

5.1 Lebesgue closure

Recall the definition of the essential closure on $\mathbb{R}$ introduced in [9] and called by us *Lebesgue closure*. It is easy to see that the Lebesgue closure coincides with the λ_1-closure defined in the previous section. Also recall from [9] the definition of the Lebesgue closure defined on S^1, the unit circle with center at the origin in $\mathbb{R}^2$.

According to [8, Theorem 265E], the pushforward Lebesgue measure on S^1 through the canonical map ($\theta \mapsto e^{i\theta}$) coincides with the Hausdorff measure $\mathcal{H}^1$ on S^1. As a result, the Lebesgue closure on S^1 coincides with the $\mathcal{H}^1$-closure on S^1.

5.2 Lebesgue density closures

To avoid confusion, the essential closures cl^* in Buczolich and Pfeffer's work [4] and in Fremlin's book [7], defined for each Lebesgue measurable set $A \subseteq \mathbb{R}^n$ by

$$\mathrm{cl}^* A = \left\{ x \in \mathbb{R}^n \colon \limsup_{\epsilon \to 0^+} \frac{\lambda_n(B(x, \epsilon) \cap A)}{\lambda_n(B(x, \epsilon))} > 0 \right\},$$

will be called *Lebesgue density closures*. Note that, with respect to the standard topology on $\mathbb{R}^n$, the Lebesgue density closure fails to satisfy at least the first property of essential closures in Postulate 1.

For each λ_n-density closure cl^* on the Lebesgue σ-algebra $\mathfrak{L}(\mathbb{R}^n)$, we define the *modified λ_n-density closure* of $A \in \mathfrak{L}(\mathbb{R}^n)$ by $\mathcal{E}(A) = \overline{\mathrm{cl}^* A}$. As a consequence of taking the topological closure of $\mathrm{cl}^* A$, $\mathcal{E}$ is forced to satisfy the first property of essential closures. Surprisingly, not only that $\mathcal{E}$ is an essential closure, but it can also be shown that $\mathcal{E}$ coincides with the λ_n-closure defined on $\mathfrak{L}(\mathbb{R}^n)$.

Firstly, we show that the modified λ_n-density closure and the λ_n-closure coincide on the Borel σ-algebra $\mathfrak{B}(\mathbb{R}^n)$. Let $A \subseteq \mathbb{R}^n$ be Borel measurable. For each Borel measurable set $B \subseteq \mathbb{R}^n$, define $\lambda_A(B) = \lambda_n(B \cap A)$. It is clear that λ_A is σ-finite and $\lambda_A \ll \lambda_n$ on the Borel σ-algebra. According to Theorem 2.3.8 in Ash's book [2], λ_A is differentiable with respect to λ_n. As a result,

$$D_{\lambda_A,\lambda_n}(x) = \lim_{\epsilon \to 0^+} \frac{\lambda_A(B(x,\epsilon))}{\lambda_n(B(x,\epsilon))} = \limsup_{\epsilon \to 0^+} \frac{\lambda_n(B(x,\epsilon) \cap A)}{\lambda_n(B(x,\epsilon))}$$

defines the Radon-Nikodym derivative of λ_A with respect to λ_n. By Proposition 25 and Theorem 21, we have

$$\mathcal{E}(A) = \overline{\mathrm{cl}^* A} = \operatorname{supp} D_{\lambda_A,\lambda_n} = \operatorname{supp} \lambda_A = \overline{A}^{\lambda_A}.$$

Moreover, it is straightforward to verify that $\overline{A}^{\lambda_A} = \overline{A}^{\lambda_n}$. Hence $\mathcal{E}(A) = \overline{A}^{\lambda_n}$ for each Borel measurable set $A \subseteq \mathbb{R}^n$.

Finally, we extend the result to the Lebesgue σ-algebra $\mathfrak{L}(\mathbb{R}^n)$. Let $A \subseteq \mathbb{R}^n$ be Lebesgue measurable. There is a Borel measurable set $B \subseteq \mathbb{R}^n$ such that $A \subseteq B$ and $\lambda_n(B \setminus A) = 0$. According to Lemma 475C in Fremlin's book [7], cl^* is distributive over finite unions and $\mathrm{cl}^*(E) = \emptyset$ if $\lambda_n(E) = 0$. As a result,

$$\mathrm{cl}^*(B) = \mathrm{cl}^*(A) \cup \mathrm{cl}^*(B \setminus A) = \mathrm{cl}^*(A).$$

Similarly, $\overline{A}^{\lambda_n} = \overline{B}^{\lambda_n}$. Thus $\mathcal{E}(A) = \overline{\mathrm{cl}^*(A)} = \overline{\mathrm{cl}^*(B)} = \mathcal{E}(B) = \overline{B}^{\lambda_n} = \overline{A}^{\lambda_n}$ for each Lebesgue measurable set $A \subseteq \mathbb{R}^n$.

