RESEARCH

**Real Analysis Exchange**
Vol. 39(1), 2013/2014, pp. 15–32

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# ON THE SUMS OF LOWER SEMICONTINUOUS STRONG ŚWIĄTKOWSKI FUNCTIONS

**Abstract**

The purpose of this article is to give a solution to a problem raised by A. Maliszewski in [5] by showing that any lower semicontinuous function can be represented as a sum of two lower semicontinuous strong Świątkowski functions.

We deal with the classes of real functions defined on the interval $[0,1]$. The symbols $C$, $D$, $Q$, $S^*$, $lsc$ and $usc$ stand for the class of continuous, Darboux, quasi-continuous, strong Świątkowski, lower and upper semicontinuous functions, respectively. $S^*lsc$ denotes $S^* \cap lsc$ and $Dlsc$ denotes $D \cap lsc$. $C_f$ is the set of all points of continuity of the function $f$, $D_f$ is the set of all points of discontinuity of the function $f$ and $f \restriction F$ denotes the restriction of the function $f$ to the set $F$. The set $B$ is bilaterally c-dense in the set $A$ ($A \subset_c B$) iff for each $x \in A$ the sets $(x, x+\delta) \cap B$, $(x-\delta, x) \cap B$ are nondenumerable for every $\delta > 0$.

A. Lindenbaum, in [3] provides the proof that every real function can be represented by a sum of Darboux functions. A similar result is valid for the functions of Baire 1 class. Every function of Baire 1 class can be represented by a sum of Darboux Baire 1 functions, [1]. In [5], A. Maliszewski shows that every cliquish function can be represented by a sum of a strong Świątkowski Baire one function and a strong Świątkowski function and also that every

Mathematical Reviews subject classification: Primary: 26A15, 26A21
Key words: the strong Świątkowski function, lower resp. upper semicontinuous functions
Received by the editors August 31, 2012
Communicated by: Brian Thomson
*This work was partly supported by the Slovak Grant Agency under the project No.1/0853/13.

lower semicontinuous function can be represented by a sum of Darboux quasicontinuous lower semicontinuous functions, [6]. In the remainder of this article we show the stronger assertion: $lsc = S^*lsc + S^*lsc$, that is: for an arbitrary lower semicontinuous function $f$, there exist strong Świątkowski lower semicontinuous functions $g$ and $h$ such that $f = g + h$. An analogous assertion for the class *usc* is valid, too. We begin with four lemmas:

**Lemma 1.** *Let $f$ be a lower semicontinuous function defined on $[0,1]$ and let $K$ be any closed subset of the set $C_f$. If a function $g \leq f$ is continuous and the set $A = \{x; g(x) = f(x)\} \subset C_f$, then there exists a continuous function $h$ such that $g \leq h \leq f$ and*

$$h(x) = f(x) \text{ for every } x \in A \cup K$$
$$g(x) < h(x) < f(x) \text{ for every } x \notin A \cup K.$$

PROOF. Since the function $f \in lsc$, there exists a sequence of continuous functions $f_1 \leq f_2 \leq f_3 \leq \ldots.$ which converges to the function $f$. We can demand that $f_1 < f_2 < f_3 < \ldots. \to f$. Otherwise we would replace the sequence of functions $f_n, n = 1, 2, \ldots$ by $f_n - \frac{1}{n}$.

The set $A = \{x; g(x) = f(x)\}$ is closed, because $f - g \geq 0$ and $f - g \in lsc$. Now let $I_n = (a_n, b_n), n = 1, 2, \ldots$ be the sequence of contiguous intervals of the set $A \cup K$ and let the function $f$ attain its minimum on the interval $[a_n, b_n]$ at a point $\xi_n$. Then there exists an index $i_n$ such that

$$f(\xi_n) - \frac{1}{n} < f_{i_n}(x) < f(x), \text{ for every } x \in [a_n, b_n].$$

Next we choose a sequence of points $a_{n_k}, b_{n_k}, k = 0, 1, 2, \ldots$

$$a_n \leftarrow \cdots < a_{n_2} < a_{n_1} < a_{n_0} = b_{n_0} < b_{n_1} < b_{n_2} < \cdots \to b_n.$$

Let $\{i_{n,k}\}_{k=1}^{\infty}$, $i_{n,k} \geq i_n$ be an increasing sequence of natural numbers such that

$$f_{i_{n,k}}(x) > g(x), \text{ for every } x \in [a_{n_k}, b_{n_k}]$$

and

$$f_{i_{n,k}}(a_n) > f(a_n) - \frac{1}{k} \wedge f_{i_{n,k}}(b_n) > f(b_n) - \frac{1}{k}.$$

A sequence with these property exists, because $f(x) > g(x)$ on the compact set $[a_{n_k}, b_{n_k}]$ and $f_n(x) \to f(x)$. We define a function $h$ as follows, where $n, k \in \mathbb{N}$.

$$h(x) = \begin{cases} f_{i_{n,k}}(x) + \frac{x-a_{n_{k-1}}}{a_{n_k}-a_{n_{k-1}}}(f_{i_{n,k+1}}(x) - f_{i_{n,k}}(x)) & x \in [a_{n_k}, a_{n_{k-1}}] \\ f_{i_{n,k}}(x) + \frac{x-b_{n_{k-1}}}{b_{n_k}-b_{n_{k-1}}}(f_{i_{n,k+1}}(x) - f_{i_{n,k}}(x)) & x \in [b_{n_{k-1}}, b_{n_k}] \\ f(x) & x \in A \cup K. \end{cases}$$

