RESEARCH

Real Analysis Exchange
Vol. 39(2), 2013/2014, pp. 335–344

Alireza Kamel Mirmostafaee, Department of Pure Mathematics, Centre of Excellence in Analysis on Algebraic Structures (CEAAS), School of Mathematical Sciences, Ferdowsi University of Mashhad, Mashhad 91775, Iran. email: `mirmostafaei@um.ac.ir`

QUASI-CONTINUITY OF HORIZONTALLY QUASI-CONTINUOUS FUNCTIONS

Abstract

Let X be a Baire space, Y a topological space, Z a regular space and $f : X \times Y \to Z$ be a horizontally quasi-continuous function. We will show that if Y is first countable and f is quasi-continuous with respect to the first variable, then every horizontally quasi-continuous function $f : X \times Y \to Z$ is jointly quasi-continuous. This will extend Martin's Theorem of quasi-continuity of separately quasi-continuous functions for non-metrizable range. Moreover, we will prove quasi-continuity of f for the case Y is not necessarily first countable.

1 Introduction

Let X and Y be topological spaces. A function $f : X \to Y$ is called quasi-continuous at $x \in X$ if for every neighborhoods U of x and W of $f(x)$, there is an open subset U' of U such that $f(U') \subseteq W$.

The notion of quasi-continuity was first used by Kempisty [6] to extend some results of Hahn and Baire on points of joint continuity of real-valued, separately continuous functions. This notion turned out to have an important place in the investigations of points of joint continuity and quasi-continuity of two variables functions [2, 7, 12, 13, 14, 17]. In particular, Martin [9] proved the following.

Theorem 1. *Let X be a Baire space, Y be second countable and Z be a metric space. If f is a function on $X \times Y$ to Z which is quasi-continuous with respect to each variable, then f is jointly quasi-continuous.*

Mathematical Reviews subject classification: Primary: 54C08, 54C05; Secondary: 54E52
Key words: Quasi-continuity, Horizontally quasi-continuous functions, Topological games
Received by the editors May 17, 2013
Communicated by: Manav Das

Using an idea of Bögel[1], the authors in [11] introduced the following concept.

Definition 2. *Let X, Y and Z be topological spaces, a function $f : X \times Y \to Z$ is called* horizontally quasi-continuous with respect to the second variable *at a point $(x_0, y_0) \in X \times Y$ if for each neighborhood W of $f(x_0, y_0)$ in Z and for each product of open sets $U \times V \subset X \times Y$ containing (x_0, y_0), there are a nonempty open set $U_1 \subset U$ and a point $y_1 \in V$ such that $f(U_1 \times \{y_1\}) \subset W$. The function f is called horizontally quasi-continuous if it is horizontally quasi-continuous at each point of $X \times Y$.*

In [11], the authors generalized Martin's theorem as follows.

Theorem 3. *[11, Theorem 3]. Suppose that X is a Baire space, Y satisfies the second axiom of countability and Z is a completely regular space. If a horizontally quasi-continuous function $f : X \times Y \to Z$ is quasi-continuous with respect to the second variable, then f is jointly quasi-continuous.*

We are going to generalize the above result for the case that the range of the function is regular, but not completely regular. In section 2, we will show that in order to prove Theorem 3, we can assume that Z is the real line. This means that completely regular spaces are not suitable candidate for our purpose.

Following [15], a collection $\mathcal{B}$ of nonempty open sets in a topological space is called a pseudo-base (or π-base) for this space if any nonempty open set contains some member of $\mathcal{B}$.

Note that the Stone-Čech compactification of $\mathbb{N}$, the natural numbers, has a countable dense set of isolated points, and thus has a countable pseudo-base. Clearly, this space is not second countable. So that the class of spaces which have a countable pseudo-base is larger than the class of second countable spaces.

We will show that the result of Theorem 3 remains true when the range of f is a regular space, and second countability of Y can be replaced by existence of a countable pseudo-base for Y.

