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THE ℓ_1-DICHOTOMY THEOREM WITH RESPECT TO A COIDEAL

Abstract

In this paper we introduce, for any coideal basis $\mathcal{B}$ on the set $\mathbb{N}$ of natural numbers, the notions of a $\mathcal{B}$-sequence, a $\mathcal{B}$-subsequence of a $\mathcal{B}$-sequence, and a $\mathcal{B}$-convergent sequence in a metric space. The usual notions of a sequence, subsequence, and convergent sequence obtain for the coideal $\mathcal{B}$ of all the infinite subsets of $\mathbb{N}$. We first prove a Bolzano-Weierstrass theorem for $\mathcal{B}$-sequences: if $\mathcal{B}$ is a Ramsey coideal basis on $\mathbb{N}$, then every bounded $\mathcal{B}$-sequence of real numbers has a $\mathcal{B}$-convergent $\mathcal{B}$-subsequence; and next, with the help of this extended Bolzano-Weierstrass theorem, we establish an extension of the fundamental Rosenthal's ℓ_1-dichotomy theorem: if $\mathcal{B}$ is a semiselective coideal basis on $\mathbb{N}$, then every bounded $\mathcal{B}$-sequence of real valued functions $(f_n)_{n\in A}$ has a $\mathcal{B}$-subsequence $(f_n)_{n\in B}$, which is either $\mathcal{B}$-convergent or equivalent to the unit vector basis of $\ell_1(B)$.

1 Introduction

Coideals, the basic notion of this paper, have been studied among others, by Mathias ([10]), who considered selective and Ramsey coideals, under the name of happy families, Glasner ([6]) in connection with the Stone-Cech compactification, under the name of families with the divisible property, Furstenberg ([5]) in connection with topological dynamics and Ramsey theory, under

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the name of Ramsey families, Farah ([2]), who introduced the semiselective coideals, Bergelson and Downarowicz ([1]), as a notion of largeness in ergodic theory, under the name of partition regular families and more recently by Farmaki, Karageorgos, Koutsogiannis and Mitropoulos ([3], [4]), in connection with topological dynamics and Ramsey theory for nets, as families generated by coideal bases.

In Section 3, for a given coideal basis $\mathcal{B}$ on the set $\mathbb{N}$ of natural numbers we introduce the $\mathcal{B}$-sequences in a set X, the $\mathcal{B}$-subsequences of a given $\mathcal{B}$-sequence in X, and the $\mathcal{B}$-convergent $\mathcal{B}$-sequences in a metric (or topological) space. The usual notions of a sequence, subsequence, and convergent sequence obtain for the coideal basis $\mathcal{B}$ of all infinite subsets of $\mathbb{N}$. We note that a sequence may have subsequences which are not $\mathcal{B}$-subsequences. In Corollary 4 we extend the classical Bolzano-Weierstrass theorem as follows: if $\mathcal{B}$ is a Ramsey coideal basis on $\mathbb{N}$, then every bounded $\mathcal{B}$-sequence of real numbers has a $\mathcal{B}$-convergent $\mathcal{B}$-subsequence. With the aid of the extended Bolzano-Weierstrass theorem, we characterize (in Corollary 5) the bounded $\mathcal{B}$-sequences of real numbers which are not $\mathcal{B}$-convergent.

In Section 4 we prove the main result of the present paper, Theorem 11, an extended version of Rosenthal's ℓ_1-dichotomy theorem: if $\mathcal{B}$ is a semiselective coideal basis on the set $\mathbb{N}$, then every bounded $\mathcal{B}$-sequence $(f_n)_{n\in A}$ of real valued functions has a $\mathcal{B}$-subsequence $(f_n)_{n\in B}$ which is either $\mathcal{B}$-convergent or equivalent to the unit vector basis of $\ell_1(B)$. The fundamental Rosenthal's ℓ_1-dichotomy theorem corresponds to the coideal basis $\mathcal{B} = [\mathbb{N}]$ of all the infinite subsets of $\mathbb{N}$. Granted the remarkable applications of the classical Rosenthal's ℓ_1-dichotomy theorem in several branches of mathematics, it is hoped that our extended ℓ_1-dichotomy theorem will find some interesting applications too.

2 Preliminaries

In this preliminary section we will refer to the necessary notions about coideal bases and also to a remarkable partition theorem proved by Farah ([2]) for the infinite subsets of $\mathbb{N}$ with respect to a semiselective coideal basis on $\mathbb{N}$ (Theorem 2), which is analogous to the classical Nash-Williams partition theorem and has a central role in the proofs of our results.

Notation. We denote by $\mathbb{N} = \{1, 2, ...\}$ the set of all the natural numbers. For an infinite subset M of $\mathbb{N}$ we denote by $[M]^{<\infty}$ the set of all the finite subsets of M and by $[M]$ the set of all the infinite subsets of M (considering them as increasing sequences).

We start with the notion of a coideal on the set $\mathbb{N}$ of natural numbers ([10], [2], [15], [4]).

Definition 2.1. *A non-empty subset $\mathcal{H}$ of the set $[\mathbb{N}]$ of all the infinite subsets of $\mathbb{N}$ is a* **coideal** *on $\mathbb{N}$ if it satisfies the following two properties:*

(i) If $A \cup B \in \mathcal{H}$, then either $A \in \mathcal{H}$ or $B \in \mathcal{H}$.

(ii) If $A \in \mathcal{H}$ and $A \subseteq B \subseteq \mathbb{N}$, then $B \in \mathcal{H}$.

Remarks 2.2.

(i) Obviously, a family $\mathcal{H} \subseteq [\mathbb{N}]$ is a coideal on $\mathbb{N}$ if its complement is an ideal on $\mathbb{N}$.

(ii) A nonprincipal ultraflter on $\mathbb{N}$ is a coideal on $\mathbb{N}$ closed under finite intersections.

(iii) The set $[\mathbb{N}]$ of all the infinite subsets of $\mathbb{N}$ is a coideal on $\mathbb{N}$, which is not an ultrafilter.

(iv) A subset of $[\mathbb{N}]$ is a coideal on $\mathbb{N}$ if and only if it is a union of ultrafilters on $\mathbb{N}$ ([6], Proposition 1.1.).

(v) The set $\mathcal{H}_d = \{A \in [\mathbb{N}] \ : \ d^*(A) := \limsup_n \frac{|A \cap \{1,2,\ldots,n\}|}{n} > 0\}$ is a coideal on $\mathbb{N}$.

(vi) According to van der Waerden's Theorem [16], the set AP of all the infinite subsets of $\mathbb{N}$ that contain arbitrarily long arithmetic progressions is a coideal on $\mathbb{N}$.

We now define the coideal bases on $\mathbb{N}$ which generates coideals on $\mathbb{N}$ ([3]).