The function $h$ is continuous on every contiguous interval $I_n = (a_n, b_n)$ , $n = 1, 2, \ldots$ of the set $A \cup K$ and for each $x \in \left[a_{n_k}, a_{n_{k-1}}\right]$

$$f(x) > f_{i_{n,k+1}}(x) \geq h(x) \geq f_{i_{n,k}}(x)$$

holds. Therefore,

$$f(x) > h(x) \geq f_{i_{n,k}}(x), \ \forall x \in \left(a_n, a_{n_{k-1}}\right], \ \forall k \in \mathbb{N}$$

and because $a_n \in C_f$, it follows that

$$f(a_n) = \lim_{x \to a_n^+} f(x) \geq \lim_{x \to a_n^+} h(x) \geq \lim_{x \to a_n^+} f_{i_{n,k}}(x) = f_{i_{n,k}}(a_n) > f(a_n) - \frac{1}{k}.$$

Consequently

$$\lim_{x \to a_n^+} h(x) = f(a_n) = h(a_n) \ \text{ and similarly } \lim_{x \to b_n^-} h(x) = h(b_n),$$

that is, the function $h$ is continuous on every interval $[a_n, b_n]$ , $n = 1, 2, \ldots$ .

In order to prove the continuity of the function $h$ on the interval $[0, 1]$, it is sufficient, by the construction of $h$, to show that $h$ is continuous at an arbitrary point $x_0 \in K \cup A$. Let a sequence $x_j$, $j = 1, 2, \ldots$ converge to the point $x_0$. Since the restriction $f \restriction A \cup K = h \restriction A \cup K$ is continuous, it can be assumed that $x_j \in I_{n(j)}$, $j = 1, 2, \ldots$ and because $h \restriction [a_n, b_n]$ is continuous for each $n \in \mathbb{N}$, we can assume that $n(j) \to \infty$. For each $j$ there exists a point $\xi_{n(j)} \in \left[a_{n(j)}, b_{n(j)}\right]$, such that

$$f\left(\xi_{n(j)}\right) - \frac{1}{n(j)} < h(x_j) < f(x_j).$$

$x_0 \in A \cup K \subseteq C_f$ implies

$$h(x_0) = f(x_0) = \lim_{j \to \infty} f(\xi_j) - \frac{1}{n(j)} \leq \lim_{n \to \infty} h(x_j) \leq \lim_{n \to \infty} f(x_j) = f(x_0),$$

and therefore

$$h(x_0) = \lim_{j \to \infty} h(x_j).$$

The inequality

$$g(x) < h(x) < f(x) \text{ for every } x \notin A \cup K$$

follows directly from the definition of the function $h$, because

$$g(x) < f_{i_n}(x) < f_{i_{n,k}}(x) \le h(x) \le f_{i_{n,k+1}}(x) < f(x),$$
$$x \in \left[a_{n_k}, a_{n_{k-1}}\right] \cup \left[b_{n_{k-1}}, b_{n_k}\right].$$

□

**Definition 2.** *Let $P$ be a perfect set. We say that the function $f$ is from the class $K(P)$ iff $f$ is constant on every contiguous interval of the set $P$ and we denote by $CK(P)$ the class $C \cap K(P)$.*

**Remark 3.** *The class $CK(P)$ has the following properties.*
*Let $P_1, P$ be perfect sets, $P_1 \subset_c P$, and let $\alpha, \beta$ be real numbers, then*

$$f \in CK(P_1) \Rightarrow f \in CK(P)$$
$$f, g \in CK(P) \Rightarrow \alpha f + \beta g \in CK(P).$$

**Lemma 4.** *Let $f \ge 0$ be a continuous function, $K$ a closed set, $P$ a nowhere dense perfect set and $K \subset_c P$. Then there exists a function $g \in CK(P)$ such that $0 \le g \le f$ and $g(x) = f(x)$ for every $x \in K$.*

Proof. Let $(a, b)$ be a contiguous interval of the set $K$ and let the function $f$ attain its minimum on the closed contiguous interval $I = [a, b]$ at a point $c$. The set $I \cap \{x; f(x) = f(c)\}$ is closed. Then there exist

$$\min(I \cap \{x; f(x) = f(c)\}) \text{ and } \max(I \cap \{x; f(x) = f(c)\}).$$

If $a = \min(I \cap \{x; f(x) = f(c)\})$, we set $a_0 = a$. On the other hand, if we have that $a < \min(I \cap \{x; f(x) = f(c)\})$, we choose

$$a_0 \in P,\ a < a_0 < \min(I \cap \{x; f(x) = f(c)\}),$$

such that $a_0$ is the left boundary point of some contiguous interval of the set $P$. Such a point exists, because $a \in K \subset_c P$. Analogously, we define a point $b_0 \in P$. If $\max(I \cap \{x; f(x) = f(c)\}) = b$ then $b_0 = b$ and if $\max(I \cap \{x; f(x) = f(c)\}) < b$, we choose

$$b_0 \in P,\ \max(I \cap \{x; f(x) = f(c)\}) < b_0 < b,$$

where $b_0$ is the right boundary point of some contiguous interval of the set $P$.