Moreover, we will prove that the set of points of joint continuity of such a function is a residual subset of its domain provided that Z is second countable.

In 1976, Gruenhage introduced a class of topological spaces, called W-spaces. It is known that every first countable space is a W-space, but the converse is not true in general [4, 8]. In section 3, we apply a topological game argument to show that if X is a Baire space, Y is a W-space and $f : X \times Y \to Z$ is a horizontally quasi-continuous function which is continuous with respect to the second variable, where Z is a regular space, then f is jointly quasi-continuous. It follows that if $X \times Y$ is also a Baire space, then f is jointly continuous on a dense G_δ subset of its domain.

2 Quasi-continuity of horizontally quasi-continuous functions

In this section, we first show that in order to prove Theorem 3, we may assume that Z is equal to $\mathbb{R}$. Then we will extend this result for regular space Z. We also obtain points of joint continuity of separately quasi-continuous functions in some special cases.

Theorem 4. *Let X and Y be topological spaces. The following assertions are equivalent:*

(1) Every horizontally quasi-continuous function which is quasi-continuous with respect to the second variable from $X \times Y$ to $\mathbb{R}$ is quasi-continuous.

(2) Every horizontally quasi-continuous function which is quasi-continuous with respect to the second variable from $X \times Y$ to a completely regular space Z is quasi-continuous.

Proof. Since $\mathbb{R}$ is completely regular (1) follows from (2). Suppose that (1) holds and f is a horizontally quasi-continuous function which is quasi-continuous with respect to the second variable from $X \times Y$ into a completely regular space Z. If f is not quasi-continuous at some point $(a,b) \in X \times Y$, then by the definition, we can find neighborhoods U, V and G of a, b and $f(a,b)$, respectively, such that $f(U' \times V') \nsubseteq G$ for each nonempty pair of open sets (U', V') with $U' \times V' \subset U \times V$. Since Z is completely regular, there is a continuous function $g : Z \to [0,1]$ such that $g(f(a,b)) = 1$ and $g(z) = 0$ for each $z \in G^c$. Let $G_1 = \{z \in Z : g(z) > \frac{1}{2}\} \subset G$. Since g is continuous, G_1 is open in Z and $f(a,b) \in G_1$. Applying (1) for $g \circ f : X \times Y \to \mathbb{R}$, we can find nonempty open sets U' and V' of U and V, respectively, such that $g \circ f(U' \times V') \subset (\frac{1}{2}, 1]$. However, by our assumption, there is some $(x_0, y_0) \in U' \times V'$ such that $f(x_0, y_0) \in G^c$. It follows that $\frac{1}{2} < g \circ f(x_0, y_0) = 0$. This contradiction proves our result. □

In order to give a generalization of Theorem 3, we need to the following result.

Lemma 5. *[10, Lemma 2] Let X, Y and Z be topological spaces and let $f : X \times Y \to Z$ be horizontally quasi-continuous. If U and V are open subsets of X and Y, respectively, $A \subseteq X$ and $U \subseteq \overline{A}$, then $f(U \times V) \subseteq \overline{f(A \times V)}$.*

Theorem 6. *Let X be a Baire space, let Y have a countable π-base and let Z be a regular space. If $f : X \times Y \to Z$ is a horizontally quasi-continuous function which is quasi-continuous with respect to the second variable, then f is jointly quasi-continuous.*

