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TUBES ABOUT FUNCTIONS AND MULTIFUNCTIONS

Abstract

We provide a characterization of lower semicontinuity for multifunctions with values in a metric space $\langle Y, d\rangle$ which, in the special case of single-valued functions, says that a function is continuous if and only if for each $\varepsilon > 0$, the ε-tube about its graph is an open set. Applications are given, one of which provides a novel understanding of the Open Mapping Theorem from functional analysis. We also give a related but more complicated characterization of upper semicontinuity for multifunctions with closed values in a metrizable space.

1 Introduction

Let $\langle X, \tau\rangle$ be a topological space. If $\langle f_n\rangle$ is a sequence of real-valued functions on X and $f : X \to \mathbb{R}$, then the uniform convergence of $\langle f_n\rangle$ to f is described graphically as follows: given any ε-tube about the graph of f with $\varepsilon > 0$

$$\text{Tube}\,(\,f, \varepsilon\,) = \{(x, y) : x \in X, y \in \mathbb{R} \text{ and } |\,y - f(x)\,| < \varepsilon\},$$

then the graph of f_n must lie in the tube for large enough n. (see Figure 1).

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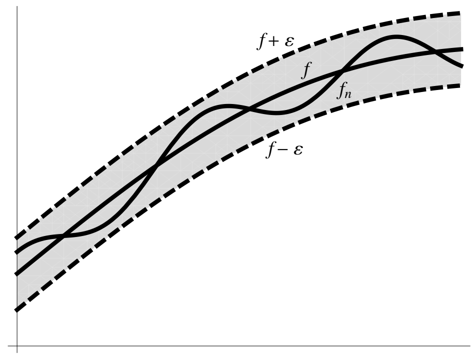

Figure 1: Uniform convergence and ε-tubes.

If the target space $\mathbb{R}$ is replaced by a general metric space $\langle Y, d\rangle$, then the same description of uniform convergence applies replacing $\mathbb{R}$ by Y and $|\, y - f(x)\,|$ by $d(y, f(x))$ in the definition of Tube $(\,f, \varepsilon\,)$. It is tempting to call this metric tube a tubular neighborhood, and indeed this is literally the case if $f : X \to Y$ is continuous. In fact, f is continuous if and only each such metric tube about its graph is an open subset of the product space.

By a *multifunction* Γ from X to Y, denoted by $\Gamma : X \rightrightarrows Y$, we mean a single-valued function from X to the nonempty subsets of Y. One can construct metric tubes about the graph of a multifunction Γ from X to Y in an analogous way, and we show here that each metric tube is open if and only if the multifunction is lower semicontinuous in the usual sense. As a special case, continuity of a single-valued function $f : X \to Y$ is characterized by the condition that each ε-tube $\{(x, y) \in X \times Y : d(y, f(x)) < \varepsilon\}$ is open in $X \times Y$. We give several applications of our general characterization of lower semicontinuity, one of which leads to an alternative understanding of the Open Mapping Theorem from functional analysis.

We also prove a related characterization of upper semicontinuity for multifunctions with closed values in a metrizable space Y, which yields as a special case this unusual characterization of continuity for a single-valued function f: for each metric d on Y compatible with its topology, and for each $\varepsilon > 0$, the "anti-tube" $\{(x, y) \in X \times Y : d(y, f(x)) > \varepsilon\}$ is open in $X \times Y$.

2 Preliminaries

All topological spaces are assumed to contain at least two points. If $\langle Y, d\rangle$ is a metric space and $y \in Y$ and $\varepsilon > 0$, then the open ε-ball about y will be denoted by $B_d(y, \varepsilon)$. If A is a nonempty subset of Y, then as in [2], we call $\cup_{a\in A} B_d(a, \varepsilon) = \{y \in Y : \exists a \in A \text{ with } d(y, a) < \varepsilon\}$ the *ε-enlargement* of A.