In the case $a < a_0$, we define a function $f_1$ on the interval $[a, a_0]$. Since $f(x) > f(c)$ for every $x \in [a, a_0]$, there exists a constant $M > 0$, such that $f(x) > f(a) - M > f(c)$ on the interval $[a, a_0]$. The function $f$ is continuous at the point $a$. Then a sequence of points $a_0 > a_1 > a_2 > \ldots . \to a$ exists such that

$$f(x) > f(a) - \frac{M}{n},\ \forall x \in [a, a_{n-1}],\ n = 1, 2, \ldots .$$

Let $f_1(a) = f(a)$ and let the graph of $f_1$ be linear segments that join points $[a_0, f(c)], [a_1, f(a) - M], \left[a_2, f(a) - \frac{M}{2}\right], \ldots, \left[a_n, f(a) - \frac{M}{n}\right], \ldots$ . Apparently, the function $f_1$ is continuous and decreasing over the interval $[a, a_0]$. Moreover,

$$f(c) \le f_1(x) < f(x),\ \forall x \in (a, a_0],$$

because

$$f_1(x) < f(a) - \frac{M}{n} < f(x),\ \forall x \in [a_n, a_{n-1}],\ n = 1, 2, \ldots .$$

Now, let $Q_I = \{q_i\}_{i=1}^{\infty}$ be the sequence of all rational numbers of the interval $(f(c), f(a))$, and let $\mathcal{I} = \{I_n, n = 1, 2, \ldots\}$, $I_n \subset I$, be the sequence of all closed contiguous intervals of the set $[a, a_0] \cap P$. Since $[a, a_0] \cap P$ is the perfect set and $a, a_0 \in [a, a_0] \cap P$ then it is evident that

$$I_i \cap I_j = \emptyset,\ \forall I_i, I_j \in \mathcal{I},\ i \ne j,\ i, j \in \{1, 2, \ldots\}$$

and

$$\{a, a_0\} \cap I_i = \emptyset,\ \forall i \in \{1, 2, \ldots\}.$$

The set $\mathcal{I}$ is ordered, $I_i < I_j,\ I_i, I_j \in \mathcal{I}$, it means that $\max I_i < \min I_j$. We define a mapping

$$G : \{I_n, n = 1, 2, \ldots\} \to Q_I$$

inductively.

In the first step, let $G(I_1) = q_{i_1}$, where $q_{i_1}$ is the first member of the sequence $Q_I$ from which follows the condition

$$f_1(x) > q_{i_1},\ \forall x \in I_1.$$

Such a $q_{i_1}$ exists, because $\min\{f_1(x), x \in I_1\} > f(c)$ and the set $Q_I$ is dense in the interval $(f(c), f(a))$. Denote $Q_1$ the sequence, which is created by excluding the element $q_{i_1}$ from the sequence $Q_I$.

In the $n$-th step define the mapping $G$ on the set $\{I_1, I_2, \ldots, I_n\}$, such that

$$G(I_j) = q_{i_j} \in Q_I,\ 1 \le j \le n,$$

$$\forall j \in \{1,2,\dots,n\} \Rightarrow f_1(x) > q_{i_j},\ \forall x \in I_j,$$

$$I_m < I_l \Rightarrow G(I_m) = q_{i_m} > q_{i_l} = G(I_l),\ \forall m,l \in \{1,2,\dots,n\}$$

and let $Q_n$ be the sequence which we get from the sequence $Q_I$ by excluding $q_{i_1}, q_{i_2}, \dots, q_{i_n}$.

In the $n+1$-th step we set $G(I_{n+1}) = q_{i_{n+1}}$, such that $q_{i_{n+1}}$ is the first member of the sequence $Q_n$, that satisfies the following conditions:

$$f_1(x) > q_{i_{n+1}},\ \forall x \in I_{n+1},$$

and

$$\text{if } m,l \in \{1,2,\dots,n\} \wedge I_m < I_{n+1} < I_l \Rightarrow q_{i_m} > q_{i_{n+1}} > q_{i_l}.$$

Such $q_{i_{n+1}}$ exists, because

$$I_m < I_{n+1} < I_l \Rightarrow q_{i_m} > q_{i_l} \wedge \min\{f_1(x), x \in I_{n+1}\} > q_{i_l}$$

and the set $Q_n$ is dense in $(f(c), f(a))$. Denote $Q_{n+1}$ the sequence that we get by excluding the element $q_{i_{n+1}}$ from the sequence $Q_n$.

Next, we show that the mapping $G : \{I_n, n = 1,2,\dots\} \to Q_I$ is one to one mapping. With respect to the construction of $G$ it is sufficient to show that $G$ map the set $\mathcal{I}$ onto the set $Q_I$: $G(\mathcal{I}) = Q_I$. Suppose that $G(\mathcal{I}) \subsetneqq Q_I$. Then there exists a sequence $Q_n$ and $q_0 \in Q_I \setminus G(\mathcal{I})$ such that $q_0$ is the first member of the sequence $Q_n$. Let

$$I_{\max} = \max\left\{I_j; 1 \le j \le n \wedge q_{i_j} > q_0\right\}$$

and

$$I_{\min} = \min\left\{I_j; 1 \le j \le n \wedge q_{i_j} < q_0\right\}.$$

For every $x \in I_{\max}$ the inequality $f_1(x) > q_0$ is valid. Because the function $f_1$ is decreasing, then $\max I_{\max} < f_1^{-1}(q_0)$ and therefore there exists an infinite number of intervals $I_k \in \{I_{n+1}, I_{n+2}, \dots..\}$ such that

$$I_{\max} < I_k < I_{\min} \wedge f_1(x) > q_0, \forall x \in I_k.$$

If $I_{k_0}$ is the first such interval in the sequence $\{I_{n+1}, I_{n+2}, \dots..\}$, then according to the definition of the mapping $G$, $G(I_{k_0}) = q_0$. This contradicts the assumption $q_0 \in Q_I \setminus G(\mathcal{I})$.

Let $g$ be the function defined on the interval $[a, a_0)$ by

$$g(x) = \sup\{G(I), I \in \mathcal{I} \wedge x < \min I\}.$$

If $b_0 < b$ then, in a similar way, we define a function $g$ on the interval $(b_0, b]$ and let

$$g(x) = f(c) \text{ for every } x \in [a_0, b_0].$$

It is easy to verify that the function $g$ is continuous on the interval $[a,b]$, $0 \leq f(c) \leq g(x) \leq f(x)$, $\forall x \in [a,b]$, $f(a) = g(a)$, $f(b) = g(b)$ and moreover, $g$ is constant on each contiguous interval $I$ of the set $[a,b] \cap P$.