Definition 2.3. *A non-empty subset $\mathcal{B}$ of the set $[\mathbb{N}]$ is a* **coideal basis** *on $\mathbb{N}$, if it satisfies only the following property:*

> *If $A \cup B \in \mathcal{B}$, then there exists $C \in \mathcal{B}$ such that either $C \subseteq A$ or $C \subseteq B$.*

Equivalently, $\mathcal{B}$ is a coideal basis on $\mathbb{N}$ if the family

$$\mathcal{L}_{\mathcal{B}} = \{A \in [\mathbb{N}] \ : \ \textit{there exists } B \in \mathcal{B} \ \textit{ with } \ B \subseteq A\}$$

is a coideal on $\mathbb{N}$.

Remarks 2.4.

(i) Obviously, a coideal $\mathcal{H}$ on $\mathbb{N}$ is a coideal basis on $\mathbb{N}$ and $\mathcal{L}_{\mathcal{H}} = \mathcal{H}$.

(ii) The set $[2\mathbb{N}]$ of all the infinite subsets of $\mathbb{N}$ that contain only even numbers is a coideal basis on $\mathbb{N}$ but it is not a coideal.

In Definitions 2.5, 2.6, 2.9 below we will define the classes of Ramsey, selective and semiselective coideal bases on $\mathbb{N}$ respectively. The Ramsey ultrafilters on $\mathbb{N}$ were defined in [8]. Mathias in [10] defined the selective coideals on $\mathbb{N}$ under the name of happy families, and later Farah in [2] introduced the semiselective coideals on $\mathbb{N}$ (see also [15]).

Notation. Let $A \in [\mathbb{N}]$ and $n \in \mathbb{N}$. We set,

$$[A]^n = \{(x_1, \dots, x_n) : x_1 < \dots < x_n \in A\}, \quad \textit{and} \quad A \setminus n = \{m \in A : n < m\}.$$

Definition 2.5. *A coideal basis $\mathcal{B}$ on $[\mathbb{N}]$ is* **Ramsey** *if for every $n, r \in \mathbb{N}$ and for every $A \in \mathcal{L}_{\mathcal{B}}$ with $[A]^n = C_1 \cup \dots \cup C_r$, there exist $B \in \mathcal{B}, B \subseteq A$ and $1 \leqslant i_0 \leqslant r$ such that $[B]^n \subseteq C_{i_0}$.*

Definition 2.6. *A coideal basis $\mathcal{B}$ on $\mathbb{N}$ is* **selective***, if for every decreasing sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{L}_{\mathcal{B}}$, there exists $B \in \mathcal{L}_{\mathcal{B}}$ such that $B \setminus n \subseteq A_n$ for every $n \in B$.*

A way to generate selective coideals on $\mathbb{N}$ is via the following theorem of Mathias [10] (see also [15]).

Theorem 1. *Let $\mathcal{A}$ be an infinite subset of the set $[\mathbb{N}]$ such that $A \cap B$ is finite for every pair A, B of distinct elements of $\mathcal{A}$. Let $\mathcal{H}$ be the set of all the infinite subsets of $\mathbb{N}$ that cannot be covered up to a finite set by finitely many members of $\mathcal{A}$. Then, $\mathcal{H}$ is a selective coideal on $\mathbb{N}$.*

Example 2.7. *Let $\mathcal{P} = \{p_n : n \in \mathbb{N}\}$ be an enumeration of all the prime natural numbers. For each $n \in \mathbb{N}$ we set*

$$A_n = \{k \in \mathbb{N} : \textit{ the least prime divisor of } k \textit{ is equal to } p_n\}.$$

Obviously the set A_n is infinite for every $n \in \mathbb{N}$ and $A_n \cap A_m = \emptyset$ for every $n, m \in \mathbb{N}$ with $n \neq m$. According to Theorem 1, the set

$$\mathcal{H}_p = \{A \in [\mathbb{N}] \ : A \setminus \bigcup_{n\in F} A_n \textit{ is infinite for every finite subset } F \textit{ of } \mathbb{N}\}$$

is a selective coideal on $\mathbb{N}$.

We will now define the notion of a semiselective coideal basis on $\mathbb{N}$, as found in Todorcevic's book ([15]). This definition is equivalent to the one originally given by Farah in [2]. In order to define the semiselective coideal bases on $\mathbb{N}$ (according to [15], [3]) we need the following definition:

Definition 2.8. *Let $\mathcal{B}$ be a coideal basis on $\mathbb{N}$ and $A \in \mathcal{L}_{\mathcal{B}}$. A subset $\mathcal{R}$ of $\mathcal{L}_{\mathcal{B}}$ has the* **dense-open** *property in $\mathcal{B}$ (respectively has the* **dense-open** *property in $\mathcal{B}$ on A) if it satisfies the following two conditions:*

(i) For every $B \in \mathcal{L}_{\mathcal{B}}$, (respectively for every $B \in \mathcal{L}_{\mathcal{B}}$, $B \subseteq A$) there exists $C \in \mathcal{R}$ with $C \subseteq B$.

(ii) If $C \in \mathcal{R}$ and $B \in \mathcal{L}_{\mathcal{B}}$ with $B \subseteq C$, then $B \in \mathcal{R}$.

Definition 2.9. *A coideal basis $\mathcal{B}$ on $\mathbb{N}$ is* **semiselective** *if it has the following two properties:*

(i) For every sequence $(\mathcal{R}_n)_{n\in\mathbb{N}}$ of subsets of $\mathcal{L}_{\mathcal{B}}$ with the dense-open property in $\mathcal{B}$ and for every $A \in \mathcal{L}_{\mathcal{B}}$ (alternatively, for every $A \in \mathcal{L}_{\mathcal{B}}$ and every sequence $(\mathcal{R}_n)_{n\in\mathbb{N}}$ of subsets of $\mathcal{L}_{\mathcal{B}}$ with the dense-open property in $\mathcal{B}$ on A) there exists $B \in \mathcal{L}_{\mathcal{B}}$ with $B \subseteq A$ such that for each $n \in \mathbb{N}$ there exist $C \in \mathcal{R}_n$ and a finite subset F of B such that $B \setminus F \subseteq C$.

(ii) For every $A \in \mathcal{L}_{\mathcal{B}}$ and every disjoint partition $A = \bigcup_{n=1}^{\infty} F_n$ of A, where F_n is finite for every $n \in \mathbb{N}$, there exists $B \in \mathcal{L}_{\mathcal{B}}$ with $B \subseteq A$ such that the set $B \cap F_n$ has at most one element for all $n \in \mathbb{N}$.

Remarks 2.10.

(i) A selective coideal basis on $\mathbb{N}$ is obviously semiselective. The inverse implication does not hold (see [2]).

(ii) A semiselective coideal basis on $\mathbb{N}$ is Ramsey, according to [2] (see also [15], [3]). The inverse implication does not hold (see [2], [15]).

(iii) The coideal $[\mathbb{N}]$ of all the infinite subsets of $\mathbb{N}$ is selective, so it is semiselective and consequently Ramsey.