Suppose $\langle X, \tau\rangle$ and $\langle Y, \sigma\rangle$ are topological spaces. A function from X to Y is continuous provided the inverse image of each open subset of Y is open in X. For a multifunction Γ from X to Y there are two natural ways to define the inverse image of a subset V of Y under Γ: either the set of all $x \in X$ such that $\Gamma(x)$ *intersects* V or the set of all $x \in X$ such that $\Gamma(x)$ is *contained in* V. These lead to two distinct continuity concepts for multifunctions. For $V \subseteq Y$ put

$$\Gamma^-(V) := \{x \in X : \Gamma(x) \cap V \neq \emptyset\} \text{ and } \Gamma^+(V) := \{x \in X : \Gamma(x) \subseteq V\}.$$

We call Γ *lower semicontinuous* (resp. *upper semicontinuous*) provided $\Gamma^-(V)$ (resp. $\Gamma^+(V)$) is open whenever V is open [1, 4, 7, 8]. If $f : X \to Y$ is single-valued and $\Gamma(x) = \{f(x)\}$, then these are the same, so continuity of f is equivalent both to lower semicontinuity and to upper semicontinuity of $x \rightrightarrows \{f(x)\}$.

Recall that a single-valued function $f : X \to (-\infty, \infty]$ is called *lower semicontinuous* provided for each $\alpha \in \mathbb{R}, \{x \in X : f(x) > \alpha\}$ is open in X. Dually, $f : X \to [-\infty, \infty)$ is called *upper semicontinuous* provided for each $\alpha \in \mathbb{R}, \{x \in X : f(x) < \alpha\}$ is open in X. We confine our subsequent remarks to the former class of functions, leaving it to the reader to construct/recall analagous assertions for the latter class.

For analysts, f is lower semicontinuous on X if and only if at each $x_0 \in X$, whenever $\langle x_\lambda\rangle_{\lambda\in\Lambda}$ is a net convergent to x_0, we have $\liminf_{\lambda\in\Lambda} f(x_\lambda) \geq f(x_0)$. Lower semicontinuity is characterized by the following geometric condition: the *epigraph* of f defined by epi $(f) := \{(x, \alpha) : x \in X, \alpha \in \mathbb{R} \text{ and } \alpha \geq f(x)\}$ is a closed subset of $X \times \mathbb{R}$ [2, p. 14].

For real-valued functions, lower semicontinuity can be described in terms of the semicontinuity of two natural associated multifunctions. The proof of the following folk-theorem is left to the interested reader (cf. [8, p. 174]).

Proposition 2.1. *Let $\langle X, \tau\rangle$ be a topological space and let $f : X \to \mathbb{R}$. The following conditions are equivalent:*

1. *f is lower semicontinuous;*
2. *$x \rightrightarrows (-\infty, f(x)]$ is lower semicontinuous;*

3. $x \rightrightarrows [f(x), \infty)$ *is upper semicontinuous.*

If $\Gamma : X \rightrightarrows Y$ is a multifunction, its *graph* is defined by

$$\mathrm{Gr}\,(\,\Gamma\,) := \{(x, y) : x \in X, y \in Y \text{ and } y \in \Gamma(x)\}.$$

(Note that this is a subset of $X \times Y$ and is not the same as the graph of Γ as a function from X to the power set of Y.) Any subset of $X \times Y$ whose projection on X is surjective is the graph of a multifunction from X to Y.

Suppose now that $\langle Y, \sigma\rangle$ is metrizable with understood compatible metric d. With a slight abuse of notation, we use the same d for the distance from a point in Y to a subset of Y. If $\Gamma : X \rightrightarrows Y$ and $\varepsilon > 0$, we define the *enlargement multifunction* $\Gamma_\varepsilon : X \rightrightarrows Y$ by the formula $\Gamma_\varepsilon(x) = B_d(\Gamma(x), \varepsilon) = \{y \in Y : d(y, \Gamma(x)) < \varepsilon\}$. For the *metric tube of radius* ε about the graph of Γ, we will use the graph of Γ_ε.

$$\begin{aligned}\mathrm{Gr}\,(\,\Gamma_\varepsilon\,) &= \{(x, y) \in X \times Y : y \in \Gamma_\varepsilon(x)\}\\ &= \{(x, y) \in X \times Y : d(y, \Gamma(x)) < \varepsilon\}\end{aligned}$$

If $f : X \to Y$ is single valued and $\Gamma(x) = \{f(x)\}$, then this is the same as $\mathrm{Tube}\,(\,f, \varepsilon\,)$ from the introduction. As we shall see, while each enlargement multifunction has open values, it need need not have an open graph.