Now let the function $g$ be defined on each contiguous interval of the set $K$ in the same manner as above and let

$$g \upharpoonright K = f \upharpoonright K.$$

Such a function $g$ is continuous on the interval $[0,1]$. It is sufficient to show that the function $g$ is continuous at an arbitrary point $x_0 \in K$. Let a sequence $x_j, j = 1, 2, \ldots$ converge to the point $x_0$. The restriction $g \upharpoonright K$ is a continuous function and $g$ is a continuous function on each closed contiguous interval of the set $K$. Applying the same reasoning as in Lemma 1, it can be assumed that $x_j \in I_{n_j}, j = 1, 2, \ldots$, where $I_{n_j} = \left(a_{n_j}, b_{n_j}\right)$ is a sequence of contiguous intervals of the set $K$ and $n_j \to \infty$. If the function $f$ attains its minimum over $\left[a_{n_j}, b_{n_j}\right]$ at a point $c_{n_j}$, then the same holds for the function $g$ and $f\left(c_{n_j}\right) = g\left(c_{n_j}\right)$. The sequence $c_{n_j}, j = 1, 2, \ldots$ again converges to the point $x_0$ and

$$\begin{aligned} f(x_0) &= \lim_{j\to\infty} f\left(c_{n_j}\right) = \lim_{j\to\infty} g\left(c_{n_j}\right) \leq \lim_{j\to\infty} g(x_j) \\ &\leq \lim_{j\to\infty} f(x_j) = f(x_0) = g(x_0). \end{aligned}$$

That is,

$$\lim_{j\to\infty} g(x_j) = g(x_0).$$

The function $g$, as defined above, satisfies the assertion of Lemma 4, because it is continuous, constant on each interval contiguous of the set $P$, $0 \leq g \leq f$ and $g(x) = f(x)$ for every $x \in K$. □

**Lemma 5.** *Let $f$ be a lower semicontinuous function defined on $[0,1]$ and let $K$ be any closed subset of the set $C_f$. If a function $g \leq f$ is continuous, the set $A = \{x; g(x) = f(x)\} \subset C_f$ and the set $A \cup K$ is nowhere dense in $[0,1]$, then there exists a perfect set $P \subset C_f$, nowhere dense in $[0,1]$, and a function $h \in CK(P)$ such that*

$$A \cup K \subset_c P$$

$$h(x) = f(x) \text{ for every } x \in A \cup K$$

$$g(x) < h(x) < f(x) \text{ for every } x \notin A \cup K.$$

PROOF. If $(a,b)$ is a contiguous interval of the set $A\bigcup K$, then, for an arbitrary $x\in(a,b)$,

$$g(x)<h(x)<f(x).$$

We will show that there exist sequences $a_i,b_i\in C_f$, $a_i\downarrow a^+$, $b_i\uparrow b^-$, $i=1,2,\ldots,$ $a_1=b_1$, such that

$$\max\{g(x),x\in[a_{i+1},a_i]\}<\min\{h(x),x\in[a_{i+1},a_i]\}$$

$$\max\{h(x),x\in[a_{i+1},a_i]\}<\min\{f(x),x\in[a_{i+1},a_i]\}$$

for each interval $[a_{i+1},a_i]$ and that the same is true for each interval $[b_i,b_{i+1}]$.

Let $\{x_n\}_{n=0}^{\infty}$, $\{y_n\}_{n=0}^{\infty}$ be arbitrary sequences of points $x_n,y_n\in C_f$,

$$a\leftarrow\cdots<x_2<x_1<x_0=y_0<y_1<y_2<\cdots\rightarrow b.$$

Since $f-h\in lsc,\ h-g\in C$ and

$$\forall x\in[x_{n+1},x_n]:\ f(x)-h(x)>0\wedge h(x)-g(x)>0,$$

then there exists $\varepsilon_n>0$, such that

$$\forall x\in[x_{n+1},x_n]:\ f(x)-h(x)>\varepsilon_n\wedge h(x)-g(x)>\varepsilon_n.$$

The functions $h$ and $g$ are uniformly continuous on the interval $[x_{n+1},x_n]$, so there exists $\delta_n>0$, such that for every $x,y\in[x_{n+1},x_n]$ it holds

$$|x-y|<\delta_n\Rightarrow|h(x)-h(y)|<\frac{1}{3}\varepsilon_n\wedge|g(x)-g(y)|<\frac{1}{3}\varepsilon_n.$$

Now we choose an arbitrary finite sequence of points
$a_i^n\in C_f, i\in\{0,1,2,\ \ldots\ ,k_n\}$,

$$x_{n+1}=a_{k_n}^n<a_{k_n-1}^n<\ldots.<a_2^n<a_1^n<a_0^n=x_n,$$

such that

$$\left|a_{i+1}^n-a_i^n\right|<\delta_n\ \text{ for every } i\in\{0,1,2,\ \ldots\ ,k_n-1\}.$$

Let the function $f$ attain its minimum on the interval $\left[a_{i+1}^n,a_i^n\right]$ at a point $\xi_i^n$ and the function $h$ at a point $\eta_i^n$, that is

$$f(\xi_i^n)=\min\left\{f(x);\,x\in\left[a_{i+1}^n,a_i^n\right]\right\}\wedge h(\eta_i^n)=\min\left\{h(x);\,x\in\left[a_{i+1}^n,a_i^n\right]\right\}.$$

