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OUTER MEASURES ON THE REAL LINE BY WEAK SELECTIONS

Abstract

A weak selection on an infinite set X is a function $f : [X]^2 \to X$ such that $f(F) \in F$ for each $F \in [X]^2 := \{E \subseteq X : |E| = 2\}$. If $f : [X]^2 \to X$ is a weak selection and $x, y \in \mathbb{R}$, then we say that $x <_f y$ if $f(\{x, y\}) = x$ and $x \leq_f y$ if either $x = y$ or $x <_f y$. Given a weak selection f on X and $x, y \in X$, we let $(x, y]_f = \{z \in X : x <_f z \leq_f y\}$. If $f : [\mathbb{R}]^2 \to \mathbb{R}$ is a weak selection and $A \subseteq \mathbb{R}$, then we define

$$\lambda_f^*(A) := \inf\Big\{\sum_{n\in\mathbb{N}} |b_n - a_n| \,:\, A \subseteq \bigcup_{n\in\mathbb{N}} (a_n, b_n]_f\Big\}$$

if there exists a countable cover by semi open f-intervals of A, and if there is not such a cover, then we say that $\lambda_f^*(A) = +\infty$. This function $\lambda_f^* : \mathcal{P}(\mathbb{R}) \longrightarrow [0, +\infty]$ is an outer measure on the real line $\mathbb{R}$ which generalizes the Lebesgue outer measure. In this paper, we show several interesting properties of these kind of outer measures.

1 Preliminaries

The Euclidian (standard) order on the real numbers will be denoted by $\leq$. The Lebesgue outer measure will be denoted by λ^* and $\mathcal{M}$ and $\mathcal{N}$ will stand

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for the family of all Lebesgue measurable sets and the family of all null (zero measure) sets, respectively. The cardinality of the real line shall be denoted by $\mathfrak{c}$.

For an infinite set X, we let $[X]^2 = \{F \subseteq X : |F| = 2\}$. A function $f : [X]^2 \to X$ is called a *weak selection* if $f(F) \in F$ for all $F \in [X]^2$. The most common example of a weak selection on the real line is the Euclidian weak selection $f_E : [\mathbb{R}]^2 \to \mathbb{R}$ given by

$$f_E(\{x, y\}) = x \quad \text{iff} \quad x < y,$$

for each $\{x, y\} \in [\mathbb{R}]^2$. The weak selections have been recently studied in topology (see for instance [2], [4], [5], [6], [7], [9], [11], [10] and [12]). One important property of the weak selections is that each one of them generates a topology which has many strong topological properties like complete regularity and Hausdorff property (see [4], [7] and [11]). In this article we shall only consider weak selections defined on $\mathbb{R}$.

For a weak selection $f : [X]^2 \to X$ and $\{x, y\} \in [X]^2$, we say $x <_f y$ if $f(\{x, y\}) = x$, and for $x, y \in \mathbb{R}$ we define $x \leq_f y$ if either $x <_f y$ or $x = y$. This relation $\leq_f$ is reflexive, antisymmetric and linear, but it could fail to be transitive: To see this let us define $f : [\mathbb{R}]^2 \to \mathbb{R}$ by $f(\{0, 1\}) = 0$, $f(\{1, 2\}) = 1$, $f(\{0, 2\}) = 2$ and for the other points we keep the Euclidian order. It is evident that $\leq_f$ is not transitive.

If f is a weak selection and $a, b \in \mathbb{R}$, then the f-intervals are

$$(a, b)_f := \Big\{x \in X : a <_f x <_f b\Big\},$$
$$(a, b]_f := \Big\{x \in X : a <_f x <_f b\Big\},$$
$$(a, \to)_f := \Big\{x \in X : a <_f x\Big\}, \text{ etc.}$$

For the Euclidian weak selection f_E we just write (a, b), $(a, b]$, $(a, \to)$ etc.. In the notation (a, b) we shall understand that $a < b$. Meanwhile, in the general notation $(a, b)_f$ we do not require that $a <_f b$ since the relation $<_f$ is not always transitive.

Definition 1.1. Let f be a weak selection and $r \in \mathbb{R}$. We say that r is f-*minimal* if $r <_f x$, for every x in $\mathbb{R} \setminus \{r\}$. If $x <_f r$ for all $x \in \mathbb{R} \setminus \{r\}$, then r is called f-*maximal*.

We remark that every weak selection f admits at most one f-minimal point and also at most one f-maximal point.