Proof. Let $(x_0, y_0) \in X \times Y$, and let U, V and G be neighborhoods of x_0, y_0 and $f(x_0, y_0)$, respectively. Since Z is regular, we can find an open subset G_1 of G such that

$$f(x_0, y_0) \in G_1 \subseteq \overline{G_1} \subseteq G.$$

Thanks to horizontal quasi-continuity of f, we can find a nonempty open subset U_1 of U and $y_1 \in V$ such that $f(U_1 \times \{y_1\}) \subseteq G_1$. Let $\mathcal{B} = \{V_n : n \geq 1\}$ be a countable π-base for Y. For each $n \geq 1$, define $A_n = \emptyset$ if $V_n \nsubseteq V$ and let

$$A_n = \{x \in U_1 : f_x(V_n) \subseteq G_1\},$$

if $V_n \subseteq V$. Clearly, $\bigcup_{n=1}^{\infty} A_n \subseteq U_1$. Let $x \in U_1$. Then $f(x, y_1) \in G_1$. Since f is quasi-continuous with respect to the second variable, we can find a nonempty open subset W of V such that $f(x, y) \in G_1$ for each $y \in W$. Choose some $V_m \in \mathcal{B}$ such that $V_m \subseteq W$. Then $f(x, y) \in G_1$ for each $y \in V_m$. Hence, $U_1 = \bigcup_{n=1}^{\infty} A_n$. Since X is a Baire space, there is some $k \in \mathbb{N}$ such that $O = \operatorname{int}(\overline{A_k}) \neq \emptyset$. Let $U' = U_1 \bigcap O$, $V' = V_k$ and $A_0 = U' \cap A_k$. Since $O \subseteq \overline{A_k}$, we have $O \subseteq \overline{A_k \cap O}$. Therefore, $A_k \cap O \neq \emptyset$. But

$$\emptyset \neq A_k \cap O \subseteq U_1 \cap O = U'.$$

Hence, U' is a nonempty open subset of X. Moreover, we have

$$U' = U_1 \cap (O \cap \overline{A_k}) \subseteq \overline{A_k} \quad \text{and} \quad f(A_k \times V') \subseteq G_1.$$

Therefore, by Lemma 5,

$$f(U' \times V') \subseteq \overline{f(A_k \times V')} \subseteq \overline{G_1} \subseteq G.$$

This proves our result. □

The following result shows that in some special cases, the points of continuity of a quasi-continuous function is a residual subset of its domain.

Theorem 7. *If f is a quasi-continuous function from a topological space X into a second countable space Z, then f is continuous on a residual subset of X. In particular, when X is a Baire space, f is continuous on a dense G_δ subset of its domain.*

PROOF. Let $\{G_n : n \geq 1\}$ be a countable base for Z. For each $n \geq 1$, let A_n denote the set of all $x \in X$ such that $f(x) \in G_n$, but $f(U) \nsubseteq G_n$ for each neighborhood U of x.

Clearly, $\bigcup_{n=1}^{\infty} A_n$ is the set of points of discontinuity of f. We will show that $\mathrm{int}(\overline{A_n}) = \emptyset$ for each $n \geq 1$. For some $n \geq 1$, let U be a nonempty open subset of $\overline{A_n}$ and take some point $x \in U \cap A_n$. By the definition, $f(x) \in G_n$. By quasi-continuity of f, we can find a nonempty open subset U_1 of U such that $f(U_1) \subseteq G_n$. Hence, $U_1 \cap A_n = \emptyset$. This contradiction proves our claim. Therefore, f is continuous on the residual set $X \setminus \bigcup_{n=1}^{\infty} A_n$.

Since for each $n \geq 1$, $D_n = X \setminus (\overline{A_n})$ is a dense open subset of X, if X is a Baire space, then $D = \bigcap_{n=1}^{\infty} D_n$ is a dense G_δ subset of X. Clearly, f is continuous at each point of D. □

The following result follows immediately from Theorems 6 and 7.