Every $x\in\left[a_{i+1}^n,a_i^n\right]$ satisfies the condition $|x-a_i^n|<\delta_n$. Then

$$h(x)<h(a_i^n)+\frac{1}{3}\varepsilon_n$$

and because $\xi_i^n \in \left[a_{i+1}^n, a_i^n\right]$ then $|\xi_i^n - a_i^n| < \delta_n$. Therefore

$$h\left(a_i^n\right) - \frac{1}{3}\varepsilon_n < h\left(\xi_i^n\right).$$

According to the definition of $\varepsilon_n$, we have $f\left(\xi_i^n\right) - h\left(\xi_i^n\right) > \varepsilon_n$. Consequently, using the foregoing inequalities, we have that $\forall x \in \left[a_{i+1}^n, a_i^n\right]$,

$$f\left(\xi_i^n\right) > h\left(\xi_i^n\right) + \varepsilon_n > h\left(a_i^n\right) - \frac{1}{3}\varepsilon_n + \varepsilon_n > h\left(a_i^n\right) + \frac{1}{3}\varepsilon_n > h\left(x\right).$$

That is,

$$\max\left\{h\left(x\right); x \in \left[a_{i+1}^n, a_i^n\right]\right\} < \min\left\{f\left(x\right); x \in \left[a_{i+1}^n, a_i^n\right]\right\}.$$

If we use the same arguments as in the previous procedure and if $(f, h, \xi_i^n)$ is replaced by $(h, g, \eta_i^n)$, we get that $\forall x \in \left[a_{i+1}^n, a_i^n\right]$,

$$h\left(\eta_i^n\right) > g\left(\eta_i^n\right) + \varepsilon_n > g\left(a_i^n\right) - \frac{1}{3}\varepsilon_n + \varepsilon_n > g\left(a_i^n\right) + \frac{1}{3}\varepsilon_n > g\left(x\right).$$

That is,

$$\max\left\{g\left(x\right); x \in \left[a_{i+1}^n, a_i^n\right]\right\} < \min\left\{h\left(x\right); x \in \left[a_{i+1}^n, a_i^n\right]\right\}.$$

It is evident that the sequence $\left\{\{a_i^n\}_{i=0}^{k_n-1}\right\}_{n=0}^{\infty}$ converges to the point $a$ from the right hand side and satisfies the inequalities from the preface of the proof. On the interval $[y_0, b)$ we proceed analogously.

We choose a perfect nowhere dense subset $P$ of the interval $[a, b]$ such that the set $\{a_i, b_i; i = 1, 2, \dots\} \subset_c P$. Then Lemma 4 implies that there exists a continuous function $h_1$ defined on the interval $[a, b]$, such that

$$h_1\left(a\right) = h\left(a\right),\ h_1\left(b\right) = h\left(b\right),$$

and

$$h_1\left(a_i\right) = h\left(a_i\right),\ h_1\left(b_i\right) = h\left(b_i\right),$$

$$\min\left\{h\left(a_i\right), h\left(a_{i+1}\right)\right\} \le h_1\left(x\right) \le \max\left\{h\left(a_i\right), h\left(a_{i+1}\right)\right\}$$

hold for every $i = 1, 2, \dots$ and every $x \in \left(a_{i+1}, a_i\right)$. In the same manner,

$$\min\left\{h\left(b_i\right), h\left(b_{i+1}\right)\right\} \le h_1\left(x\right) \le \max\left\{h\left(b_i\right), h\left(b_{i+1}\right)\right\}$$

for every $x \in \left(b_i, b_{i+1}\right)$. Moreover, the function $h_1$ is constant on every contiguous interval of the set $P$. Naturally, since the set $\{a, b, a_i, b_i; i = 1, 2, \dots\} \subset_c C_f$, according to Lemma 2 in [8] we can choose $P \subset C_f$. If we replace the function $h$ by the function of type $h_1$ on every contiguous interval of the set $A \cup K$, we obtain the assertion of Lemma 5. □

The class of strong Świątkowski functions was defined by T. Mańk and T. Świątkowski in [7] .

**Definition 6.** *We say that $f$ is a strong Świątkowski function if, whenever $a < b$ and $y$ is a number between $f(a)$ and $f(b)$, then there exists an $x_0 \in (a,b) \cap C_f$ such that $f(x_0) = y$.*

**Lemma 7.** *Suppose that a sequence of continuous functions $s_1 \leq s_2 \leq s_3 \leq \dots$ converges on $[0,1]$ to the function $s$. For a double sequence of positive real numbers $(\delta_n, \varepsilon_n)$, $n = 1, 2, \dots,$ $(\delta_n, \varepsilon_n) \to (0,0)$ and a sequence of closed sets $F_1 \subseteq F_2 \subseteq F_3 \subseteq \dots,$ we consider the following properties:*

*(i)* $|x_1 - x_2| < \delta_n \Rightarrow |s_n(x_1) - s_n(x_2)| < \varepsilon_n$

*(ii)* $x \in [0,1] \Rightarrow \operatorname{dist}(x, F_n) < \delta_n$

*(iii)* $s_{n+1}/F_n = s_n/F_n$*, for every* $n = 1, 2, \dots$

*(iv)* $F = F_1 \cup F_2 \cup F_3 \cup \dots \subset C_s$

*(v) if $x_o \in D_s$, there are sequences $x_i, y_i \in F, i = 1, 2, \dots,$ $x_i \uparrow x_0$, $y_i \downarrow x_0$, such that*

$$s(x_i) \leq s(x_0), \ s(y_i) \leq s(x_0) \text{ and } s(x_i) \to s(x_0), \ s(y_i) \to s(x_0).$$