3 Results

We immediately characterize those multifunctions Γ for which each metric tube about $\mathrm{Gr}\,(\,\Gamma\,)$ is open.

Theorem 3.1. *Let $\langle X, \tau\rangle$ be a topological space and let $\langle Y, d\rangle$ be a metric space. For a multifunction Γ from X to Y, the following conditions are equivalent.*

1. Γ *is lower semicontinuous;*

2. $\forall \varepsilon > 0$*, the enlargement multifunction* Γ_ε *has open graph;*

3. $\forall \varepsilon > 0$*, the enlargement multifunction* Γ_ε *is lower semicontinuous;*

4. $(x, y) \mapsto d(y, \Gamma(x))$ *is an upper semicontinuous function on* $X \times Y$*.*

Proof. (1) $\Longrightarrow$ (2): Suppose Γ is lower semicontinuous and $\varepsilon > 0$. Fix $(x_0, y_0) \in \mathrm{Gr}\,(\,\Gamma_\varepsilon\,)$. Then there is a $y \in \Gamma(x_0)$ such that $\delta = d(y, y_0) < \varepsilon$. Let $\lambda = (\varepsilon - \delta)/2$. By lower semicontinuity of Γ, there is an open neighborhood

U of x_0 such that $x \in U \implies \Gamma(x) \cap B_d(y, \lambda) \neq \emptyset$. We claim that the neighborhood $U \times B_d(y_0, \lambda)$ of (x_0, y_0) is contained in $\mathrm{Gr}\,(\,\Gamma_\varepsilon\,)$.

To see this, fix (x_1, y_1) in $U \times B_d(y_0, \lambda)$. By the choice of U, there is a y_2 in $\Gamma(x_1)$ with $d(y_2, y) < \lambda$. We compute

$$d(y_1, y_2) \leq d(y_1, y_0) + d(y_0, y) + d(y, y_2) < \lambda + \delta + \lambda \leq \varepsilon.$$

This shows that $d(y_1, \Gamma(x_1)) < \varepsilon$ as required.

$\underline{(2) \implies (3)}$: This follows readily from the observation that if $V \subseteq Y$, then

$$\Gamma_\varepsilon^-(V) = \{x \in X : \Gamma_\varepsilon(x) \cap V \neq \emptyset\} = \pi_x \left(\mathrm{Gr}\,(\,\Gamma_\varepsilon\,) \cap \pi_y^{-1}(V)\right)$$

where π_x and π_y are the projections of $X \times Y$ onto X and Y respectively. If V is open, then $\pi_y^{-1}(V)$ is open since π_y is continuous. Since $\mathrm{Gr}\,(\,\Gamma_\varepsilon\,)$ is open by hypothesis and π_x is an open mapping, $\pi_x \left(\mathrm{Gr}\,(\,\Gamma_\varepsilon\,) \cap \pi_y^{-1}(V)\right)$ is open. Thus $\Gamma_\varepsilon^-(V)$ is open and Γ_ε is lower semicontinuous as claimed.

$\underline{(3) \implies (1)}$: Fix $x_0 \in X, y_0 \in \Gamma(x_0)$ and $\varepsilon > 0$. It suffices to show that there is an open neighborhood U of x_0 such that $\Gamma(x) \cap B_d(y_0, \varepsilon) \neq \emptyset$ whenever $x \in U$. Since $y_0 \in \Gamma_{\varepsilon/2}(x_0)$, the lower semicontinuity of the $(\varepsilon/2)$-enlargement multifunction gives an open neighborhood U of x_0 such that $x \in U \implies \Gamma_{\varepsilon/2}(x) \cap B_d(y_0, \varepsilon/2) \neq \emptyset$. Thus, for each x in U there is a y in $\Gamma(x)$ such that $d(y, B_d(y_0, \varepsilon/2)) < \varepsilon/2$, and so $d(y, y_0) < \varepsilon$.