(iv) The coideal AP (Remarks 2.2(v)) of all the subsets of $\mathbb{N}$ that contain arbitrarily long arithmetic progressions is not semiselective, and consequently it is not selective.

(v) The coideal $\mathcal{H}_p$, defined in Example 2.7 is selective, so it is semiselective and consequently Ramsey.

(vi) An ultrafilter is Ramsey if and only if it is semiselective and if and only if it is selective ([9]).

Now we will recall a partition theorem for the set $[\mathbb{N}]$ of all the infinite subsets of $\mathbb{N}$ with respect to a semiselective coideal bases on $\mathbb{N}$, which follows from a more general result proved by Farah in [2]. The analogous theorem with respect to the coideal basis $[\mathbb{N}]$ is the classical Nash-Williams partition theorem.

We say that a subset $\mathcal{U}$ of $[\mathbb{N}]$ is pointwise closed if, identifying each element of $[\mathbb{N}]$ with an element of $\{0,1\}^{\mathbb{N}}$, the set $\mathcal{U}$ is a closed subset of $[\mathbb{N}]$ in the relative topology of $\{0,1\}^{\mathbb{N}}$.

Theorem 2 (Farah, [2])**.** *Let $\mathcal{B}$ be a semiselective coideal basis on $\mathbb{N}$. For every pointwise closed subset $\mathcal{U}$ of $[\mathbb{N}]$ and every $A \in \mathcal{L}_{\mathcal{B}}$ there exists $B \in \mathcal{L}_{\mathcal{B}}$, $B \subseteq A$ such that*

$$\text{either } [B] \subseteq \mathcal{U} \text{ or } [B] \subseteq [\mathbb{N}] \setminus \mathcal{U}.$$

3 Sequences and convergence with respect to a coideal basis

For a given coideal basis $\mathcal{B}$ on $\mathbb{N}$, we define the $\mathcal{B}$-sequences in a set X as functions from an element of $\mathcal{L}_{\mathcal{B}}$ to X and analogously we define the $\mathcal{B}$-subsequences of a given $\mathcal{B}$-sequence. Consequently, we define the $\mathcal{B}$-convergent $\mathcal{B}$-sequences in a metric space.

For a given Ramsey coideal basis $\mathcal{B}$ on $\mathbb{N}$, we prove (in Corollary 4) that every $\mathcal{B}$-sequence of real numbers has a monotone $\mathcal{B}$-subsequence and consequently that every bounded $\mathcal{B}$-sequence of real numbers has a $\mathcal{B}$-convergent $\mathcal{B}$-subsequence, extending the fundamental Bolzano-Weierstrass theorem, corresponding to the coideal basis $\mathcal{B} = [\mathbb{N}]$. Finally, applying Corollary 4 we characterize (in Proposition 5) the bounded $\mathcal{B}$-sequences of real numbers which are not $\mathcal{B}$-convergent as those that have two $\mathcal{B}$-subsequences which $\mathcal{B}$-converge to different limits.

We apply these results in the proof of our main result in the next section.

Definition 3.1. *Let X be a nonempty set and $\mathcal{B}$ a coideal basis on $\mathbb{N}$. A **$\mathcal{B}$-sequence in** X is a function $a : A \to X$ from an element A of the coideal $\mathcal{L}_{\mathcal{B}}$ generated by $\mathcal{B}$ to the set X and it is denoted by $(a_n)_{n \in A}$.*

Remarks 3.2.

(i) Every sequence in a set X is a $\mathcal{B}$-sequence in X for every coideal basis $\mathcal{B}$ on $\mathbb{N}$, since $\mathbb{N} \in \mathcal{L}_{\mathcal{B}}$.

(ii) Let $\mathcal{B}$ be a coideal basis on $\mathbb{N}$. Every $\mathcal{B}$-sequence in a set X is a $[\mathbb{N}]$-sequence in X (Remark 2.2(ii)).

For a given coideal basis $\mathcal{B}$ on $\mathbb{N}$, we will define the $\mathcal{B}$-subsequences of a $\mathcal{B}$-sequence in a set X.

Definition 3.3. *Let $\mathcal{B}$ be a coideal basis on $\mathbb{N}$ and $(a_n)_{n\in A}$ a $\mathcal{B}$-sequence in a set X. The $\mathcal{B}$-sequence $(a_n)_{n\in B}$ in X is called a* ***$\mathcal{B}$-subsequence of*** *$(a_n)_{n\in A}$ if $B \subseteq A$.*

Let $\mathcal{B}$ be a coideal basis on $\mathbb{N}$. We will say that the subsequence $(a_{k_n})_{n\in\mathbb{N}}$ of a sequence $(a_n)_{n\in\mathbb{N}}$ corresponds to the $\mathcal{B}$-subsequence $(a_n)_{n\in B}$ of $(a_n)_{n\in\mathbb{N}}$ if $B = \{k_n : n \in \mathbb{N}\} \in \mathcal{L}_{\mathcal{B}}$. If $\mathcal{L}_{\mathcal{B}}$ is different from $[\mathbb{N}]$, then every sequence has subsequences which do not correspond to $\mathcal{B}$-subsequences (Remark 3.4(ii) below).

On the other hand if $(a_n)_{n\in B}$ is a $\mathcal{B}$-subsequence of a sequence $(a_n)_{n\in\mathbb{N}}$, then the subsequence $(a_{k_n})_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$, where $B = \{k_n : n \in \mathbb{N}\}$ with $k_n < k_{n+1}$ for every $n \in \mathbb{N}$, corresponds to $(a_n)_{n\in B}$.

Remarks 3.4.

(i) A subsequence of a given sequence $(a_n)_{n\in\mathbb{N}}$ corresponds to a $[\mathbb{N}]$-subsequence of $(a_n)_{n\in\mathbb{N}}$ (Remark 2.2(ii)).

(ii) Let $\mathcal{B}$ be a coideal basis on $\mathbb{N}$. If $\mathcal{L}_{\mathcal{B}}$ is different from $[\mathbb{N}]$, then every sequence $(a_n)_{n\in\mathbb{N}}$ in a set X has subsequences which do not correspond to $\mathcal{B}$-subsequences of $(a_n)_{n\in\mathbb{N}}$. Indeed, if $A \in [\mathbb{N}]\setminus\mathcal{L}_{\mathcal{B}}$ and $A = \{k_n : n \in \mathbb{N}\}$ with $k_n < k_{n+1}$ for every $n \in \mathbb{N}$, then the subsequence $(a_{k_n})_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ does not correspond to a $\mathcal{B}$-subsequence of $(a_n)_{n\in\mathbb{N}}$.
For example, let the coideal basis $\mathcal{H}_d$ (Remark 2.2(iv)) and $\mathcal{P}$ the set of prime natural numbers. If $\mathcal{P} = \{k_n : n \in \mathbb{N}\}$ with $k_n < k_{n+1}$ for every $n \in \mathbb{N}$, then the subsequence $(a_{k_n})_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ does not correspond to an H_d-subsequence of $(a_n)_{n\in\mathbb{N}}$.