Corollary 8. *Let X be a Baire space, let Y have a countable π-base and let Z be a regular second countable space. If $f : X \times Y \to Z$ is a horizontally quasi-continuous function which is quasi-continuous with respect to the second variable, then f is jointly continuous on a residual subset of $X \times Y$. In particular, if $X \times Y$ is Baire, then f is jointly continuous on a dense G_δ subset of $X \times Y$.*

Definition 9. *Let $\{\mathcal{G}_n : n \geq 1\}$ be a sequence of open covers of a topological space Z. This sequence is called a weak development for Z if $G_n \in \mathcal{G}_n$ for each $n \geq 1$ and $z \in \bigcap_{n=1}^{\infty} \mathcal{G}_n$ implies that the set $\{\bigcap_{k=1}^{n} G_k : n \geq 1\}$ is a base at z. A topological space with a weak development is called a weakly developable space.*

Corollary 10. *Let X be a Baire space, let Y have a countable π-base and let Z be a regular weakly developable space. If $f : X \times Y \to Z$ is a horizontally quasi-continuous function which is quasi-continuous with respect to the second variable and $X \times Y$ is Baire, then f is jointly continuous on a dense G_δ subset of $X \times Y$.*

PROOF. In [5, Theorem 4.1] it is shown that every quasi-continuous function from a Baire space into a weakly developable space is continuous on a dense G_δ subset of its domain, so that the result follows from Theorem 6.

□

3 Topological games and horizontally quasi-continuous functions

In order to state the main result of this section, we need the following topological games. We begin with the Banach-Mazur game [3, 16] which characterizes Baire spaces.

The Banach-Mazur game $\mathcal{BM}(X)$. Let (X, τ) be a topological space. The Banach-Mazur game $\mathcal{BM}(X)$ is played between two players α and β as follows:

Player β starts a game by selecting a nonempty open set U_1 of X; then player α chooses a non-empty open set $V_1 \subset U_1$. When (U_i, V_i) , $1 \leq i \leq n-1$, have been defined, player β picks a nonempty open set $U_n \subset V_{n-1}$ and α answers to his/her move by selecting a nonempty open set $V_n \subset U_n$. The player α wins the game if $(\bigcap_{n=1}^{\infty} V_n)$ is not empty. Otherwise, the player β is said to have won the play.

By a strategy for player α in $\mathcal{BM}(X)$, we mean a sequence of mappings $s = \{s_n\}$, which is defined inductively as follows:

The domain of s_1 is the set of all open sets and it assigns to each nonempty open subset U of X a nonempty open set $V \subset U$. For $n > 1$, the domain of s_n is the set of all partial plays $(U_1, s_1(U_1), \ldots, U_n)$ and it assigns to such a partial play a nonempty open set V_n which is also a subset of U_n.

An *s-play* is a play in which α selects his/her moves according to the strategy s. The strategy s for the player α is said to be a *winning strategy* if every s-play is won by α. A space X is called *α-favorable* if there exists a winning strategy for α in $\mathcal{BM}(X)$. It is known that a topological space X is Baire if and only if the player β does not have a winning strategy in $\mathcal{BM}(X)$.

In 1976, Gruenhage [4] defined a generalization of first countable spaces by means of a two person game:

The topological game $\mathcal{G}(Y, y_0)$. Let Y be a topological space and $y_0 \in Y$. The topological game $\mathcal{G}(Y, y_0)$ is played by two players $\mathcal{O}$ and $\mathcal{P}$ as follows. In step $n \geq 1$, $\mathcal{O}$ selects a neighborhood H_n of y_0 and $\mathcal{P}$ responds by choosing a point $y_n \in H_n$. We say $\mathcal{O}$ wins the game $g = (H_n, y_n)_{n \geq 1}$ if $y_n \to y_0$. If

$$g_1 = (H_1, y_1), \ \cdots, g_n = (H_1, y_1, \cdots, H_n, y_n)$$

are the first "n" moves of some play (of the game), we call g_n the n^{th} (*partial play*) of the game. As above, a strategy s and winning strategy for one of the players can be defined.

We call $y \in Y$ a W-point in Y if $\mathcal{O}$ has a winning strategy in the game $\mathcal{G}(Y, y)$. A space Y in which each point of Y is a W-point is called a W-space.

We also define Y to be a w-space if for each $y \in Y$, $\mathcal{P}$ fails to have a winning strategy in $\mathcal{G}(Y, y)$. It is known that every first countable space is a W-space [4, Theorem 3. 3]. However, the converse in not true in general [8, Example 2. 7].