5.3 Lower density operators

The essential interiors int^* in Buczolich and Pfeffer's work [4] and in Fremlin's book [7] are lower density operators. In general, lower density operators are defined as follows.

Let Ω be a σ-algebra over a set X and $\mathcal{P} \subseteq \Omega$ be a σ-ideal. For $A, B \in \Omega$, we denote $A \sim B$ when the symmetric difference $A \triangle B$ is in the σ-ideal $\mathcal{P}$.

Definition 16 ([11, p. 207])**.** A *lower density operator* on $(X, \Omega, \mathcal{P})$ is a unary operation $\Phi\colon \Omega \to \Omega$ satisfying the following conditions for all $A, B \in \Omega$:

1. If $A \sim B$, then $\Phi(A) = \Phi(B)$;
2. $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$;
3. $\Phi(\emptyset) = \emptyset$ and $\Phi(X) = X$;
4. $A \sim \Phi(A)$.

For more details on lower density operators, see the classical book of Lukeš, Malý and Zajíček [11]. According to Lemma 475C in Fremlin's book [7],

$$\mathrm{cl}^*(A) = \mathrm{int}^*(A^c)^c, \tag{4}$$

for each measurable set A. Motivated by the above relation, we derive a result on the essential closure operators induced by lower density operators.

Theorem 32. *Let Φ and $\mathcal{E}$ be unary operations on a σ-algebra Ω over X satisfying*

$$\mathcal{E}(A) = \Phi(A^c)^c \quad \textit{for all } A \in \Omega. \tag{5}$$

Then Φ is a lower density operator on $(X, \Omega, \mathcal{P})$ if and only if

1. *$\mathcal{E}$ is a σ-non-essential essential closure operator on (X, Ω),*
2. *$A \sim \mathcal{E}(A)$ for all $A \in \Omega$,*
3. *$\mathcal{P} = \mathcal{N}_\Omega(\mathcal{E})$,*
4. *$\mathcal{E}(X) = X$.*

Proof. Assume that Φ is a lower density operator on $(X, \Omega, \mathcal{P})$. By (5), we have

- $\mathcal{E}(\emptyset) = \Phi(X)^c = \emptyset$,
- $\mathcal{E}(A \cup B) = \Phi(A^c \cap B^c)^c = \mathcal{E}(A) \cup \mathcal{E}(B)$ for all $A, B \in \Omega$, and
- $\mathcal{E}(\mathcal{E}(A)) = \Phi(\Phi(A^c))^c = \Phi(A^c)^c = \mathcal{E}(A)$ for all $A \in \Omega$.

Hence $\mathcal{E}$ is an essential closure operator.

For each $A \in \Omega$, $A \sim \Phi(A^c)^c = \mathcal{E}(A)$ because $A^c \sim \Phi(A^c)$. Consequently, if A_n is non-essential for each $n \in \mathbb{N}$, then $\bigcup_{n=1}^\infty A_n \sim \bigcup_{n=1}^\infty \mathcal{E}(A_n) = \emptyset$. Thus $\mathcal{E}(\bigcup_{n=1}^\infty A_n) = \Phi(X)^c = \emptyset$. Hence $\mathcal{E}$ is σ-non-essential.

If $A \in \mathcal{P}$, then $A \sim \emptyset$. Therefore, $\mathcal{E}(A) = \emptyset$. Conversely, if $\mathcal{E}(A) = \emptyset$, then $A \sim \Phi(A^c)^c = \emptyset$, which implies that $A \in \mathcal{P}$. Hence $\mathcal{P} = \mathcal{N}_\Omega(\mathcal{E})$. Finally, $\mathcal{E}(X) = \Phi(\emptyset)^c = X$.

To prove the converse, assume that conditions 1, 2, 3 and 4 hold. Since $\mathcal{E}$ is σ-non-essential, $\mathcal{P}$ is a σ-ideal. Let $A, B \in \Omega$.

- Since $\mathcal{E}(A) = \Phi(A^c)^c$, $\Phi(A) = \mathcal{E}(A^c)^c$.
- If $A \sim B$, then $A \setminus B$ and $B \setminus A$ are non-essential since $\mathcal{P} = \mathcal{N}_\Omega(\mathcal{E})$. Thus $\mathcal{E}(A \setminus B) = \emptyset = \mathcal{E}(B \setminus A)$. Therefore,

  $$\mathcal{E}(A^c) = \mathcal{E}(A^c \cap B^c) \cup \mathcal{E}(B \setminus A) = \mathcal{E}(A^c \cap B^c) \cup \mathcal{E}(A \setminus B) = \mathcal{E}(B^c).$$

  Hence $\Phi(A) = \Phi(B)$.
- That $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$ follows directly from the assumption that $\mathcal{E}(A \cup B) = \mathcal{E}(A) \cup \mathcal{E}(B)$.
- Obviously, $\Phi(\emptyset) = \emptyset$ and $\Phi(X) = X$.
- Since $A^c \sim \mathcal{E}(A^c)$, $\Phi(A) = \mathcal{E}(A^c)^c \sim A$.