$\underline{(2) \iff (4)}$: Finally, conditions (2) and (4) are equivalent since a function $g : X \times Y \to [0, \infty)$ is upper semicontinuous if and only if $\{(x, y) : g(x, y) < \varepsilon\}$ is open in $X \times Y$ for each positive ε. Apply this with $g(x, y) = d(y, \Gamma(x))$. $\square$

As noted above this yields a pleasant continuity result for single-valued functions.

Corollary 3.2. *Suppose $\langle X, \tau \rangle$ is a topological space and $\langle Y, d \rangle$ is a metric space. Then a single-valued function $f : X \to Y$ is continuous if and only if* $\mathrm{Tube}\,(\,f, \varepsilon\,)$ *is open in $X \times Y$ for every positive ε.*

Condition (4) of Theorem 3.1 can be considerably weakened and still yield lower semicontinuity of Γ: $\forall y \in Y, x \mapsto d(y, \Gamma(x))$ is upper semicontinuous on X (see, e.g. [6, p. 352 and p. 360]).

While an upper semicontinuous multifunction with closed values from a topological space X to a regular topological space Y has closed graph (see [7, p. 78] or [8, p. 175]), a lower semicontinuous multifunction with open values need not have open graph, even if the target space is metrizable. Thus, the implication (3) $\Rightarrow$ (2) in Theorem 3.1 is not trivial.

Example 3.3. Let $X = Y = \mathbb{R}$, and define $\Gamma : X \rightrightarrows Y$ by

$$\Gamma(x) = \begin{cases} \mathbb{R} & \text{if } x = 0 \\ \mathbb{R} \setminus \{0\} & \text{otherwise} \end{cases}$$

Evidently, Γ is lower semicontinuous with open values, yet its graph,

$$\mathbb{R}^2 \setminus (\{x : x \neq 0\} \times \{0\}),$$

contains no neighborhood of $(0, 0)$.

As is easy to see, in complete generality, a multifunction $x \rightrightarrows \Gamma(x)$ is lower semicontinuous if and only if $x \rightrightarrows \mathrm{cl}(\Gamma(x))$ is lower semicontinuous [7, p. 85]. One can wonder if conditions (1) and (3) of Theorem 3.1 remain equivalent if we redefine $\Gamma_\varepsilon(x)$ to be $\{y \in Y : d(y, \Gamma(x)) \leq \varepsilon\}$. This fails, even for single-valued functions.

Example 3.4. Let $X = [0, 1]$ and $Y = [0, 1] \cup [2, \infty)$, with the Euclidean metric. The identity function $f(x) = x$ is continuous, but $x \rightrightarrows \{y : d(y, f(x)) \leq 1\}$ fails to be lower semicontinuous because $d(2, f(1)) \leq 1$. Of course, the graph of the auxiliary multifunction fails to be open as well.

Let $\langle X, \tau\rangle$ and $\langle Y, \sigma\rangle$ be topological spaces, and let $f : X \to Y$ be surjective. It is well-known that f is an open mapping if and only if the preimage multifunction $y \rightrightarrows f^{-1}(\{y\})$ is a lower semicontinuous multifunction [7, p. 82]. When X is metrizable, Theorem 3.1 allows us to express openness of f in terms of preimages of singleton subsets of Y as follows.

Theorem 3.5. *Let $\langle X, \rho\rangle$ be a metric space and $\langle Y, \sigma\rangle$ be a topological space. Suppose $f : X \to Y$ maps X onto Y. Then f is an open mapping if and only if $\{(x, y) \in X \times Y : \rho(x, f^{-1}(\{y\})) < \varepsilon\}$ is an open subset of $X \times Y$ for each positive ε.*

As important as any result regarding the openness of single-valued mappings is the Open Mapping Theorem from functional analysis [5, 10]: Let X and Y be Banach spaces (real or complex) and let $T : X \to Y$ be a continuous linear surjection. Then T is an open mapping. From Theorem 3.5, we get the following interpretation of this classical result.