Let us make some comments concerning the f-intervals. It is clear that if $r \in \mathbb{R}$ is f-maximal for some weak selection f, then $(r, s)_f = \emptyset$ for all

$s \in \mathbb{R} \setminus \{r\}$, and if r is f-minimal, then $(s,r)_f = \emptyset$ for all $s \in \mathbb{R} \setminus \{r\}$. Now, we shall define a weak selection f to see that the f-intervals and the Euclidian intervals could be quite different one from the other: Consider the Euclidean interval (a,b), $d \notin (a,b)$ and its middle point c. Define

$$f(\{x,y\}) := \begin{cases} x & \text{if} \quad x \in (a,c] \quad \text{and} \quad y = a \\ a & \text{if} \quad x = a \quad \text{and} \quad y = d \\ d & \text{if} \quad x = b \quad \text{and} \quad y = d \\ x & \text{if} \quad x < y \quad \text{otherwise} \end{cases}$$

for each $\{x,y\} \in [\mathbb{R}]^2$. Then we have that $(a,b)_f = (c,b) \cup \{d\}$ and hence $(a,b) \nsubseteq (a,b)_f \nsubseteq (a,b)$. Also we can define a weak selection f so the f-interval $(0,1)_f$ has only one single point:

$$f(\{x,y\}) := \begin{cases} x & \text{if} \quad y = 0 \quad \text{and} \quad x \in (0, \frac{1}{2}) \\ 1 & \text{if} \quad x = 1 \quad \text{and} \quad y \in (\frac{1}{2}, 1) \\ x & \text{if} \quad x < y \quad \text{otherwise} \end{cases}$$

for each $\{x,y\} \in [\mathbb{R}]^2$. It is evident that $(0,1)_f = \{\frac{1}{2}\}$.

As it is mentioned in the Abstract, the main purpose of this article is to introduce new outer measures on the real line by slightly generalizing the Lebesgue outer measure and by using weak selections. By analyzing several examples, we shall see how the analytical properties of these outer measures interact with the combinatorial properties of the weak selection. In the first chapter, we give some general properties of our outer measures. In the second chapter, several examples are given comparing them with the known properties of the Lebesgue outer measure. We are completely sure that this king of outer measures will provide several nice examples of new measure spaces with some exotic properties.

2 Outer measures by weak selections

The concept of outer measure is due to C. Carathéodory and it is defined as follows.

Definition 2.1. Let X be an infinite set. A set function $\mu : \mathcal{P}(X) \to [0, \infty)$ is called an outer measure if it satisfies the following properties:

1. $\mu(\emptyset) = 0$.
2. If $A, B \subseteq X$ and $A \subseteq B$, then $\mu(A) \subseteq \mu(B)$.
3. If $(A_n)_{n\in\mathbb{N}}$ is a sequence of subsets of X, $\mu(\bigcup_{n\in\mathbb{N}} A_n) \leq \sum_{n\in\mathbb{N}} \mu(A_n)$.

For basic properties of outer measure we refer the reader to the books [1], [3] and [8]. Recall that the Lebesgue outer measure uses the length of the Euclidian intervals and countable covers by them to estimate the outer measure of an arbitrary subset of $\mathbb{R}$. By using the basic idea of the definition of the Lebesgue outer measure, we introduce the following notion.

Definition 2.2. Let f be a weak selection on the real line $\mathbb{R}$. The *outer measure* induced by f is the set function $\lambda_f^* : \mathcal{P}(\mathbb{R}) \to [0, +\infty]$ defined, for every $A \subseteq \mathbb{R}$, by

$$\lambda_f^*(A) := \inf \Big\{ \sum_{n\in\mathbb{N}} |b_n - a_n| \,:\, A \subseteq \bigcup_{n\in\mathbb{N}} (a_n, b_n]_f \Big\}$$

if there exists a countable cover of A by semi open f-intervals, and if there is not a countable cover of this form, then we say $\lambda_f^*(A) = +\infty$.

Clearly, for the Euclidian weak selection f_E we have that $\lambda^* = \lambda_{f_E}^*$. The proof of the following theorem is similar to the one for the Lebesgue outer measure.

Theorem 2.3. *For every weak selection f, λ_f^* is an outer measure on $\mathbb{R}$.*

The outer measure induced by a weak selection f will be called f-*outer measure*, and $\mathcal{M}_f$ will denote the family of all λ_f^*- measurable sets and $\mathcal{N}_f$ will denote the family of all λ_f^*-null sets.

It is well known that the singleton sets of the reals are null sets. For the outer measures introduced in Definition 2.2 a singleton set can have f-outer measure equal to either 0 or $+\infty$, for any weak selection f. We shall prove this assertion in the next theorem, but first we need to state an easy lemma.

Lemma 2.4. *Let f be a weak selection and $r \in \mathbb{R}$.*

1. *If $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ is a sequence that converges to r and $a_n <_f r$ holds for each $n \in \mathbb{N}$, then $\lambda_f^*(\{r\}) = 0$.*
2. *If $a <_f r$ and $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ is a sequence that converges to a such that $r <_f a_n$ for all $n \in \mathbb{N}$, then $\lambda_f^*(\{r\}) = 0$.*
3. *If $r <_f a$ and $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ is a sequence that converges to a such that $a_n <_f r$ for all $n \in \mathbb{N}$, then $\lambda_f^*(\{r\}) = 0$.*