Hence Φ is a lower density operator on $(X, \Omega, \mathcal{P})$. $\square$

Corollary 33. *Let Φ be a lower density operator on $(X, \Omega, \mathcal{P})$. Define $\mathcal{E}$ by equation* (5)*. Then $\mathcal{E}$ is an essential closure operator on (X, Ω). Moreover, the induced topology τ_Φ is a compatible topology for $\mathcal{E}$.*

Proof. According to Theorem 32, $\mathcal{E}$ is an essential closure operator on (X, Ω). Recall from [11, Proposition 6.37] that $\mathcal{B}_\Phi = \{A \in \Omega\colon A \subseteq \Phi(A)\}$ is an open base for τ_Φ. To show that τ_Φ is a compatible topology for $\mathcal{E}$, it suffices to show that, with respect to τ_Φ, i) $\mathcal{E}(A)$ is closed and ii) $\mathcal{E}(A) \subseteq \operatorname{cl}(A)$ for all $A \in \Omega$.

Observe that $\Phi(A)$ is open in τ_Φ since $\Phi(A) \in \mathcal{B}_\Phi$ for all $A \in \Omega$. Thus $\mathcal{E}(A)$ is closed in τ_Φ. Moreover, for each $A \in \Omega$,

$$\begin{aligned}
\operatorname{int}(A) &= \bigcup \{O \in \tau_\Phi \colon O \subseteq A\} \\
&= \bigcup \{G \in \Omega\colon G \subseteq \Phi(G) \text{ and } G \subseteq A\} \\
&\subseteq \bigcup \{G \in \Omega\colon G \subseteq \Phi(G) \subseteq \Phi(A)\} \\
&\subseteq \bigcup \{G \in \Omega\colon G \subseteq \Phi(A)\} \\
&\subseteq \Phi(A).
\end{aligned}$$

Consequently, $\mathcal{E}(A) \subseteq \operatorname{cl}(A)$ for each $A \in \Omega$. Therefore, τ_Φ is a compatible topology for $\mathcal{E}$. $\square$

The following corollary is an immediate application. Recall that each cl^* is not an essential closure with respect to the standard topology. However, with a suitable topology, it turns into one.

Corollary 34. *Each Lebesgue density closure* cl^* *is an essential closure on* $(\mathbb{R}^n, \tau_{\mathrm{int}^*}, \mathfrak{L}(\mathbb{R}^n))$.

Proof. This follows from Corollary 33 and equation (4). □

5.4 Stochastic closures

We will call essential closures defined in [16] *stochastic closures* to avoid confusion. It has been verified that these stochastic closures are indeed essential closures. The next question is whether these essential closures are strong and σ-non-essential, and if they are, what are their corresponding submeasures?

For each integer $1 \le d \le n$, define $\mathcal{S}_d \colon \mathcal{P}([0,1]^n) \to [0,\infty]$ as follows:

$$\mathcal{S}_d(A) = \sum_W \lambda_d^*(\pi_W(A))$$

where the sum is taken over all d-dimensional standard subspaces (i.e., subspaces spanned by a collection of standard basis elements) W of $\mathbb{R}^n$. It is easy to verify that $\mathcal{S}_d$ is an outer measure, hence a submeasure, on $[0,1]^n$. Moreover, it is easy to see that for each $d \in \mathbb{N}$, the d-stochastic closure coincides with the $\mathcal{S}_d$-closure, hence strong and σ-non-essential.

5.5 Prevalence

The concept of prevalent sets is a measure-theoretic approach to defining what it means for a statement to hold "almost everywhere" in a possibly infinite-dimensional complete metric vector space. It was observed in [10] that the concept of prevalent sets extends the concept of Lebesgue almost everywhere in finite-dimensional Euclidean spaces. It is well known that there is no non-trivial translation-invariant measure in infinite-dimensional spaces. So we ask whether there is something weaker, e.g., a non-trivial translation-invariant submeasure whose submeasure zero sets are exactly the shy sets; i.e., the complements of the prevalent sets. Via the theory of essential closures, such a submeasure can be constructed. Let us recall some basic properties of shy sets. Let $A, A_1, A_2, \ldots$ be shy sets and v be a vector. Then the following hold:

1. $A + v$ is shy;

2. $B \subseteq A$ implies B is shy;

3. $\bigcup_{n=1}^{\infty} A_n$ is shy.

Observe that, with a suitable underlying σ-algebra, the collection of shy sets satisfies the properties in Definition 7. In the sequel, let V be a hereditarily Lindelöf complete metric vector space.