*We make the following inference about the function $s$ :*

1. *From properties (i)–(iii), it follows that the function $s \in Dlsc$,*
2. *From properties (i)–(iv) it follows that the function $s \in QDlsc$, and*
3. *From properties (i)–(v) it follows that the function $s \in S^*lsc$.*

Proof. Evidently $s \in lsc$. Let *(i)–(iii)* be satisfied. Then it is sufficient to show ([2]) that for an arbitrary $x_0 \in [0,1]$, there exist sequences $x_n \uparrow x_0$ and $y_n \downarrow x_0$ such that

$$\lim_{n\to\infty} s(x_n) = \lim_{n\to\infty} s(y_n) = s(x_0).$$

Naturally, if $x_0 = 0$ or $x_0 = 1$, we consider only one of these. Given the assumptions of Lemma 7, for every $n = 1, 2 \dots$ there exist $x_n < x_0 < y_n$, $x_n, y_n \in F_n$ such that $|x_n - x_0| < 2\delta_n$, $|y_n - x_0| < 2\delta_n$ and $s(x_n) = s_n(x_n)$, $s(y_n) = s_n(y_n)$. Moreover $|s_n(x_n) - s_n(x_0)| < 2\varepsilon_n$ and $|s_n(y_n) - s_n(x_0)| < 2\varepsilon_n$. Since $(\delta_n, \varepsilon_n) \to (0,0)$ and $s_n(x_0) \to s(x_0)$, the inequality

$$\begin{aligned} |s(x_n) - s(x_0)| = |s_n(x_n) - s(x_0)| &\leq |s_n(x_n) - s_n(x_0)| + |s_n(x_0) - s(x_0)| \\ &< 2\varepsilon_n + |s_n(x_0) - s(x_0)| \end{aligned}$$

implies that $s(x_n) \to s(x_0)$ and analogously $s(y_n) \to s(x_0)$.

Let *(i)–(iv)* be satisfied. Because in the sequences above $x_n, y_n \in C_s$, the assertion that $s \in Qlsc$ directly follows from Lemma 3.4. in [4].

Let *(i)–(v)* be satisfied and let $a < b$ and $y$ be a number between $s(a)$ and $s(b)$. We assume that $s(a) > y > s(b)$ and denote $x_0 = \min\{x \in [a,b]; s(x) \leq y\}$. Such $x_0$ exists, because the function $s \in lsc$, $y > s(b)$ and thus the set $\{x \in [a,b]; s(x) \leq y\}$ is not empty and closed. Evidently $s(x_0) = y$, since opposite case leads to a contradiction with the Darboux property of the function $s$. The point $x_0 \in C_s$. In the case $x_0 \in D_s$, *(v)* implies the existence of a point $x_1 < x_0$ such that $s(x_1) \leq s(x_0) = y$, which contradicts to $x_0 = \min\{x \in [a,b]; s(x) \leq y\}$. We proceed analogously when $s(a) < y < s(b)$. □

**Theorem 8.** *Let $f$ be a lower semicontinuous function. Then there are strong Świątkowski lower semicontinuous functions $g$ and $h$ such that $f = g + h$.*

PROOF. The function $f \in lsc$. Without loss of generality we may consider $f > 0$ and the existence of sequence of continuous functions $0 < f_1^0 < f_2^0 < f_3^0 < \cdots \to f$. According to Lemma 5, we can construct a sequence of nowhere dense perfect sets $P_n \subset C_f$ and a sequence of functions $f_n \in CK(P_n)$, $n = 1, 2, \ldots$ such that $f_n^0 < f_n < f_{n+1}^0$ and $P_n \subset_c P_{n+1}$. Therefore, let

$$P_1 \subset_c P_2 \subset_c P_3 \subset_c \cdots \subset C_f \subset [0,1]$$

to be a sequence of nowhere dense perfect sets and let

$$0 < f_1 < f_2 < f_3 < \ldots,\ f_n \in CK(P_n), n = 1, 2, \ldots,$$

be a sequence of functions which converges on $[0,1]$ to the function $f$. Let

$$D_f = \bigcup_{n=1}^{\infty} D_n,\ \text{where } D_1 \subset D_2 \subset D_3 \subset \ldots$$

are closed sets. We denote

$$P = \bigcup_{n=1}^{\infty} P_n.$$

Let $\varepsilon_n$, $n = 1, 2, \ldots$ be a sequence of positive real numbers, $\varepsilon_n \to 0$.

In the first step we define

$$f_1 = f_1^* = h_1 = h_1^*,\ g_1 = g_1^* = 0.$$

The functions $h_1$, $g_1$ are uniformly continuous on $[0,1]$. Then for given $\varepsilon_1 > 0$, there exists $\delta_1 > 0$ such that for every $x_1, x_2 \in [0,1]$ it holds

$$| x_1 - x_2 | < \delta_1 \Rightarrow | g_1(x_1) - g_1(x_2) | < \varepsilon_1 \wedge \ | h_1(x_1) - h_1(x_2) | < \varepsilon_1 .$$

Let $F_1 \subset C_f$ be a finite set, such that

$$\operatorname{dist}(x, F_1) < \delta_1, \text{ for every } x \in [0,1] .$$

If $I_1^k, k = 1, 2, \ldots$ is the sequence of all contiguous intervals of the set $P_1$, only for a finite number of intervals $I_1^k$ it holds that $I_1^k \cap D_1 \neq \emptyset$. In the case $I_1^k \cap D_1 \neq \emptyset$, we may choose a finite set $F_1$ such that the boundary points of interval $I_1^k$ are from the set $F_1$. Let $K_1$ be a finite subset of the set $C_f \setminus P$, such that:

1. $K_1 \cap F_1 = \emptyset$ and $\operatorname{dist}(x, K_1) < \delta_1, \text{ for every } x \in [0,1]$.
2. If $I_1^k \cap D_1 \neq \emptyset$ then $K_1 \cap I_1^k \neq \emptyset$, $\min(K_1 \cap I_1^k) < \min(I_1^k \cap D_1)$ and $\max(I_1^k \cap D_1) < \max\left(K_1 \cap I_1^k\right)$.