Corollary 3.6. *Let X and Y be Banach spaces and suppose $T : X \to Y$ is a continuous linear surjection. For each $y \in Y$, select $x_y \in X$ with $T(x_y) = y$. Then $\{(x, y) \in X \times Y : d(x, x_y + \ker(T)) < \varepsilon\}$ is an open subset of $X \times Y$ for each positive ε.*

By the Bartle-Graves Theorem [1, Theorem 4], which itself is a corollary of the comprehensive selection theorem of E. Michael (see, e.g., [2, 9, 7]), the selection $y \mapsto x_y$ as described above can be chosen to be continuous.

When T is a nontrivial linear functional defined on a general normed linear space, then T is automatically surjective and open [10, p. 38]. Assuming T is in addition continuous, each level set of T is a closed hyperplane in X. By the well-known formula of Ascoli [2, p. 5], for each $x \in X$ and each scalar α, we have $d(x, x_\alpha + \ker(T)) = |\, T(x) - \alpha \,| \,/\, \| \, T \, \|$. Thus, we can treat this special case by:

Corollary 3.7. *Let X be a normed linear space over a field of scalars $\mathbb{F}$ (either $\mathbb{R}$ or $\mathbb{C}$), and let T be a nontrivial continuous linear functional on X. Then $\{(x, \alpha) \in X \times \mathbb{F} : |\, T(x) - \alpha \,| < \varepsilon\}$ is an open subset of $X \times \mathbb{F}$ for each positive ε.*

PROOF. Let α be a fixed scalar. Then $|\, T(x) - \alpha \,| < \varepsilon$ if and only if $d(x, x_\alpha + \ker(T)) < \varepsilon / \| \, T \, \|$. $\square$

We next use Theorem 3.1 to prove a well-known optimization result often attributed to C. Berge (see [4, p. 115] or [2, p. 202]).

Proposition 3.8. *Let $\langle X, \tau \rangle$ be a topological space and $\langle Y, d \rangle$ be a metric space. Suppose $f : X \times Y \to \mathbb{R}$ is lower semicontinuous and $\Gamma : X \rightrightarrows Y$ is lower semicontinuous. Define $M : X \to (-\infty, \infty]$ by $M(x) = \sup\{f(x, y) : y \in \Gamma(x)\}$. Then M is lower semicontinuous.*

PROOF. Fix $x_0 \in X$ and let $\alpha < M(x_0)$ be arbitrary. Choose $y_0 \in \Gamma(x_0)$ with $f(x_0, y_0) > \alpha$. Then, by lower semicontinuity of f at (x_0, y_0), choose an open neighborhood U of x_0 and $\varepsilon > 0$ such that $(x, y) \in U \times B_d(y_0, \varepsilon) \implies f(x, y) > \alpha$. By condition (2) of Theorem 3.1, we can find a neighborhood $U_1 \subseteq U$ of x_0 and $\varepsilon_1 \in (0, \varepsilon)$ such that

$$U_1 \times B_d(y_0, \varepsilon_1) \subseteq \{(x, y) : d(y, \Gamma(x)) < \varepsilon\}.$$

If $x \in U_1$, then $(x, y_0) \in U_1 \times B_d(y_0, \varepsilon_1)$ so that there is a y in $\Gamma(x)$ with $d(y, y_0) < \varepsilon$, ensuring that $(x, y) \in U \times B_d(y_0, \varepsilon)$ as well. Then $M(x) \geq f(x, y) > \alpha$ as required. $\square$

As stated in Proposition 2.1, a real-valued function f is lower semicontinuous if and only if $\Gamma_f : X \rightrightarrows \mathbb{R}$ defined by $\Gamma_f(x) = (-\infty, f(x)]$ is a lower semicontinuous multifunction. For arbitrary f, the metric tube of radius ε about $\mathrm{Gr}\,(\Gamma_f)$ is easily seen to be $\{(x, \alpha) \in X \times \mathbb{R} : \alpha < f(x) + \varepsilon\}$ which is the complement of epi $(f + \varepsilon)$ in $X \times \mathbb{R}$. Thus, if we knew that just one metric tube about the graph of Γ_f were open, this would already ensure lower semicontinuity of f.