Theorem 2.5. *For every weak selection f and for every $r \in \mathbb{R}$ we have that*

$$\lambda_f^*(\{r\}) = \begin{cases} 0 \\ +\infty \end{cases}$$

Proof. Clearly, if r is f-minimal, then $r \notin (a,b]_f$ for every $a, b \in \mathbb{R}$. Thus, in this case, we have that $\lambda_f^*(\{r\}) = +\infty$. If r is f-maximal, then take a sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathbb{R} \setminus \{r\}$ that converges to r. As $a_n <_f r$ for each n in $\mathbb{N}$, by Lemma 2.4 (1), we obtain that the f-outer measure of $\{r\}$ is zero. Suppose that we can find two points $a, b \in \mathbb{R}$ so that $r \in (a,b)_f$. Assume the negation of Lemma 2.4(2). Then there is $\delta > 0$ such that $c <_f r$, for each $c \in [a-\delta, a+\delta]$. We have to consider the following two cases:

Case I. $a < r$. We know that $a + \delta <_f r$ and we may assume, in the Euclidian order, that $a + \delta < r$. Consider the set

$$A = \Big\{ s \in \mathbb{R} \,:\, a < s < r \quad \text{and} \quad s <_f r \Big\},$$

which is not void since $(a, a+\delta] \subseteq A$. Let $d = \sup A$. If $d = r$, then we are done by clause (1) of Lemma 2.4. Suppose that $d < r$. Then choose two sequences $(c_n)_{n\in\mathbb{N}} \subseteq (\leftarrow, d)$ with $c_n \to d$ and $c_n <_f r$ for every $n \in \mathbb{N}$, and $(d_n)_{n\in\mathbb{N}} \subseteq (d, r)$ with $d_n \to d$. Notice that $r <_f d_n$ for every $n \in \mathbb{N}$. Then, $r \in (c_n, d_n]_f$, for all $n \in \mathbb{N}$, and hence $\lambda_f^*(\{r\}) = 0$.

Case II. $r < a$. We may assume that $r < a - \delta$ and consider the set

$$B = \Big\{ s \in \mathbb{R} \,:\, r < s < a \quad \text{and} \quad s <_f r \Big\},$$

which is not empty since $[a-\delta, a) \subseteq B$. Let $c = \inf B$. If $c = r$, then we are done by clause (1) of Lemma 2.4. Suppose that $r < c$. Choose two sequences $(c_n)_{n\in\mathbb{N}} \subseteq (c, a)$ with $c_n \to c$ and $c_n <_f r$ for every $n \in \mathbb{N}$, and $(d_n)_{n\in\mathbb{N}} \subseteq (r, c)$ with $d_n \to c$. Then, we must have that $r \in (c_n, d_n]_f$, for all $n \in \mathbb{N}$, and hence we obtain that $\lambda_f^*(\{r\}) = 0$. □

We list some direct consequences of Theorem 2.5.

Corollary 2.6. *Let f a weak selection and $r \in \mathbb{R}$.*

1. *If r is f-maximal, then $\lambda_f^*(\{r\}) = 0$.*
2. *r is f-minimal if and only if $\lambda_f^*(\{r\}) = +\infty$.*

In particular, if there is an f-minimal point in $\mathbb{R}$, then $\lambda_f^*(\mathbb{R}) = +\infty$.

Corollary 2.7. *If $X \subseteq \mathbb{R}$ is countable and f is any weak selection, then*

$$\lambda_f^*(X) = \left\{ \begin{array}{c} 0 \\ +\infty \end{array} \right.$$

Corollary 2.8. *For every weak selection f, each countable subset of $\mathbb{R}$ is λ_f^*-measurable.*

Corollary 2.9. *For every weak selection f, $|\mathcal{M}_f| \geq \mathfrak{c}$.*

Unfortunately, we could not answer the following questions.

Question 2.10. Is there a weak selection f such that $|\mathcal{M}_f| = \mathfrak{c}$?

Question 2.11. Given an arbitrary weak selection f, is there $A \in \mathcal{M}_f$ such that $|A| = |A \setminus \mathbb{R}| = \mathfrak{c}$?

We shall see next that the Lebesgue outer measure coincides with an outer measure induced by a weak selection that only changes the Euclidian order on a countable subset of the reals.

Theorem 2.12. *Let f be a weak selection. Assume that there is a countable subset M of $\mathbb{R}$ such that*

$$f(\{x,y\}) = x \quad \textit{if and only if} \quad x < y \quad \textit{and} \quad |\{x,y\} \cap M| \leq 1,$$

for every $\{x,y\} \in [\mathbb{R}]^2$. Then $\lambda_f^*(A) = \lambda^*(A)$ for all $A \subseteq \mathbb{R}$.