(iii) Let $\mathcal{B}$ be a coideal basis on $\mathbb{N}$. If $(a_n)_{n\in B}$ is a $\mathcal{B}$-subsequence of the $\mathcal{B}$-sequence $(a_n)_{n\in A}$ and $(a_n)_{n\in C}$ is a $\mathcal{B}$-subsequence of $(a_n)_{n\in B}$, then $(a_n)_{n\in C}$ is a $\mathcal{B}$-subsequence of $(a_n)_{n\in A}$.

For a given coideal basis $\mathcal{B}$ on $\mathbb{N}$, we will define the $\mathcal{B}$-convergent $\mathcal{B}$-sequences in a metric space. More generally, one could define the $\mathcal{B}$-convergent $\mathcal{B}$-sequences in an arbitrary topological space.

Definition 3.5. *Let (X, d) be a metric space and $\mathcal{B}$ a coideal basis on $\mathbb{N}$. A $\mathcal{B}$-sequence $(a_n)_{n\in B}$ in X* ***$\mathcal{B}$-converges*** *to the element a of X if for every $\varepsilon > 0$ the set*

$$\{n \in B : d(a_n, a) \geqslant \varepsilon\}$$

*does not contain an element of $\mathcal{B}$ (i.e. does not belong to $\mathcal{L}_{\mathcal{B}}$). Also, the $\mathcal{B}$-sequence $(a_n)_{n\in A}$ in (X, d) is $\mathcal{B}$-***convergent** *if it $\mathcal{B}$-converges to an element a of X.*

According to the previous definition, a $\mathcal{B}$-sequence $(a_n)_{n\in A}$ of real numbers $\mathcal{B}$-converges to the real number a if for every $\varepsilon > 0$ the set $\{n \in A : |a_n - a| \geqslant \varepsilon\}$ does not contain an element of $\mathcal{B}$ (i.e. does not belong to $\mathcal{L}_{\mathcal{B}}$).

Remarks 3.6.

1. A sequence $(a_n)_{n\in\mathbb{N}}$ in a metric space (X, d) $[\mathbb{N}]$-converges to a if and only if $(a_n)_{n\in\mathbb{N}}$ converges to a. Moreover, a $\mathcal{B}$-sequence $(a_n)_{n\in A}$ in a metric space (X, d), where $\mathcal{B}$ is a coideal basis on $\mathbb{N}$, $[\mathbb{N}]$-converges to a if and only if the sequence $(a_{k_n})_{n\in\mathbb{N}}$ converges to a, where $\{k_n : n \in \mathbb{N}\} = A$ and $k_n < k_{n+1}$ for every $n \in \mathbb{N}$.

2. Let a coideal basis $\mathcal{B}$ on $\mathbb{N}$. If a $\mathcal{B}$-sequence $(a_n)_{n\in A}$ in a metric space $[\mathbb{N}]$-converges to $a \in X$, then $(a_n)_{n\in A}$ $\mathcal{B}$-converges to a.

3. Let the coideal basis $\mathcal{H}_d$ (Remark 2.2(iv)) on $\mathbb{N}$ and the sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers with $a_n = \begin{cases} 1, & \text{if } n \text{ is not prime} \\ 0, & \text{if } n \text{ is prime.} \end{cases}$

   The sequence $(a_n)_{n\in\mathbb{N}}$ $\mathcal{H}_d$-converges to 1, but it is not a convergent sequence of real numbers.

4. Let $\mathcal{B}_1, \mathcal{B}_2$ be coideal bases on $\mathbb{N}$ with $\mathcal{B}_1 \subseteq \mathcal{B}_2$ and $(a_n)_{n\in A}$ a $\mathcal{B}_1$-sequence in a metric space. If $(a_n)_{n\in A}$ $\mathcal{B}_2$-converges to $a \in X$, then $(a_n)_{n\in A}$ $\mathcal{B}_1$-converges to a.

5. Let $\mathcal{B}$ be a coideal basis on $\mathbb{N}$ and $(a_n)_{n\in A}$ a $\mathcal{B}$-sequence in a metric space. The $\mathcal{B}$-sequence $(a_n)_{n\in A}$ $\mathcal{B}$-converges to a if and only if every $\mathcal{B}$-subsequence of $(a_n)_{n\in A}$ $\mathcal{B}$-converges to a.

6. Let $\mathcal{B}$ be a coideal basis on $\mathbb{N}$ and $(a_n)_{n\in\mathbb{N}}$ a sequence in a metric space. If a $\mathcal{B}$-subsequence $(a_n)_{n\in B}$ of $(a_n)_{n\in\mathbb{N}}$ is $\mathcal{B}$-convergent, the subsequence $(a_{k_n})_{n\in\mathbb{N}}$ corresponding to $(a_n)_{n\in B}$ is not necessarily $\mathcal{B}$-convergent.

   Indeed, let the coideal basis $[2\mathbb{N}]$ (Remark 2.4(ii)) and the sequence $(a_n)_{n\in\mathbb{N}}$ of real numbers with $a_n = \begin{cases} 2, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}$

   The $[2\mathbb{N}]$-subsequence $(a_n)_{n\in B}$ of $(a_n)_{n\in\mathbb{N}}$, where $B = \{n \in \mathbb{N} : n > 1\}$, $[2\mathbb{N}]$-converges to 2, but the subsequence $(a_{k_n})_{n\in\mathbb{N}}$ corresponding to

$(a_n)_{n\in B}$, where $k_{2n-1} = 2n$ and $k_{2n} = 2n+1$ for every $n \in \mathbb{N}$, is not $[2\mathbb{N}]$-convergent.

Analogously, we define, for a coideal basis $\mathcal{B}$ on $\mathbb{N}$, the $\mathcal{B}$-sequences of real valued functions which are pointwise $\mathcal{B}$-convergent.