We continue to the second step. According to Lemma 5, there exist a nowhere dense perfect set $P_2^*$ and a function $f_2^* \in CK(P_2^*)$ such that

$$\begin{gathered} F_1 \cup P_2 \subset_c P_2^* \subset C_f \setminus K_1, \\ f_2^*(x) = f(x) \text{ for every } x \in F_1, \\ \max\{f_1^*(x), f_2(x)\} < f_2^*(x) < f(x) \text{ for every } x \notin F_1. \end{gathered}$$

Denote $I_2^k, k = 1, 2, \ldots$ the sequence of all contiguous closed intervals of the set $P_2^*$. The sets $F_1$, $K_1$ are finite, $F_1 \cap K_1 = \emptyset$. Because $F_1 \subset_c P_2^*$ then $I_2^k \cap F_1 = \emptyset$ for each $k = 1, 2, \ldots$ . We know that $I_2^k \cap (D_1 \cup K_1) \neq \emptyset$ holds only for finite number of intervals $I_2^k$. Let the set $\left\{I_2^{k_1}, I_2^{k_2}, \ldots, I_2^{k_m}\right\}$ consist of all of these intervals. The set $F_1$ and the set $\bigcup I_2^{k_i}, i = 1, 2, \ldots, m$ are closed and disjoint. Then the function

$$g_2^*(x) = \begin{cases} 0 & \text{if } x \in \bigcup I_2^{k_i}, i = 1, 2, \ldots, m \\ f_2^*(x) - f_1^*(x) & \text{if } x \in F_1 \end{cases}$$

is continuous on the closed set $F_1 \cup \left(\bigcup I_2^{k_i}, i = 1, 2, \ldots, m\right)$. According to the Tietze theorem there is a continuous extension of the function $g_2^*$ on $[0,1]$. Since $f_2^* - f_1^*$ is a continuous function and $0 \leq f_2^* - f_1^*$ then there exists a continuous extension $g_2^*$ such that $0 \leq g_2^* \leq f_2^* - f_1^*$. Consequently, by Lemma 4, there exists a continuous function $g_2^*$, $0 \leq g_2^* \leq f_2^* - f_1^*$ such that

$$g_2^*\left(I_2^k\right) = 0, \text{ if } I_2^k \cap (D_1 \cup K_1) \neq \emptyset,$$
$$g_2^* \in CK\left(P_2^*\right),$$
$$g_2^*(x) = f_2^*(x) - f_1^*(x) \text{ for every } x \in F_1.$$

We define the function $h_2^*$ by the equation

$$f_2^* - f_1^* = g_2^* + h_2^*$$

and the functions $g_2$ and $h_2$:

$$g_2 = g_1 + g_2^*, \; h_2 = h_1 + h_2^*.$$

The functions $h_2$, $g_2$ are uniformly continuous on $[0,1]$, then for given $\varepsilon_2 > 0$, there exists $\delta_2 > 0$ such that for every $x_1, x_2 \in [0,1]$ it holds

$$| x_1 - x_2 | < \delta_2 \Rightarrow | g_2(x_1) - g_2(x_2) | < \varepsilon_2 \wedge \; | h_2(x_1) - h_2(x_2) | < \varepsilon_2.$$

Let $F_2 \subset C_f \setminus K_1$ be a finite set, $F_1 \subset F_2$, such that for every $x \in [0,1]$

$$\operatorname{dist}(x, F_2) < \frac{1}{2}\delta_2.$$

Again, we may choose a set $F_2$ such that if $I_2^k \cap D_2 \neq \emptyset$, then the boundary points of the interval $I_2^k$ are from the set $F_2$. Let $K_2 \supset K_1$ be a finite subset of the set $C_f \setminus P \cup P_2^*$, such that:

1. $K_2 \cap F_2 = \emptyset$, and $\operatorname{dist}(x, K_2) < \frac{1}{2}\delta_2$, for every $x \in [0,1]$.
2. If $I_2^k \cap D_2 \neq \emptyset$ then $K_2 \cap I_2^k \neq \emptyset$, $\min(K_2 \cap I_2^k) < \min(I_2^k \cap D_2)$ and $\max(I_2^k \cap D_2) < \max\left(K_2 \cap I_2^k\right)$.

By induction, for every $n = 2, 3, 4, \ldots$ can be found nowhere dense perfect set $P_n^*$, $P_{n-1}^* \subset_c P_n^*$, a continuous function $f_n^* \in CK\left(P_n^*\right)$:

$$F_{n-1} \cup P_n \subset_c P_n^* \subset C_f \setminus K_{n-1}, \tag{1}$$
$$f_n^*(x) = f(x) \text{ for every } x \in F_{n-1}, \tag{2}$$
$$\max\left\{f_{n-1}^*(x), f_n(x)\right\} < f_n^*(x) < f(x) \text{ for every } x \notin F_{n-1}, \tag{3}$$

and a continuous function $g_n^*, 0 \leq g_n^* \leq f_n^* - f_{n-1}^*$ such that

$$g_n^*\left(I_n^k\right) = 0, \text{ if } I_n^k \cap (D_{n-1} \cup K_{n-1}) \neq \emptyset, \tag{4}$$
$$g_n^* \in CK\left(P_n^*\right), \tag{5}$$
$$g_n^*(x) = f_n^*(x) - f_{n-1}^*(x) \text{ for every } x \in F_{n-1} \tag{6}$$