The uninitiated might guess that upper semicontinuity of a multifunction with values in a metric space $\langle Y, d\rangle$ is characterized by the lower semicontinuity of $(x,y) \mapsto d(y,\Gamma(x))$ on $X \times Y$. However, continuity of $(x,y) \mapsto d(y,\Gamma(x))$ is not enough to ensure upper semicontinuity of Γ, even if Γ has closed graph.

Example 3.9. Let $X = \mathbb{R}$ equipped with the usual topology and equip $Y = \mathbb{R}^2$ with the Euclidean metric d. Define $\Gamma : X \rightrightarrows Y$ by $\Gamma(x) = \{(x,\beta) : \beta \in \mathbb{R}\}$. Then $d((\alpha,\beta),\Gamma(x)) = |x-\alpha|$ and so $(x,(\alpha,\beta)) \mapsto d((\alpha,\beta),\Gamma(x))$ is continuous. However Γ fails to be upper semicontinuous because with $V = \{(\alpha,\beta) : \alpha < e^{\beta}\}$, we have $\Gamma^+(V) = (-\infty, 0]$.

Upper semicontinuity for multifunctions is a much stronger notion than lower semicontinuity. What is required to obtain upper semicontinuity for a multifunction with closed values is lower semicontinuity of $(x,y) \mapsto d(y,\Gamma(x))$ on $X \times Y$ with respect to each compatible metric d for Y. Our next result is anticipated by a characterization of the Vietoris topology by Beer, Lechicki, Levi and Naimpally [3]; indeed our main construction is based on techniques found in [3].

First, we introduce a well-studied weaker upper semicontinuity condition that coincides with upper semicontinuity as we have defined it for multifunctions with compact values (see, e.g., [2, 7]).

Definition 3.10. Let Γ be a multifunction from a topological space $\langle X, \tau\rangle$ to a metric space $\langle Y, d\rangle$. We call Γ *d-Hausdorff upper semicontinuous* if for each $x_0 \in X$ and each $\varepsilon > 0$, there exists a neighborhood U of x_0 such that for each $x \in U$ we have $\Gamma(x) \subseteq B_d(\Gamma(x_0), \varepsilon)$.

Theorem 3.11. *Let $\langle X, \tau\rangle$ be a topological space and let $\langle Y, \sigma\rangle$ be a metrizable topological space. Suppose $\Gamma : X \rightrightarrows Y$ is a multifunction such that $\Gamma(x)$ is a closed subset of Y for every x in X. Then the following conditions are equivalent:*

1. *Γ is upper semicontinuous;*
2. *Γ is d-Hausdorff upper semicontinuous for all metrics d which are compatible with σ;*
3. *$(x,y) \mapsto d(y,\Gamma(x))$ is a lower semicontinuous function on $X \times Y$ for each metric d compatible with σ;*
4. *for each metric d compatible with σ and each $\varepsilon > 0$, the set $\{(x,y) : d(y,\Gamma(x)) > \varepsilon\}$ is open in $X \times Y$.*

PROOF. $\underline{(1) \implies (2)}$: This is trivial because an enlargement of a set is an open neighborhood of the set.

$\underline{(2) \implies (3)}$: Let d be a compatible metric and suppose $\alpha < d(y_0, \Gamma(x_0))$. If $d(y_0, \Gamma(x_0)) = 0$, then $\{(x, y) : d(y, \Gamma(x)) > \alpha\} = X \times Y$. If not, then $d(y_0, \Gamma(x_0)) > 0$ and we may assume that $\alpha > 0$ as well. Put $\varepsilon = (d(y_0, \Gamma(x_0)) - \alpha)/3$. Since $B_d(\Gamma(x_0), \varepsilon)$ is an open neighborhood of $\Gamma(x_0)$, by d-Hausdorff upper semicontinuity of Γ we can find an open neighborhood U of x_0 such that $x \in U \implies \Gamma(x) \subseteq B_d(\Gamma(x_0), \varepsilon)$. A routine calculation now shows that whenever $x \in U$ and $y \in B_d(y_0, \varepsilon)$, we have $d(y, \Gamma(x)) > \alpha$.