Definition 3.7. *Let $\mathcal{B}$ be a coideal basis on $\mathbb{N}$, $A \in \mathcal{L}_\mathcal{B}$ and $f_n : X \to \mathbb{R}$ be a function from a set X to the set of real numbers for every $n \in A$. The $\mathcal{B}$-sequence $(f_n)_{n\in A}$* ***pointwise $\mathcal{B}$-converges*** *to the function $f : X \to \mathbb{R}$ if and only if the $\mathcal{B}$-sequence $(f_n(x))_{n\in A}$ of real numbers $\mathcal{B}$-converges to $f(x)$ for every $x \in X$.*

We will prove that every $\mathcal{B}$-sequence of real numbers has a monotone $\mathcal{B}$-subsequence in case $\mathcal{B}$ is a Ramsey coideal basis on $\mathbb{N}$, extending the fundamental Bolzano-Weierstrass theorem, which corresponds to the particular case $\mathcal{B} = [\mathbb{N}]$.
We note that, for a given coideal basis $\mathcal{B}$ on $\mathbb{N}$, a $\mathcal{B}$-sequence $(a_n)_{n\in A}$ of real numbers is increasing (respectively decreasing) if $a_n \leq a_m$ (respectively $a_n \geq a_m$) for every $n, m \in A$ with $n < m$.
Notation. For an infinite subset M of $\mathbb{N}$ we set

$$[M]^2 = \{(n,m) \in M \times M : n < m\}.$$

Proposition 3. *Let $\mathcal{B}$ be a Ramsey coideal basis on $\mathbb{N}$. Every $\mathcal{B}$-sequence of real numbers has a monotone $\mathcal{B}$-subsequence.*

PROOF. Let $(a_n)_{n\in A}$ be a $\mathcal{B}$-sequence of real numbers. We set

$$\mathcal{F} = \{(n,m) \in [A]^2 : a_n < a_m\}.$$

Since $\mathcal{B}$ is a Ramsey coideal basis on $\mathbb{N}$, there exists $B \in \mathcal{B}, B \subseteq A$ such that

$$\textit{either}\ \ [B]^2 \subseteq \mathcal{F}\ \ \textit{or}\ \ [B]^2 \subseteq [A]^2 \setminus \mathcal{F}.$$

In case $[B]^2 \subseteq \mathcal{F}$, the $\mathcal{B}$-subsequence $(a_n)_{n\in B}$ of $(a_n)_{n\in A}$ is increasing, while in case $[B]^2 \subseteq [A]^2 \setminus \mathcal{F}$, the $\mathcal{B}$-subsequence $(a_n)_{n\in B}$ of $(a_n)_{n\in A}$ is decreasing. □

From Proposition 3 and Remark 3.6(i) follows that every bounded $\mathcal{B}$-sequence of real numbers, where $\mathcal{B}$ is a Ramsey coideal basis on $\mathbb{N}$, has a monotone, $[\mathbb{N}]$-convergent $\mathcal{B}$-subsequence, which, according to Remark 3.6(ii), is also $\mathcal{B}$-convergent.

Corollary 4. *Let $\mathcal{B}$ be a Ramsey coideal basis on $\mathbb{N}$. Every bounded $\mathcal{B}$-sequence of real numbers has a monotone, $[\mathbb{N}]$-convergent and consequently $\mathcal{B}$-convergent $\mathcal{B}$-subsequence.*

Using Corollary 4, we will characterize, in case $\mathcal{B}$ is a Ramsey coideal basis on $\mathbb{N}$, the bounded $\mathcal{B}$-sequences of real numbers which are not $\mathcal{B}$-convergent.

Corollary 5. *Let $\mathcal{B}$ be a Ramsey coideal basis on $\mathbb{N}$ and $(a_n)_{n\in A}$ a bounded $\mathcal{B}$-sequence of real numbers which is not $\mathcal{B}$-convergent. Then, there exist two $\mathcal{B}$-subsequences of $(a_n)_{n\in A}$ which $\mathcal{B}$-converge to different limits.*

Proof. The $\mathcal{B}$-sequence $(a_n)_{n\in A}$ of real numbers is bounded, so, according to Corollary 4, $(a_n)_{n\in A}$ has a monotone $\mathcal{B}$-subsequence $(a_n)_{n\in B_1}$ which $\mathcal{B}$-converges to a real number a.
Since the $\mathcal{B}$-sequence $(a_n)_{n\in A}$ is not $\mathcal{B}$-convergent to a, there exists $\varepsilon > 0$ such that

$$\{n \in A : |a_n - a| \geqslant \varepsilon\} = \{n \in A : a_n \geqslant a + \varepsilon\} \cup \{n \in A : a_n \leqslant a - \varepsilon\} \in \mathcal{L}_{\mathcal{B}}.$$

Hence, either

$$\{n \in A : a_n \geqslant a + \varepsilon\} \in \mathcal{L}_{\mathcal{B}} \text{ or } \{n \in A : a_n \leqslant a - \varepsilon\} \in \mathcal{L}_{\mathcal{B}}.$$

Without loss of generality we assume that $C = \{n \in A : a_n \geqslant a + \varepsilon\} \in \mathcal{L}_{\mathcal{B}}$.

The $\mathcal{B}$-sequence $(a_n)_{n\in C}$ of real numbers is bounded, so, according to Proposition 3, it has a monotone $\mathcal{B}$-subsequence $(a_n)_{n\in B_2}$. The monotone and bounded $\mathcal{B}$-sequence $(a_n)_{n\in B_2}$ of real numbers $[\mathbb{N}]$-converges to a real number b with $b \geqslant a + \varepsilon$, according to Remark 3.6(i). According to Remark 3.6(ii), the $\mathcal{B}$-sequence $(a_n)_{n\in B_2}$ $\mathcal{B}$-converges to b. Since $B_2 \in \mathcal{L}_{\mathcal{B}}$ and $B_2 \subseteq C \subseteq A$, $(a_n)_{n\in B_2}$ is a $\mathcal{B}$-subsequence of $(a_n)_{n\in A}$ which $\mathcal{B}$-converges to a real number b with $b \neq a$.

Hence, there exist two $\mathcal{B}$-subsequences $(a_n)_{n\in B_1}$ and $(a_n)_{n\in B_2}$ of $(a_n)_{n\in A}$ which $\mathcal{B}$-converge to the real numbers a, b respectively with $b \neq a$. □

4 The ℓ_1-dichotomy principle with respect to a coideal basis

In this section, using the results of the previous sections we will prove, in Theorem 11 below, that for a given semiselective coideal basis $\mathcal{B}$ on $\mathbb{N}$, every bounded $\mathcal{B}$-sequence of real valued functions $(f_n)_{n\in A}$ has a $\mathcal{B}$-subsequence $(f_n)_{n\in B}$ which is either $\mathcal{B}$-convergent or equivalent to the unit vector basis of $\ell_1(B)$.

In the particular case where $\mathcal{B} = [\mathbb{N}]$, Theorem 11 coincides with the fundamental ℓ_1- dichotomy theorem of Rosenthal ([13]) given below.

Theorem 6 (Rosenthal, [13]). *Let $(f_n)_{n\in\mathbb{N}}$ be a bounded sequence of functions from an infinite set X to the set of real numbers. Then, there exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that either $(f_{n_k})_{k\in\mathbb{N}}$ is pointwise convergent, or $(f_{n_k})_{k\in\mathbb{N}}$ is equivalent to the unit vector basis of ℓ_1.*

For a given coideal basis $\mathcal{B}$ on $\mathbb{N}$, we will define now the $\mathcal{B}$-convergent and the independent sequences of pairs of disjoint subsets of an infinite set X.