Theorem 35. *There exists a finite non-trivial translation-invariant submeasure on V whose submeasure zero sets are exactly the shy sets.*

Proof. The σ-algebra generated by the open subsets and the shy subsets of V will be called the *prevalence σ-algebra* and denoted by $\mathfrak{L}(V)$. According to Hunt et al. [10], the collection of shy sets on V satisfies the properties in Definition 7 with respect to $\mathfrak{L}(V)$. By Theorem 8, there exists a unique σ-non-essential strong essential closure whose collection of non-essential sets is exactly the collection of shy sets. We call the induced essential closure the *prevalence closure.*

By Theorem 20, the prevalence closure induces a submeasure on $\mathfrak{L}(V)$. Note that an induced submeasure is not unique. We call such a submeasure a *prevalence submeasure.* Moreover, by Theorem 18, the collection of non-essential sets, which is the collection of shy sets, is exactly the collection of prevalence submeasure zero sets. In addition, it is worth mentioning that the space V is essentially closed with respect to the prevalence closure. This is due to the fact that non-empty open sets are not shy, hence are of positive prevalence submeasure.

To conclude, we have a prevalence submeasure on $\mathfrak{L}(V)$ whose prevalence submeasure zero sets are exactly the shy sets on V. Moreover, it is straightforward to verify that the prevalence closure commutes with the translations. However, a prevalence submeasure is generally not translation-invariant. Nevertheless, there is a special prevalence submeasure that is translation-invariant.

In the proof of Theorem 20, the normalized submeasure obtained from the prevalence closure will be called the *normalized prevalence submeasure* and denoted by μ_p. For each vector $v \in V$, observe that $\mu_p(A+v) = 0$ if and only if $A + v$ is shy, which in turn is valid if and only if A is shy. Equivalently, $\mu_p(A) = 0$. Since μ_p assumes the value of either 0 or 1, the normalized prevalence submeasure μ_p is translation-invariant. $\square$

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References

[1] E. de Amo, M. Díaz Carrillo and J. Fernández-Sánchez, *Measure-preserving functions and the independence copula*, Mediterr. J. Math., **8(3)** (2011), 431–450.

[2] R. B. Ash, *Probability and Measure Theory*, Second ed. With contributions by Catherine Doléans-Dade, Harcourt/Acad. Press, Burlington, MA, 2000.

[3] H. Bauer, *Measure and Integration Theory*, Walter de Gruyter, Berlin, 2001.

[4] Z. Buczolich and W. F. Pfeffer, *Variations of additive functions*, Czech. Math. J., **47(3)** (1997), 525–555.

[5] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley & Sons Ltd., West Sussex, 1990.

[6] G. A. Fredricks, R. B. Nelsen and J. A. Rodríguez-Lallena, *Copulas with fractal supports*, Insurance. Math. Econom., **37(1)** (2005), 42–48.

[7] D. H. Fremlin, *Measure Theory: Topological Measure Spaces*, Vol. 4, Torres Fremlin, Colchester, 2006.

[8] D. H. Fremlin, *Measure Theory: Broad Foundations*, Vol. 2, Second ed., Torres Fremlin, Colchester, 2010.

[9] F. Gesztesy, K. A. Makarov and M. Zinchenko, *Essential closures and AC spectra for reflectionless CMV, Jacobi, and Schrödinger operators revisited*, Acta Appl. Math., **103(3)** (2008), 315–339.

[10] B. R. Hunt, T. Sauer and J. A. Yorke, *Prevalence: A translation-invariant "almost every" on infinite-dimensional spaces*, Bull. Amer. Math. Soc., **27(2)** (1992), 217–238.

[11] J. Lukeš, J. Malý and L. Zajíček, *Fine topology methods in real analysis and potential theory*, Lecture Notes in Math., Vol. 1189, Springer-Verlag, Berlin, 1986.

[12] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge Univ. Press, Cambridge, 1995.

[13] J. R. Munkres, *Topology*, Second ed., Prentice Hall Inc., Upper Saddle River, NJ, 2000.

[14] R. B. Nelsen, *An Introduction to Copulas*, Second ed., Lecture Notes in Stat., Vol. 139, Springer, New York, 2006.

[15] M. Ondreját, *Existence of global martingale solutions to stochastic hyperbolic equations driven by a spatially homogeneous Wiener process*, Stoch. Dyn., **6(1)** (2006), 23–52.

[16] P. Ruankong and S. Sumetkijakan, *Essential closures and supports of multivariate copulas*, Int. J. Approx. Reason., **54(6)** (2013), 762–768.

[17] W. Trutschnig, *On a strong metric on the space of copulas and its induced dependence measure*, J. Math. Anal. Appl., **384(2)** (2011), 690–705.