Proof. First, notice that $(a,b]_f = (a,b]$ whenever $a, b \notin M$. On the other hand, we known that if $B \subseteq \mathbb{R}$, then

$$\lambda^*(B) = \inf \Big\{ \sum_{n\in\mathbb{N}} |y_n - x_n| \,:\, B \subseteq \bigcup_{n\in\mathbb{N}} (x_n, y_n] \text{ and } \forall n \in \mathbb{N}(x_n, y_n \in \mathbb{R} \setminus M) \Big\}.$$

So, $\lambda_f^*(B) \leq \lambda^*(B)$ for every $B \subseteq \mathbb{R}$. Fix $z \in M$. For each $n \in \mathbb{N}$, choose two points $a_n, b_n \in \mathbb{R} \setminus M$ so that

$$z \in (a_n, b_n] \quad \text{and} \quad b_n - a_n < \frac{1}{2^n}.$$

It then follows that

$$\lambda_f^*(\{z\}) \leq \lambda_f^*\big((a_n, b_n]\big) = \lambda_f^*\big((a_n, b_n]_f\big) \leq \frac{1}{2^n},$$

for every $n \in \mathbb{N}$. Thus, $\lambda_f^*(\{z\}) = 0$ for each $z \in M$. So, $\lambda_f^*(M) = 0$ and hence the set M is λ_f^*-null. Now, fix $A \subseteq \mathbb{R}$. Applying the Carathéodory's condition to M, we obtain that

$$\lambda_f^*(A) = \lambda_f^*(A \cap M) + \lambda_f^*(A \setminus M) = \lambda_f^*(A \setminus M).$$

To finish the proof we only need to show that the f-outer measure of $A \setminus M$ is equal to its Lebesgue outer measure. We already know that $\lambda_f^*(A \setminus M) \leq \lambda^*(A \setminus M)$. Assume that $\lambda_f^*(A \setminus M) < \lambda^*(A \setminus M)$. Then, we can find $\epsilon > 0$ and a countable cover of $A \setminus M$ by f-intervals $\{(x_n, y_n]_f : n \in \mathbb{N}\}$ satisfying

$$\sum_{n\in\mathbb{N}} |y_n - x_n| + \epsilon < \lambda^*(A \setminus M).$$

For each $n \in \mathbb{N}$ choose ϵ_n for which $\epsilon_n < \frac{\epsilon}{2^{n+1}}$ and $x_n - \epsilon_n$, $y_n + \epsilon_n \notin M$. Observe that $(x_n, y_n]_f \setminus M \subseteq (x_n - \epsilon_n, y_n + \epsilon_n]$ for all $n \in \mathbb{N}$. Hence,

$$A \setminus M \subseteq \bigcup_{n\in\mathbb{N}} (x_n - \epsilon_n, y_n + \epsilon_n]$$

and so

$$\lambda^*(A \setminus M) \leq \sum_{n\in\mathbb{N}} |y_n - x_n| + \sum_{n\in\mathbb{N}} \frac{\epsilon}{2^n} = \sum_{n\in\mathbb{N}} |y_n - x_n| + \epsilon < \lambda^*(A \setminus M),$$

but this is a contradiction. Thus, we have proved that $\lambda_f^*(A) = \lambda^*(A \setminus M)$. Therefore, $\lambda_f^*(A) = \lambda^*(A)$. □

3 Examples

In this section, we give several examples of weak selections that provide several very interesting outer measures on $\mathbb{R}$.

In the first example, we shall see that an Euclidean interval can have infinite f-outer measure for a suitable weak selection f.

Example 3.1. Let (a, b) an Euclidean interval where $a < b$. We shall use transfinite induction to define the weak selection f. To do that we enumerate all sequences of $\mathbb{R}$ as $\left\{S_\xi : \xi < \mathfrak{c}\right\}$ and each S_ξ as $\left\{x_n^\xi : n \in \mathbb{N}\right\}$. Take $r_0 \in (a, b) \setminus S_0$ and define $r_0 <_f x_n^0$, for all $n \in \mathbb{N}$. Let $\theta < \mathfrak{c}$ and suppose that for each $\xi < \theta$ we have carefully chosen a real number $r_\xi \in \mathbb{R}$. Fix

$$r_\theta \in (a, b) \setminus \left[\bigcup_{\xi\leq\theta} S_\xi \cup \left\{r_\xi : \xi < \theta\right\}\right]$$

and define $r_\theta <_f x_n^\xi$ for each $n \in \mathbb{N}$ and for each $\xi \leq \theta$. The weak selection f will preserve the Euclidean order in the rest of the two point sets which were not considered above. Let $\{(a_n, b_n]_f : n \in \mathbb{N}\}$ be a countable family of f-intervals. Pick $\theta < \mathfrak{c}$ so that $x^\theta_{2n} = a_n$ and $x^\theta_{2n+1} = b_n$ for all $n \in \mathbb{N}$. By definition, we know that $r_\theta <_f x^\theta_n$, for every $n \in \mathbb{N}$, and so

$$r_\theta \in (a,b) \setminus \left[\bigcup_{n=0}^{\infty} (a_n, b_n]_f \right].$$

This shows that (a,b) cannot be cover by a countable family of semi open f-intervals. Therefore, $\lambda^*_f\big((a,b)\big) = +\infty$.