Definition 4.1. *Let $\mathcal{B}$ be a coideal basis on $\mathbb{N}$. We say that a $\mathcal{B}$-sequence $(Y_n, Z_n)_{n\in A}$ of pairs of subsets of an infinite set X such that $Y_n \cap Z_n = \emptyset$ for every $n \in A$:*

*(i) is $\mathcal{B}$-**convergent**, if for every $x \in X$ either*

$\mathcal{F}_x^A$=$\{n \in A : x \in Y_n\}$ *does not contain an element of* $\mathcal{B}$ (*i.e.* $\mathcal{F}_x^A \notin \mathcal{L}_\mathcal{B}$) *or*

$\mathcal{G}_x^A = \{n \in A : x \in Z_n\}$ *does not contain an element of* $\mathcal{B}$ (*i.e.* $\mathcal{G}_x^A \notin \mathcal{L}_\mathcal{B}$).

*(ii) is **independent**, if for every choice of finite, disjoint subsets F, G of A*

$$\left(\bigcap_{n\in F} Y_n\right) \cap \left(\bigcap_{n\in G} Z_n\right) \neq \emptyset.$$

It is obvious that if a $\mathcal{B}$-sequence $(Y_n, Z_n)_{n\in A}$ is $\mathcal{B}$-convergent (respectively independent), then every $\mathcal{B}$-subsequence of $(Y_n, Z_n)_{n\in A}$ is $\mathcal{B}$-convergent (respectively independent).

Now we will prove that, if $\mathcal{B}$ is a semiselective coideal basis on $\mathbb{N}$, then every $\mathcal{B}$-sequence of pairs of disjoint subsets of an infinite set X has a $\mathcal{B}$-subsequence which is either $\mathcal{B}$-convergent or independent.

Proposition 7. *Let $\mathcal{B}$ be a semiselective coideal basis on $\mathbb{N}$ and $(Y_n, Z_n)_{n\in A}$ a $\mathcal{B}$-sequence of pairs of subsets of an infinite set X such that $Y_n \cap Z_n = \emptyset$ for every $n \in \mathbb{N}$. Then, there exists a $\mathcal{B}$-subsequence $(Y_n, Z_n)_{n\in B}$ of $(Y_n, Z_n)_{n\in A}$ such that either $(Y_n, Z_n)_{n\in B}$ is $\mathcal{B}$-convergent, or $(Y_n, Z_n)_{n\in B}$ is independent.*

PROOF. We suppose that all the $\mathcal{B}$-subsequences $(Y_n, Z_n)_{n\in B}$, for every $B \in \mathcal{L}_\mathcal{B}$, $B \subseteq A$, of $(Y_n, Z_n)_{n\in A}$ are not $\mathcal{B}$-convergent.

For notational purposes we will denote by $(-1)Y_n$ the set Z_n for every $n \in A$. Considering an infinite subset of $\mathbb{N}$ as an increasing sequence in $\mathbb{N}$, for each $k \in \mathbb{N}$ we set

$$\mathcal{U}_k = \{N = (n_i)_{i\in\mathbb{N}} \in [\mathbb{N}] : \bigcap_{i=1}^{k} (-1)^i Y_{n_i} \neq \emptyset\}.$$

The subsets $\mathcal{U}_k$ of $[\mathbb{N}]$ for every $k \in \mathbb{N}$ are pointwise closed, as identifying each element of $[\mathbb{N}]$ with an element of $\{0,1\}^{\mathbb{N}}$, the sets $\mathcal{U}_k$ are closed subset of $[\mathbb{N}]$ in the relative topology of $\{0,1\}^{\mathbb{N}}$. Consequently $\mathcal{U} = \bigcap_{k\in\mathbb{N}} \mathcal{U}_k$ is also a pointwise closed subset of $[\mathbb{N}]$.

We apply Theorem 2 for the partition family $\mathcal{U}$ of $[\mathbb{N}]$ and the semiselective coideal basis $\mathcal{B}$. So, we get the existence of a $B \in \mathcal{L}_{\mathcal{B}},\ B \subseteq A$ such that either $[B] \subseteq \mathcal{U}$ or $[B] \subseteq [\mathbb{N}] \setminus \mathcal{U}$.

We assume that there exists $B \in \mathcal{L}_{\mathcal{B}},\ B \subseteq A$ such that $[B] \subseteq [\mathbb{N}]\setminus\mathcal{U}$. Since the $\mathcal{B}$-sequence $(Y_n, Z_n)_{n\in B}$ is not $\mathcal{B}$-convergent there exists $x_0 \in X$ such that

$$C_1 = \{n \in B : x_0 \in Y_n\} \in \mathcal{L}_{\mathcal{B}} \quad \text{and} \quad C_2 = \{n \in B : x_0 \in Z_n\} \in \mathcal{L}_{\mathcal{B}}.$$

Hence, $x_0 \in Y_n$ for every $n \in C_1$ and $x_0 \in (-1)Y_n$ for every $n \in C_2$. So,

$$x_0 \in (\cap_{n\in C_1} Y_n) \bigcap (\cap_{n\in C_2}(-1)Y_n).$$

We construct an infinite subset C of B such that $C = \{c_i : i \in \mathbb{N}\}$ with $c_i < c_{i+1}$, $c_{2i} \in C_1$ and $c_{2i-1} \in C_2$ for every $i \in \mathbb{N}$. Then,

$$x_0 \in \bigcap_{i\in\mathbb{N}} (-1)^i Y_{c_i},$$

and consequently $C = (c_i)_{i\in\mathbb{N}} \in \mathcal{U}$. Thus, $C \in \mathcal{U} \cap [B]$. This is a contradiction, since we assumed that $[B] \subseteq [\mathbb{N}] \setminus \mathcal{U}$.

Hence, there exists $B \in \mathcal{L}_{\mathcal{B}},\ B \subseteq A$ such that $[B] \subseteq \mathcal{U}$. Let $B = \{b_i : i \in \mathbb{N}\}$ with $b_i < b_{i+1}$ for every $i \in \mathbb{N}$. Since $B \in \mathcal{L}_{\mathcal{B}}$ we have that either

$$B_0 = \{b_{2n} : n \in \mathbb{N}\} \in \mathcal{L}_{\mathcal{B}} \ \textit{ or } \ B_1 = \{b_{2n-1} : n \in \mathbb{N}\} \in \mathcal{L}_{\mathcal{B}}.$$

Let $B_0 \in \mathcal{L}_{\mathcal{B}}$. We will prove that the $\mathcal{B}$-sequence $(Y_n, Z_n)_{n\in B_0}$ is independent. Indeed, let F, G be finite, disjoint subsets of B_0. There exists $N = \{b_{k_i} : i \in \mathbb{N}\} \in [B]$ with $k_i < k_{i+1}$ for every $i \in \mathbb{N}$ such that

$$F \subseteq \{b_{k_i} : i \textit{ even}\},\ G \subseteq \{b_{k_i} : i \textit{ odd}\}.$$