$\underline{(3) \iff (4)}$: As with upper semicontinuity, lower semicontinuity of the map $(x, y) \mapsto d(y, \Gamma(x))$ means that $\{(x, y) : d(y, \Gamma(x)) > \varepsilon\}$ is open for each positive ε. Thus, statements (3) and (4) are equivalent.

$\underline{(3) \implies (1)}$: We want to show that $\Gamma^-(V)$ is open whenever V is open in Y. This is trivial if $V = Y$ or if $\Gamma^-(V) = \emptyset$. Otherwise, let $x_0 \in \Gamma^-(V)$ and $y_0 \in Y \setminus V$ be arbitrary. Let d be a compatible metric for the topology of Y such that $d(y_1, y_2) \leq 1$ whenever $\{y_1, y_2\} \subseteq Y$, and let h be a Urysohn function for $\Gamma(x_0)$ and $Y \setminus V$, say h takes $\Gamma(x_0)$ to one and h takes $Y \setminus V$ to zero.

Define a second compatible metric ρ on Y by the formula

$$\rho(y_1, y_2) := d(y_1, y_2) + |\, h(y_1) - h(y_2)\,|,$$

and compute

$$\rho(y_0, \Gamma(x_0)) = \inf_{y \in \Gamma(x_0)} (d(y, y_0) + |\, h(y) - h(y_0)\,|) = d(y_0, \Gamma(x_0)) + 1 > 1.$$

By lower semicontinuity of $(x, y) \mapsto \rho(y, \Gamma(x))$ at (x_0, y_0) - in fact, by the lower semicontinuity of $x \mapsto \rho(y_0, \Gamma(x))$ at x_0 alone - we can find a neighborhood U of x_0 such that $x \in U \implies \rho(y_0, \Gamma(x)) > 1$. But then we have $\rho(y, y_0) > 1$ for each y in $\Gamma(x)$ and so $y \in V$ whenever $y \in \Gamma(x)$. We have shown that $x_0 \in U \subseteq \Gamma^-(V)$ as required. $\square$

Note that condition (4) of our last theorem can be recast as a statement about "closed" metric tubes: for each metric d compatible with σ and each $\varepsilon > 0$, the tube $\{(x, y) : d(y, \Gamma(x)) \leq \varepsilon\}$ is closed in $X \times Y$.

We note that openness of $\{(x, y) : d(y, \Gamma(x)) > \varepsilon\}$ for each positive ε and *some* compatible metric d implies $\{(x, y) : d(y, \Gamma(x)) > 0\}$ is open, so that if Γ in addition has closed values, then Γ has closed graph.

Condition (3) of Theorem 3.11 does not imply condition (1) without the assumption that Γ has closed values. Further, condition (2) does not imply condition (1) without this assumption. Consider again the multifunction Γ :

$\mathbb{R} \rightrightarrows \mathbb{R}$ given in Example 3.3, but now interchanging the defining formulas to obtain $x \rightrightarrows \mathbb{R} \setminus \{0\}$ if $x = 0$, and $x \rightrightarrows \mathbb{R}$ otherwise. Let us denote this new multifunction by Δ. For each metric d compatible with the usual topology of the real line and for each $(x, y) \in \mathbb{R}^2$, we have $d(y, \Delta(x)) = 0$. Further, for each $\varepsilon > 0$ and each $x \in \mathbb{R}$, we have $B_d(\Delta(x), \varepsilon) = \mathbb{R}$ so that Δ is d-Hausdorff upper semicontinuous. But $\Gamma^+(\mathbb{R} \setminus \{0\})$ is not open.