Now we shall see that the f-outer measures are not in general translation invariant.

Example 3.2. Fix $r \in \mathbb{R}$ and consider the weak selection f given by

$$f(\{x,y\}) := \begin{cases} r \quad \text{if} \quad x \in \mathbb{R} \setminus \{r\} \quad \text{and} \quad y = r \\ \\ x \quad \text{if} \quad x < y \quad \text{and} \quad x \neq r \neq y \end{cases}$$

for each $\{x,y\} \in [\mathbb{R}]^2$. As the point r is f-minimal, by Corollary 2.6, the f-outer measure of the set $\{r\}$ is infinite. Therefore, by Theorem 2.5, we have that

$$\lambda^*_f(\{r\} + 1) = \lambda^*_f(\{r+1\}) = 0 < +\infty = \lambda^*_f(\{r\}).$$

We shall continue giving examples of f-outer measures that satisfy some unusual properties when we compare them with the properties of the Lebesgue outer measure.

Example 3.3. If the point $a \in \mathbb{R}$ is f-maximal for some weak selection f, then $\lambda^*_f((a,b)_f) = 0$ since $(a,b)_f = \emptyset$, for every $b \in \mathbb{R} \setminus \{a\}$.

The most trivial outer measure on an infinite set is the one taking the constant value 0 everywhere. But by using weak selections we can collapse the outer measure of an Euclidian open interval to 0 as it is shown in the next example.

Example 3.4. The f-outer measure of an Euclidean open interval could be zero for some weak selection f.

Fix an Euclidean open interval (a,b). Without loss of generality, we may assume that $\frac{1}{n+1} < a$ for each $n \in \mathbb{N}$. Our weak selection f is defined by

$$f(\{x,y\}) := \begin{cases} x & \text{if} \quad x \in (a,b) \quad \text{and} \quad y = \frac{1}{n+1} \\ y & \text{if} \quad x \in (a,b) \quad \text{and} \quad y = -\frac{1}{n+1} \\ x & \text{if} \quad x < y \quad \text{otherwise} \end{cases}$$

for each $\{x,y\} \in [\mathbb{R}]^2$ and for each $n \in \mathbb{N}$. By the definition of the weak selection, we have $(a,b) \subseteq \left(-\frac{1}{n+1}, \frac{1}{n+1}\right]_f$, for every $n \in \mathbb{N}$. Hence, $\lambda_f^*\big((a,b)\big) \leq \frac{2}{n+1}$, for each $n \in \mathbb{N}$. Therefore, $\lambda_f^*\big((a,b)\big) = 0$.

Example 3.5. The f-outer measure of an Euclidean interval could be positive and smaller than its Lebesgue outer measure.

To see this consider the weak selección f defined by

$$f(\{x,y\}) := \begin{cases} y & \text{if} \quad x = \frac{1}{3} \quad \text{and} \quad y \in \left(\frac{1}{3}, \frac{2}{3}\right] \\ x & \text{if} \quad x < y \quad \text{otherwise} \end{cases}$$

for each $\{x,y\} \in [\mathbb{R}]^2$. Notice that $\left(\frac{1}{3}, \frac{2}{3}\right]_f \cap \left(\frac{1}{3}, \frac{2}{3}\right] = \emptyset$. We claim that

$$(0,1] = (0, \frac{1}{3}]_f \cup (\frac{2}{3}, 1]_f.$$

Indeed, fix $r \in (0,1]$.

- If $r \in (0, \frac{1}{3}]$, then $0 <_f r$ and $r <_f \frac{1}{3}$. So, $r \in (0, \frac{1}{3}]_f$.
- If $r \in (\frac{2}{3}, 1]$, then $\frac{2}{3} <_f r$ and $r <_f 1$. Hence, $r \in (\frac{2}{3}, 1]_f$.
- If $r \in (\frac{1}{3}, \frac{2}{3}]$, then $0 <_f r$ and $r <_f \frac{1}{3}$. That is, $r \in (0, \frac{1}{3}]_f$.

Thus we have shown that $(0,1] \subseteq (0, \frac{1}{3}]_f \cup (\frac{2}{3}, 1]_f$. Evidently, by definition,

$$(0, \frac{1}{3}]_f \cup (\frac{2}{3}, 1]_f \subseteq (0,1].$$

This shows our claim.

Since $(0,\frac{1}{3}]_f \cup (\frac{2}{3},1]_f$ covers $(0,1]$, we must have that $\lambda^*_f\big((0,1]\big) \leq \frac{2}{3}$. On the other hand, it follows from the definition of the weak selection that

$$[(0,\frac{1}{3}) \cup (\frac{2}{3},1)] \cap (a,b)_f = [(0,\frac{1}{3}) \cup (\frac{2}{3},1)] \cap (a,b),$$

for every $a, b \in \mathbb{R}$ with $a < b$. Hence,

$$\frac{2}{3} = \lambda^*\big((0,\frac{1}{3}) \cup (\frac{2}{3},1)\big) = \lambda^*_f\big((0,\frac{1}{3}) \cup (\frac{2}{3},1)\big) \leq \lambda^*_f\big((0,1]\big).$$

Therefore, $\lambda^*_f\big((0,1]\big) = \frac{2}{3}$.