Since $N = \{b_{k_i} : i \in \mathbb{N}\} \in [B] \subseteq \mathcal{U}$, we have that

$$\bigcap_{i=1}^k (-1)^i Y_{b_{k_i}} \neq \emptyset \text{ for every } k \in \mathbb{N},$$

and consequently

$$(\bigcap_{n\in F} Y_n) \cap (\bigcap_{n\in G} Z_n) \neq \emptyset.$$

Hence, the $\mathcal{B}$-subsequence $(Y_n, Z_n)_{n\in B_0}$ of $(Y_n, Z_n)_{n\in A}$ is independent. $\square$

Notation. Let $\mathcal{B}$ be a coideal basis on $\mathbb{N}$ and $(f_n)_{n\in A}$ a bounded $\mathcal{B}$-sequence of functions from an infinite set X to the set of real numbers (there exists a real number M such that $|f_n(x)| \leq M$ for every $n \in \mathbb{N}$ and $x \in X$).
For two rational numbers p, q with $p < q$ and $n \in A$ we set

$$Y_n^p = \{x \in X : f_n(x) < p\} \qquad \text{and} \qquad Z_n^q = \{x \in X : f_n(x) > q\}$$

Obviously $Y_n^p \cap Z_n^q = \emptyset$ for every $n \in A$ and every rational numbers p, q with $p < q$. Hence, according to Proposition 7, if $\mathcal{B}$ is a semiselective coideal basis on $\mathbb{N}$, then for given rational numbers p, q with $p < q$ there exists $B \in \mathcal{L}_{\mathcal{B}},\ B \subseteq A$ such that
either the $\mathcal{B}$-sequence $(Y_n^p, Z_n^q)_{n\in B}$ is $\mathcal{B}$-convergent,
or the $\mathcal{B}$-sequence $(Y_n^p, Z_n^q)_{n\in B}$ is independent.
In the following proposition we will improve the previous dichotomy as follows:

Proposition 8. *Let $\mathcal{B}$ be a semiselective coideal basis on $\mathbb{N}$ and $(f_n)_{n\in A}$ a bounded $\mathcal{B}$-sequence of functions from an infinite set X to the set of real numbers. Then, there exists $B \in \mathcal{L}_{\mathcal{B}},\ B \subseteq A$ such that either the $\mathcal{B}$-sequences $(Y_n^p, Z_n^q)_{n\in B}$ are $\mathcal{B}$-convergent for every rational numbers p, q with $p < q$ or there exist rational numbers p, q with $p < q$ such that the $\mathcal{B}$-sequence $(Y_n^p, Z_n^q)_{n\in B}$ is independent.*

PROOF. We consider the set

$$\mathcal{P} = \{(p, q) \in \mathbb{Q} \times \mathbb{Q} : p < q\} = \{(p_1, q_1), (p_2, q_2), \dots\},$$

where $\mathbb{Q}$ is the set of rational numbers.

We assume that for every $B \in \mathcal{L}_{\mathcal{B}},\ B \subseteq A$ and for every $(p, q) \in \mathcal{P}$ the $\mathcal{B}$-sequence $(Y_n^p, Z_n^q)_{n\in B}$ is not independent. For $k \in \mathbb{N}$ we set,

$$\mathcal{R}_k = \{B \in \mathcal{L}_{\mathcal{B}} : (Y_n^{p_k}, Z_n^{q_k})_{n\in B} \ \textit{is}\ \ \mathcal{B}-\textit{convergent}\}.$$

We will first prove that the families $\mathcal{R}_k$, for every $k \in \mathbb{N}$, have the dense-open property in $\mathcal{B}$ on A (Definition 2.8). Indeed, let $k \in \mathbb{N}$.

(i) Let $C \in \mathcal{L}_{\mathcal{B}}$, $C \subseteq A$. According to our assumption, for every $D \in \mathcal{L}_{\mathcal{B}}$, $D \subseteq C$ the $\mathcal{B}$-sequence $(Y_n^{p_k}, Z_n^{q_k})_{n\in D}$ is not independent. So, according to Proposition 7 there exists $B \in \mathcal{L}_{\mathcal{B}}, B \subseteq C$ such that the $\mathcal{B}$-sequence $(Y_n^{p_k}, Z_n^{q_k})_{n\in B}$ is $\mathcal{B}$-convergent. Thus, $B \in \mathcal{R}_k$, and $B \subseteq C$.

(ii) Let $B \in \mathcal{R}_k$ and $C \in \mathcal{L}_{\mathcal{B}}$ with $C \subseteq B$. The $\mathcal{B}$-sequence $(Y_n^{p_k}, Z_n^{q_k})_{n\in C}$ is $\mathcal{B}$-convergent, since it is a $\mathcal{B}$-subsequence of $(Y_n^{p_k}, Z_n^{q_k})_{n\in B}$, which is $\mathcal{B}$-convergent. Thus, $C \in \mathcal{R}_k$.

Thus, the families $\mathcal{R}_k$, for every $k \in \mathbb{N}$, have the dense-open property in $\mathcal{B}$ on A.

Since $\mathcal{B}$ is a semiselective coideal basis (Definition 2.9) there exists $B \in \mathcal{L}_{\mathcal{B}}$, $B \subseteq A$ such that for each $k \in \mathbb{N}$ there is a $C \in \mathcal{R}_k$ and a finite subset F of B such that $B \setminus F \subseteq C$. Hence, the $\mathcal{B}$-sequences $(Y_n^{p_k}, Z_n^{q_k})_{n \in B}$ are $\mathcal{B}$-convergent for every $k \in \mathbb{N}$. □

Now we will prove that in case the first alternative of the dichotomy proved in Proposition 8 holds, it will then follow that the $\mathcal{B}$-sequence $(f_n)_{n \in A}$ has a $\mathcal{B}$-subsequence $(f_n)_{n \in B}$ which is $\mathcal{B}$-convergent.