Now, we are ready to state the main result of this section.

Theorem 11. *Let X be a Baire space, Y be a W-space, Z be a regular space and $f : X \times Y \to Z$ be a horizontally quasi-continuous function which is continuous with respect to the second variable. Then f is jointly quasi-continuous.*

PROOF. Suppose that f is not quasi-continuous at some point (x_0, y_0) in $X \times Y$. Then there are neighborhoods U, H and G of x_0, y_0 and $f(x_0, y_0)$, respectively, such that $f(U' \times H') \nsubseteq G$ for all nonempty open subsets U' of U and $V' \subseteq H'$.

Since Z is regular, we can find some open subset G_1 of Z such that

$$f(x_0, y_0) \in G_1 \subseteq \overline{G_1} \subset G.$$

By horizontal quasi-continuity of f, there is a nonempty open subset U_1 of U and $y_0' \in H$ such that $f(U_1 \times \{y_0'\}) \subseteq G$.

In order to get a contradiction, we simultaneously define a strategy $t = \{t_n\}$ for β in $\mathcal{BM}(X)$ and a strategy $s = \{s_n\}$ for the player $\mathcal{P}$ in $\mathcal{G}(Y, \{y_0'\})$ as follows.

Let $t_1(\emptyset) = U_1$ be the first move of the player β and V_1 be the answer of the player α to this movement. If H_1 is the first choice of the player $\mathcal{O}$ in $\mathcal{G}(Y, \{y_0'\})$, then by our assumption, we can find some $(x_1, y_1') \in V_1 \times (H_1 \cap H)$ such that $f(x_1, y_1') \notin G$. There is a nonempty open subset U_2 of V_1 and $y_1 \in H \cap H_1$ such that $f(U_2 \times \{y_1\}) \subseteq Z \setminus \overline{G_1}$, since f is horizontally quasi-continuous. Define $t_2(U_1, V_1) = U_2$ and $s_1(H_1) = y_1$.

In general, in step n, let the partial plays $H_1, y_1, \ldots, H_n$ and $U_1, V_1, \ldots, V_n$ be specified such that $f(U_i, \times\{y_i\}) \subseteq Z \setminus \overline{G_1}$ for each $1 \leq i \leq n$. By our assumption, $f(V_n \times (H_n \cap H)) \nsubseteq G$. Therefore, we can find some $(x_n, y_n') \in V_n \times (H_n \cap H)$ such that $f(x_n, y_n') \in Z \setminus \overline{G_1}$. By horizontal quasi-continuity of f, we can find some nonempty open subset U_{n+1} of V_n and $y_n \in H \cap H_n$ such that $f(U_n \times \{y_n\}) \subseteq Z \setminus \overline{G_1}$. Let $t_{n+1}(U_1, V_1, \ldots, V_n) = U_{n+1}$ and $s_n(H_1, \ldots, H_n) = y_n$. Hence, strategies t and s are inductively defined.

Since X is a Baire space, α wins some t-play, say $p = (U_n, V_n)_n$. Hence $\bigcap_{n \geq 1} U_n \neq \emptyset$. Let x^* be a point in $\bigcap_{n \geq 1} U_n$. Let $g = (H_n, y_n)_n$ be the corresponding s-play. Since Y is a W-space, $\mathcal{O}$ wins the game g. Hence, $y_n \to y_0'$. Thanks to continuity of $y \mapsto f(x^*, y)$ at y_0' and the fact that $f(x^*, y_0') \in G_1$, there is a neighborhood H' of y_0' such that $f(x^*, y) \in G_1$ for each $y \in H'$. Since $y_n \to y_0'$, we can find some $n_0 \in \mathbb{N}$ such that $y_n \in H'$ for each $n \geq n_0$. Hence, $f(x^*, y_n) \in G_1$ for each $n \geq n_0$. However, for each $n \geq 1$, we have $f(x^*, y_n) \in Z \setminus \overline{G_1}$. This contradiction proves our result. □

Definition 12. *Let X be a topological space and let $\mathcal{F}$ be a family of nonempty closed and separable subspaces of X. Then $\mathcal{F}$ is called rich [8] if the following conditions are satisfied:*

(i) For every separable subspace Y of X, there exists an $F \in \mathcal{F}$ such that $Y \subseteq F$.