In the next example, we give a weak selection f for which all subsets of $\mathbb{R}$ are λ^*_f-measurable.

Example 3.6. There exists a weak selection f such that $\mathbb{R} \in \mathcal{N}_f$. In particular, every subset of $\mathbb{R}$ is λ^*_f-measurable.

Construct a sequence $\big((a^i_z)_{z\in\mathbb{Z}}\big)_{i\in\mathbb{N}}$ of sequences of real numbers as follows:

First set $a^0_z = z$ for all $z \in \mathbb{Z}$. Suppose that the sequence $(a^i_z)_{z\in\mathbb{Z}}$ has been defined for $i \in \mathbb{N}$. Then, for each $z \in \mathbb{Z}$ we define a^{i+1}_{2z} as the middle point of the interval (a^i_{z+1}, a^i_{z+2}), and $a^{i+1}_{2z-1} = a^i_{z+1}$. Then we have the following properties of these sequences.

1. The sequence $(a^i_z)_{z\in\mathbb{Z}}$ is a not bounded neither below nor above, discrete in $\mathbb{R}$ and strictly increasing, for all $i \in \mathbb{N}$,
2. $a^i_{z+1} < a^{i+1}_{2z}$ for all $(i,z) \in \mathbb{N} \times \mathbb{Z}$, and
3. $a^i_{z+1} - a^i_z < \frac{1}{2^{i-1}}$ for all $(i,z) \in \mathbb{N} \times \mathbb{Z}$.

Let $A = \{a^i_z : (i,z) \in \mathbb{N} \times \mathbb{Z}\}$. By definition, we have that if $(i,z) \in \mathbb{N} \times \mathbb{Z}$, then $a^i_{z+1} < a^{i+j}_{2^j z}$ for every positive $j \in \mathbb{N}$. The weak selection $f : [\mathbb{R}]^2 \to \mathbb{R}$ is defined by:

i. For every pair $(i,z) \in \mathbb{N} \times \mathbb{Z}$ and for every real $r \in (a^i_z, a^i_{z+1}) \setminus A$ we define $a^{i+j}_{2^j z} <_f r <_f a^{i+j}_{2^j z+1}$, for each positive $j \in \mathbb{N}$.

ii. The Euclidean order is preserved in the rest of the points not being considered above.

Notice that there are neither f-minimal point nor f-maximal point. Thus, by Corollary 2.7, $\lambda^*_f(A) = 0$. Let us calculate $\lambda^*_f(\mathbb{R} \setminus A)$. Fix $(i,z) \in \mathbb{N} \times \mathbb{Z}$. We know that

$$(a^i_z, a^i_{z+1}) \setminus A \subseteq (a^{i+j}_{2^j z}, a^{i+j}_{2^j z+1}]_f,$$

for each positive $j \in \mathbb{N}$. Hence,

$$\lambda_f^*\big((a_z^i, a_{z+1}^i) \setminus A\big) \leq a_{2^j z+1}^{i+j} - a_{2^j z}^{i+j} \leq \frac{1}{2^{i+j-1}},$$

for every positive $j \in \mathbb{N}$. This implies that $\lambda_f^*\big((a_z^i, a_{z+1}^i) \setminus A\big) = 0$, for all $(i,z) \in \mathbb{N} \times \mathbb{Z}$. So,

$$\lambda_f^*\big(\mathbb{R} \setminus A\big) \leq \lambda_f^*\big(\bigcup_{z\in\mathbb{Z}}[(a_z^0, a_{z+1}^0) \setminus A]\big) \leq \sum_{z\in\mathbb{Z}} \lambda_f^*\big((a_z^0, a_{z+1}^0) \setminus A\big) = 0.$$

Therefore, $\lambda_f^*(\mathbb{R}) = 0$.

By a slight modification of the previous construction we obtain the following example.

Example 3.7. For each proper subset $N \subset \mathbb{R}$ there exists a weak selection f such that $N \in \mathcal{N}_f$ and $\lambda_f^*(\mathbb{R} \setminus N) = +\infty$.

Fix $r \in \mathbb{R} \setminus N$ and consider the sequence $\big((a_z^i)_{z\in\mathbb{Z}}\big)_{i\in\mathbb{N}}$ constructed in Example 3.6. Assume, without loss of generality, that $r \neq a_z^i \notin N$ for each $(i,z) \in \mathbb{N} \times \mathbb{Z}$, and let $A = \{a_z^i : (i,z) \in \mathbb{N} \times \mathbb{Z}\}$. Then define f as follows:

1. $r <_f s$ for all $s \in \mathbb{R} \setminus \{r\}$;
2. for every $(i,z) \in \mathbb{N} \times \mathbb{Z}$ and for every $s \in (a_z^i, a_{z+1}^i) \setminus \big(A \cup \{r\}\big)$ we define $a_{2^j z}^{i+j} <_f s <_f a_{2^j z+1}^{i+j}$, for each positive $j \in \mathbb{N}$; and
3. the Euclidean order is preserved in the rest of the points not being considered above.