Proposition 9. *Let $\mathcal{B}$ be a semiselective coideal basis on $\mathbb{N}$ and $(f_n)_{n \in B}$ a bounded $\mathcal{B}$-sequence of functions from an infinite set X to the set of real numbers. If the $\mathcal{B}$-sequences $(Y_n^p, Z_n^q)_{n \in B}$ are $\mathcal{B}$-convergent for every rational numbers p, q with $p < q$, then the $\mathcal{B}$-sequence $(f_n)_{n \in B}$ is $\mathcal{B}$-convergent.*

Proof. We suppose that the $\mathcal{B}$-sequence $(f_n)_{n \in B}$ is not $\mathcal{B}$-convergent. Then, there exists $x_0 \in X$ such that the $\mathcal{B}$-sequence $(f_n(x_0))_{n \in B}$ is not $\mathcal{B}$-convergent. According to Corollary 5 there exist $B_1, B_2 \in \mathcal{L}_{\mathcal{B}}$ with $B_1 \subseteq B$, $B_2 \subseteq B$ and $a, b \in \mathbb{R}$ with $a \neq b$ such that the $\mathcal{B}$-sequences $(f_n(x_0))_{n \in B_1}$, $(f_n(x_0))_{n \in B_2}$ to be $\mathcal{B}$-convergent to a, b respectively. We can assume that $a < b$ and let p, q be rational numbers such that $a < p < q < b$. Then,

$$\{n \in B_1 : f_n(x_0) \geqslant p\} \notin \mathcal{L}_{\mathcal{B}} \;\; \textit{and} \;\; \{n \in B_2 : f_n(x_0) \leqslant q\} \notin \mathcal{L}_{\mathcal{B}}.$$

Hence,

$$\{n \in B : f_n(x_0) < p\} \in \mathcal{L}_{\mathcal{B}} \;\; \textit{and} \;\; \{n \in B : f_n(x_0) > q\} \in \mathcal{L}_{\mathcal{B}},$$

and consequently

$$\{n \in B : x_0 \in Y_n^p\} \in \mathcal{L}_{\mathcal{B}} \;\; \textit{and} \;\; \{n \in B : x_0 \in Z_n^q\} \in \mathcal{L}_{\mathcal{B}}.$$

This is a contradiction, since the sequence $(Y_n^p, Z_n^q)_{n \in B}$ is $\mathcal{B}$-convergent. Hence, the $\mathcal{B}$-sequence $(f_n)_{n \in B}$ is $\mathcal{B}$-convergent. □

Finally we will prove that in case the second alternative of the dichotomy proved in Proposition 8 holds, it will then follow that the bounded $\mathcal{B}$-sequence $(f_n)_{n \in A}$ in the Banach space $\ell^\infty(X)$ has a $\mathcal{B}$-subsequence $(f_n)_{n \in B}$ which is equivalent to the unit vector basis of $\ell_1(B)$.

We note that a bounded $\mathcal{B}$-sequence $(x_n)_{n \in B}$ in the Banach space $(X, ||.||)$, where $\mathcal{B}$ is a coideal basis on $\mathbb{N}$, is equivalent to the unit vector basis of $\ell_1(B)$ if there exists a real positive number K such that $K\sum_{i\in H}|\lambda_i| \leqslant ||\sum_{i\in H}\lambda_i x_i||$ for every $\mathcal{B}$-sequence $(\lambda_n)_{n\in B}$ of real numbers and every nonempty finite subset H of B.

It is obvious that a $\mathcal{B}$-sequence $(a_n)_{n\in B}$ in a Banach space X is equivalent to the unit vector basis of $\ell_1(B)$ if and only if the subsequence $(a_{k_n})_{n\in\mathbb{N}}$ corresponding to $(a_n)_{n\in B}$ is equivalent to the natural basis of ℓ_1.

Proposition 10. *Let $\mathcal{B} \subseteq [\mathbb{N}]$ be a semiselective coideal basis on $\mathbb{N}$ and $(f_n)_{n\in B}$ a bounded $\mathcal{B}$-sequence of functions from an infinite set X to the set of real numbers. If there exist rational numbers p, q with $p < q$ such that the $\mathcal{B}$-sequence $(Y_n^p, Z_n^q)_{n\in B}$ is independent, then the $\mathcal{B}$-sequence $(f_n)_{n\in B}$ in the Banach space $\ell^\infty(X)$ is equivalent to the unit vector basis of $\ell_1(B)$.*

Proof. Let a $\mathcal{B}$-sequence $(\lambda_n)_{n\in B}$ of real numbers. For a finite subset H of $\mathbb{N}$ we set $F_H = \{i \in H : \lambda_i \geqslant 0\}$ and $G_H = \{i \in H : \lambda_i < 0\}$. Since the $\mathcal{B}$-sequence $(Y_n^p, Z_n^q)_{n\in B}$ is independent there exist

$$y_1 \in \left(\bigcap_{i\in F_H} X_i^p\right) \cap \left(\bigcap_{i\in G_H} Y_i^q\right) \text{ and } y_2 \in \left(\bigcap_{i\in G_H} X_i^p\right) \cap \left(\bigcap_{i\in F_H} Y_i^q\right).$$

Since $(f_n)_{n\in B}$ is bounded it is enough to prove that there exists $K > 0$ such that $K\sum_{i\in H}|\lambda_i| \leqslant ||\sum_{i\in H}\lambda_i f_i||_\infty$ for every $\mathcal{B}$-sequence $(\lambda_n)_{n\in B}$ of real numbers and every nonempty finite subset H of B. For a $\mathcal{B}$-sequence $(\lambda_n)_{n\in B}$ of real numbers and a nonempty finite subset H of B we have that

$$\sum_{i\in H}\lambda_i f_i(y_1) \leqslant p\sum_{i\in F_H}|\lambda_i| - q\sum_{i\in G_H}|\lambda_i|$$

and

$$\sum_{i\in H}\lambda_i f_i(y_2) \geqslant q\sum_{i\in F_H}|\lambda_i| - p\sum_{i\in G_H}|\lambda_i|.$$

It follows that

$$(q-p)\sum_{i=1}^{n}|\lambda_i| \leqslant |\sum_{i\in H}\lambda_i f_i(y_2) - \sum_{i\in H}\lambda_i f_i(y_1)| \leqslant 2||\sum_{i\in H}\lambda_i f_i||_\infty$$

and consequently that

$$\frac{(q-p)}{2}\sum_{i\in H}|\lambda_i| \leqslant ||\sum_{i\in H}\lambda_i f_i||_\infty.$$

Hence, $(f_n)_{n\in B}$ is equivalent to the unit vector basis of $\ell_1(B)$. □

Finally, the ℓ_1-dichotomy principle with respect to a semiselective coideal basis on $\mathbb{N}$ follows from Propositions 8, 9, and 10.

We remark that the full force of the assumption that the coideal basis is semiselective is used only in the proof of Proposition 8, while the proofs of the remaining Propositions 9 and 10 made use only of the weaker, according to Theorem 2, Nash-Williams property of the coideal.

Theorem 11. *Let $\mathcal{B} \subseteq [\mathbb{N}]$ be a semiselective coideal basis on $\mathbb{N}$ and $(f_n)_{n\in A}$ a bounded $\mathcal{B}$-sequence of functions from an infinite set X to the set of real numbers. Then, there exists a $\mathcal{B}$-subsequence $(f_n)_{n\in B}$ of $(f_n)_{n\in A}$ such that either $(f_n)_{n\in B}$ is $\mathcal{B}$-convergent, or $(f_n)_{n\in B}$ is equivalent to the unit vector basis of $\ell_1(B)$.*

We remark that in the particular case where $\mathcal{B} = [\mathbb{N}]$, a selective coideal basis on $\mathbb{N}$, Theorem 11 coincides with the fundamental ℓ_1-dichotomy theorem of Rosenthal ([13]).

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