(ii) For every increasing sequence $\{F_n\}_{n\geq 1}$ in $\mathcal{F}$, $\overline{\bigcup_{n\geq 1} F_n} \in \mathcal{F}$.

Corollary 13. *Let X be a Baire space, Y be a W-space, Z be a regular weakly developable space and $f : X \times Y \to Z$ be a horizontally quasi-continuous function which is continuous with respect to the second variable. If Y possesses a rich family $\mathcal{F}$ of Baire subspaces, then f is jointly continuous on a dense G_δ subset of $X \times Y$.*

Proof. In [8, Theorem 4], it is shown that if a W-space has a rich family of W-spaces, then for every Baire space X, the product space $X \times Y$ is Baire, so that the result follows from Theorem 11 and [5, Theorem 4.1]. □

References

[1] K. Bögel, *Uher die stetigkeit und die Schwankung yon funktionen zweiier reeller Veranderlicher,* Math. Ann., **81** (1920), 64–93.

[2] A. Bouziad and J. P. Troallic, *Lower quasicontinuity, joint continuity and related concepts*, Topology Appl., **157** (2010), 2889–2894.

[3] G. Choquet, *Lectures on Analysis*, Vol 1. W. A. Benjamin Inc., New-York (1969).

[4] G. Gruenhage, *Infinite games and generalizations of first-countable spaces*, Topology Appl., **6** (1976), 339–352.

[5] L. Hola and Z. Piotrowski, *Set of continuity of functions with values in generalized metric spaces*, Tatra Mt. Math. Publ., **42** (2009), 149–160.

[6] S. Kempisty, *Sur les fonctions quasicontinues*, Fund. Math., **19** (1932), 184–197.

[7] P. S. Kenderov, S. Kortezov S and W. B. Moors, *Continuity points of quasi-continuous mappings*, Topology Appl., **109** (2001), 321–346.

[8] P. Lin and W. B. Moors, *Rich families, W-spaces and the product of Baire spaces*, Math. Balkanica (N.S.), **22** (2008), 175-187.

[9] N. F. G. Martin, *Quasi-continuous functions on product spaces*, Duke Math. J., **28** (1961), 39–43.

[10] V. K. Maslyuchenko, V. V. Mykhailyuk and O. I. Shyshyna, *Joint continuity of horizontally quasicontinuous mappings with values in σ-metrizable spaces*, Mat. Metodi Fiz.-Mekh. Polya, **45** (2002), 42–46.

[11] V. K. Maslyuchenko and V. V. Nesterenko, *Joint continuity and quasi-continuity of horizontally quasicontinuous mappings*, Ukrainian Math. J., **52** (2000), 1952–1955.

[12] Y. Mibu, *On quasi-continuous mappings defined on product spaces*, Proc. Japan. Acad., **192** (1958), 189–192.

[13] A. K. Mirmostafaee, *Topological games and strong quasi-continuity*, Banach J. Math. Anal., **5** (2011), 131–137.

[14] A. K. Mirmostafaee, *Norm continuity of quasi-continuous mappings and product spaces*, Topology Appl., **157** (2010), 530–535.

[15] J. C. Oxtoby, *Cartesian products of Baire spaces*, Fund. Math., **49** (1961), 157–166.

[16] J. C. Oxtoby, *Measure and Category*, Grad. Texts Math. 2. Springer-Verlag (1971).

[17] Z. Piotrowski, *On the theorem of Y. Mibu and G. Debs on separate continuity*, Internat. J. Math. Math. Sci., **19** (1996), 495–500.