In connection with Examples 3.6 and 3.7, we pose the following question.

Question 3.8. Given an arbitrary weak selection f such that $0 < \lambda_f^*(\mathbb{R}) < +\infty$, is $\mathcal{P}(\mathbb{R}) \setminus \mathcal{M}_f \neq \emptyset$?

Example 3.9. Let (a,b) be an Euclidian interval. Then there are a weak selection f and $r \in \mathbb{R}$ such that $E + r \in \mathcal{M}_f$ and $\lambda^*\big(E\big) = \lambda_f^*\big(E\big)$, for every $E \subseteq (a,b)$.

Choose a positive $r \in \mathbb{R}$ so that $[-1,1] \cap (a+r, b+r) = \emptyset = [a,b] \cap (a+r, b+r)$. We define the weak selection f as follows:

$$f(\{x,y\}) := \begin{cases} x & \text{if} \quad x \in (a+r, b+r) \quad \text{and} \quad y = \frac{1}{n+1} \\[2ex] -\frac{1}{n+1} & \text{if} \quad x \in (a+r, b+r) \quad \text{and} \quad y = -\frac{1}{n+1} \\[2ex] x & \text{if} \quad x < y \quad \text{otherwise} \end{cases}$$

for every $n \in \mathbb{N}$ and for each $\{x, y\} \in [\mathbb{R}]^2$. By definition, we have that

$$(a+r, b+r) \subseteq (-\tfrac{1}{n+1}, \tfrac{1}{n+1})_f,$$

for every $n \in \mathbb{N}$. Hence, $\lambda_f^*\big((a+r, b+r)\big) = 0$ and so $\lambda_f^*\big(E + r\big) = 0$, for every $E \subseteq (a, b)$. Thus we obtain that $E + r \in \mathcal{M}_f$. Since $(x, y] \cap (a, b) = (x, y]_f \cap (a, b)$ for distinct $x, y \in \mathbb{R}$, we conclude that $\lambda^*\big(E\big) = \lambda_f^*\big(E\big)$, for each $E \subseteq (a, b)$.

As an interesting consequence of the previous example is that if E is a non-measurable subset of (a, b) (for the existence of this set see, for instance, the book [1, Th. 1.4.7]), then there are a weak selection f and $r \in \mathbb{R}$ such that $E + r \in \mathcal{M}_f$ and $\lambda^*\big(E\big) = \lambda_f^*\big(E\big)$.

We know that every Euclidean interval (a, b) is Lebesgue measurable, contrary to this fact we have the following example.

Example 3.10. There is a weak selection f such that $(0, 1)_f$ is not λ_f^*-measurable.

Our weak selection f is given by the rule:

$$f(\{x, y\}) := \begin{cases} x & \text{if} \quad x \in (3,4) \quad \text{and} \quad y = 1 \\ 0 & \text{if} \quad x \in (3,4) \quad \text{and} \quad y = 0 \\ x & \text{if} \quad x \in (2,3) \quad \text{and} \quad y = 0 \\ -1 & \text{if} \quad x \in (2,3) \quad \text{and} \quad y = -1 \\ x & \text{if} \quad x \in (2,4) \quad \text{and} \quad y = 6 \\ 5 & \text{if} \quad x \in (2,4) \quad \text{and} \quad y = 5 \\ x & \text{if} \quad x < y \quad \text{otherwise} \end{cases}$$

for each $\{x, y\} \in [\mathbb{R}]^2$. We have the following properties:

- $(2, 4) = (2, 4)_f$, $(2, 3) = (2, 3)_f$ and $(3, 4) = (3, 4)_f$.
- $(0, 1)_f = (0, 1) \cup (3, 4)$ and $(-1, 0)_f = (-1, 0) \cup (2, 3)$.
- $(5, 6)_f = (5, 6) \cup (2, 4)$.

It is not hard to show that $\lambda_f^*\big((2,4)\big) = \lambda_f^*\big((2,3)\big) = \lambda_f^*\big((3,4)\big) = 1$. If $E = (0,1)_f$ and $A = (2,4)_f$, then

$$E \cap A = [(0,1) \cup (3,4)] \cap (2,4) = (3,4) = (3,4)_f, \text{ and}$$

$$E^c \cap A = [(-\infty,0] \cup [1,3] \cup [4,+\infty)] \cap (2,4) = (2,3) = (2,3)_f.$$

So, $\lambda_f^*(E \cap A) = \lambda_f^*(E^c \cap A) = 1$. On the other hand, since $(2,4) \subseteq (5,6)_f$, we must have that $\lambda_f^*\big((2,4)_f\big) \leq 1$. Thus,

$$\lambda_f^*\big(E \cap A\big) + \lambda_f^*\big(E^c \cap A\big) = 2 > 1 = \lambda_f^*\big(A\big).$$

Therefore, $(0,1)_f$ cannot be λ_f^*-measurable.