Condition (4) of of Theorem 3.11, leads readily to alternative necessary and sufficient conditions for continuity of a single-valued function assuming values in a metrizable space. Instead of ε-tubes, we reverse the inequality and consider "anti-tubes"

$$\{(x, y) : d(y, f(x)) > \varepsilon\}.$$

Corollary 3.12. *Let $\langle X, \tau\rangle$ be a topological space and let $\langle Y, \sigma\rangle$ be a metrizable topological space. Then f is continuous if and only if for each compatible metric d for σ and each $\varepsilon > 0$, the set $\{(x, y) : d(y, f(x)) > \varepsilon\}$ is open in $X \times Y$.*

We close with an example showing that openness of each anti-tube for just one compatible metric is not enough to guarantee continuity for a function f.

Example 3.13. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(0) = 0$ and $f(x) = 1/|\,x\,|$ otherwise. Equipping the target space with the usual metric and fixing $\varepsilon > 0$, we see that $\{(x, y) : d(y, f(x)) > \varepsilon\}$ has three connected components (See Figure 2 which shows the graph of the function and the anti-tube for $\varepsilon = 3/4$).

1. $\{(x, y) : x < 0 \text{ and } y > (1/|\,x\,|) + \varepsilon\}$;
2. $\{(x, y) : x > 0 \text{ and } y > (1/|\,x\,|) + \varepsilon\}$;
3. $\{(0, y) : |\,y\,| > \varepsilon\} \cup \{(x, y) : x \neq 0 \text{ and } y < (1/|\,x\,|) - \varepsilon\}$.

Evidently the first two components are open. If we define $g : \mathbb{R} \to (-\infty, \infty]$ by

$$g(x) = \begin{cases} (1/|\,x\,|) - \varepsilon, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0 \end{cases},$$

then g is lower semicontinuous and so $\operatorname{epi}(g)$ is closed. Evidently, the third component can be expressed as $\mathbb{R}^2 \setminus (\operatorname{epi}(g) \cup \{(0, y) : -\varepsilon \leq y \leq \varepsilon\})$ which is open as well. This shows that $\{(x, y) : d(y, f(x)) > \varepsilon\}$ is open for each positive ε while f fails to be continuous.

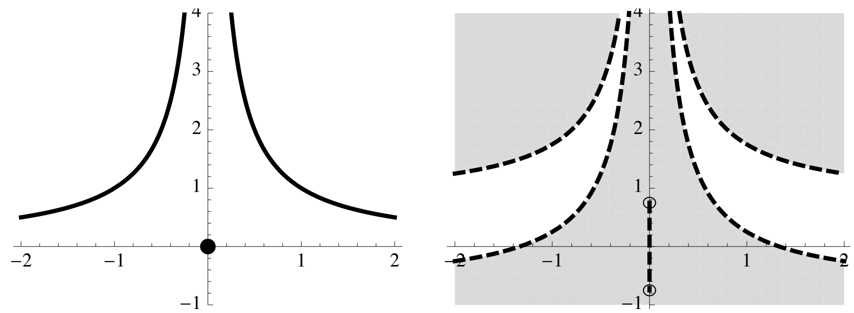

Figure 2: A discontinuous function with closed graph and open anti-tube.

References

[1] R. Bartle and L. Graves, *Mappings between function spaces*, Trans. Amer. Math. Soc., **72** (1952), 400–413.

[2] G. Beer, *Topologies on closed and closed convex sets*, Kluwer Acad. Publ., Dordrecht, 1993.

[3] G. Beer, A. Lechicki, S. Levi, and S. Naimpally, *Distance functionals and suprema of hyperspace topologies*, Ann. Mat. Pura Appl., **162** (1992), 367–381.

[4] C. Berge, *Topological spaces*, Macmillan, New York, 1963.

[5] J. Conway, *A course in functional analysis*, Springer-Verlag, New York, 1985.

[6] S. Francaviglia, A. Lechicki, and S. Levi, *Quasi-uniformization of hyperspaces and convergence of nets of semicontinuous multifunctions*, J. Math. Anal. Appl., **112** (1985), 347–370.

[7] E. Klein and A. Thompson, *Theory of correspondences*, Wiley, New York, 1984.

[8] K. Kuratowski, *Topology*, **1**, Academic Press, New York, 1966.

[9] E. Michael, *Continuous selections*, Ann. of Math., **63** (1956), 361–382.

[10] W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973.