Question 3.11. Let f be a weak selection. Given an f-interval $(a,b)_f$, is there a weak selection g such that $(a,b)_g$ is not λ_f^*-measurable ?

Next, we shall see that a subset of $\mathbb{R}$ of size $\mathfrak{c}$ can have arbitrary positive real f-outer measure by defining a suitable weak selection f.

Example 3.12. If $A \subseteq \mathbb{R}$ has size $\mathfrak{c}$ and $a,b \in \mathbb{R}$ satisfy $a < b$, then there exists a weak selection f such $\lambda_f^*\big(A\big) = b - a$.

Without loss of generality, we may assume that $A \cap (a,b) = \emptyset$. The definition of the weak selection f will be by transfinite induction. First we define

- $a <_f r <_f b$ for all $r \in A$. Next, consider the set

  $$\mathcal{R} = \Big\{S \in \mathbb{R}^\omega \,:\, S = (x_n)_{n\in\mathbb{N}} \quad \text{and} \quad \forall n \in \mathbb{N}\,(x_{2n} \neq x_{2n+1})\Big\}.$$

  Enumerate $\mathcal{R}$ as $\Big\{S_\xi \,:\, \xi < \mathfrak{c}\Big\}$ and S_ξ by $(x_n^\xi)_{n\in\mathbb{N}}$ for each $\xi < \mathfrak{c}$. Now, suppose that the weak selection f and the number $r_\xi \in A$ have been defined in a certain convenient way, for all $\xi < \theta < \mathfrak{c}$. Consider the sequence S_θ. Fix

  $$r_\theta \in A \setminus \Big[\Big\{r_\xi \,:\, \xi < \theta\Big\} \cup \Big\{x_n^\xi \,:\, \xi \leq \theta \quad \text{and} \quad n \in \mathbb{N}\Big\}\Big],$$

  This is possible, since the cardinality of this set is $\mathfrak{c}$. We define the weak selection f on the point r_θ with respect to x_n^ξ, for each $n \in \mathbb{N}$, as follows:

- If $x_{2n}^\xi \neq a$ for all $n \in \mathbb{N}$, then $r_\theta <_f x_{2n}^\xi$.
- If there is $n \in \mathbb{N}$ such that $x_{2n}^\xi = a$ and $x_{2n+1}^\xi = b$, then $a <_f r_\theta <_f b$.

- If there is $n \in \mathbb{N}$ such that $x^{\xi}_{2n} = a$ and $x^{\xi}_{2n+1} \neq b$ for all $n \in \mathbb{N}$, then $x^{\xi}_{2n+1} <_f r_\theta$.

The weak selection f will preserve the Euclidean order on the rest of the points. By the definition of the weak selection, we have that $A \subseteq (a,b]_f$ and hence $\lambda^*_f[A] \leq b-a$. Suppose that

$$A \subseteq \bigcup_{n=0}^{\infty} (a_n, b_n]_f.$$

Let $\theta < \mathfrak{c}$ such that the sequence $S_\theta = (x^{\theta}_n)_{n\in\mathbb{N}}$ satisfies $x^{\theta}_{2n} = a_n$ and $x^{\theta}_{2n+1} = b_n$, for all $n \in \mathbb{N}$. Applying the definition of f, for every $n \in \mathbb{N}$ we have the following cases:
$r_\theta \notin (x^{\xi}_{2n}, x^{\xi}_{2n+1}]_f$ if $a \neq x^{\xi}_{2n}$ and $b \neq x^{\xi}_{2n+1}$,
$r_\theta \notin (x^{\xi}_{2n}, b]_f$ if $a \neq x^{\xi}_{2n}$ and
$r_\theta \notin (a, x^{\xi}_{2n+1}]_f$ if $b \neq x^{\xi}_{2n+1}$.
Hence, it follows that if either $a \neq x^{\xi}_{2n}$ or $b \neq x^{\xi}_{2n+1}$, for all $n \in \mathbb{N}$, then

$$r_\theta \notin \bigcup_{n=0}^{\infty} (x^{\xi}_{2n}, x^{\xi}_{2n+1}]_f.$$

This shows that the only possible cover of A by f-intervals should contain the f-interval $(a,b]_f$. Therefore, $\lambda^*_f\big(A\big) = b-a$.

Corollary 3.13. *If $a, b \in \mathbb{R}$ satisfy $a < b$, then there exists a weak selection f such that $\lambda^*_f\big((0,1)\big) = b-a$.*

We end the article with an open question.

Question 3.14. Let f and g be two weak selections. If $\lambda^*_f\big((a,b]_f\big) = \lambda^*_g\big((a,b]_g\big)$, for all $a, b \in \mathbb{R}$, must $\lambda^*_f = \lambda^*_g$?

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