where $I_n^k, k = 1, 2, \ldots$ are contiguous intervals of the set $P_n^*$. We define the function $h_n^*$ by the equation

$$f_n^* - f_{n-1}^* = g_n^* + h_n^* \tag{7}$$

and the functions $g_n$ and $h_n$:

$$g_n = g_{n-1} + g_n^*, \; h_n = h_{n-1} + h_n^*. \tag{8}$$

For given $\varepsilon_n > 0$, there exists $\delta_n, 1 \geq \delta_n > 0$ such that for every $x_1, x_2 \in [0, 1]$ it holds

$$\mid x_1 - x_2 \mid < \delta_n \Rightarrow \mid g_n(x_1) - g_n(x_2) \mid < \varepsilon_n \wedge \; \mid h_n(x_1) - h_n(x_2) \mid < \varepsilon_n.$$

Let $F_n \subset C_f \setminus K_{n-1}$ be a finite set, $F_{n-1} \subset F_n$, such that for every $x \in [0, 1]$

$$\operatorname{dist}(x, F_n) < \frac{1}{n}\delta_n.$$

Again, we may choose a set $F_n$ such that if $I_n^k \cap D_n \neq \emptyset$, then the boundary points of interval $I_n^k$ are from the set $F_n$. Let $K_n \supset K_{n-1}$ be a finite subset of the set $C_f \setminus P \cup P_n^*$, such that the following two conditions hold:

1. $K_n \cap F_n = \emptyset$ and $\operatorname{dist}(x, K_n) < \frac{1}{n}\delta_n,$ for every $x \in [0, 1]$.
2. If $I_n^k \cap D_n \neq \emptyset$, then $K_n \cap I_n^k \neq \emptyset$, $\min(K_n \cap I_n^k) < \min(I_n^k \cap D_n)$ and $\max(I_n^k \cap D_n) < \max\left(K_n \cap I_n^k\right)$.

We notice that the sequences of continuous functions $g_n$ and $h_n$, $n = 1, 2, \ldots$ are nondecreasing, $f_n^* = g_n + h_n$. From the inequalities $0 < f_n < f_n^* \leq f$, it follows that the sequence $f_n^*$ converges to the function $f$. Evidently the sequences $g_n$ and $h_n$ are convergent too, $g_n \to g \in lsc$, $h_n \to h \in lsc$ and $g + h = f$. Moreover, we have sequences of closed sets

$$F_1 \subset F_2 \subset F_3 \subset \ldots \text{ and } K_1 \subset K_2 \subset K_3 \subset \ldots$$

and the double sequence $\left(\frac{1}{n}\delta_n, \varepsilon_n\right) \to (0, 0)$, such that:

(i.) If $|x_1 - x_2| < \frac{\delta_n}{n}$ then

$$\mid g_n(x_1) - g_n(x_2) \mid < \varepsilon_n \text{ and } \mid h_n(x_1) - h_n(x_2) \mid < \varepsilon_n.$$

(ii.) If $x \in [0,1]$ then $\operatorname{dist}(x, F_n) < \frac{1}{n}\delta_n$ and $\operatorname{dist}(x, K_n) < \frac{1}{n}\delta_n$.

Evidently, from (4) it follows that $g^*_{n+1}/K_n = 0$ and from (8) we see that $g_{n+1}/K_n = g_n/K_n + g^*_{n+1}/K_n$.

Moreover, from (7) and (6) it follows that

$$h^*_{n+1}/F_n = f^*_{n+1}/F_n - f^*_n/F_n - g^*_{n+1}/F_n = 0,$$

and from (8) that $h_{n+1}/F_n = h_n/F_n + h^*_{n+1}/F_n$. Putting these together we conclude that:

(iii.) $g_{n+1}/K_n = g_n/K_n$ and $h_{n+1}/F_n = h_n/F_n$.

Because the double sequence $\left(\frac{1}{n}\delta_n, \varepsilon_n\right) \to (0,0)$, the functions $h$ and $g$ satisfy conditions $(i)$–$(iii)$ of Lemma 7 and thus $g, h \in Dlsc$. Since $f, g, h \in lsc$ and $f = g + h$ it is easy to show that the set $C_f$ is the subset of the set $C_g \cap C_h$. Therefore,

(iv.)

$$K = K_1 \cup K_2 \cup K_3 \cup \cdots \subset C_f \subset C_g, \text{ and } F = F_1 \cup F_2 \cup F_3 \cup \cdots \subset C_f \subset C_h.$$

Next we prove that the functions $h$ and $g$ are strong Świątkowski functions. Because conditions $(i)$–$(iv)$ are satisfied, it is sufficient to prove that the functions $h$ and $g$ satisfy the condition $(v)$ in Lemma 7, too.

Let $x_0$ be an arbitrary point of discontinuity of the function $g$. Because $D_g \subset D_f$, there exists $n_0$ such that $x_0 \in D_{n_0} \ \wedge \ x_0 \notin D_n, \ for \ n < n_0$ and a sequence $\left\{I_n^{k_n}\right\}_{n=1}^{\infty}, \ I_1^{k_1} \supset I_2^{k_2} \supset I_3^{k_3} \supset \cdots \supset \{x_0\}$, where $I_n^{k_n}$ is a contiguous interval of the sets $P^*_n$. Each function $g^*_n, n \le n_0$ is constant on the interval $I_n^{k_n}$, and according to (4), $g^*_n/I_n^{k_n} = 0, \ for \ n > n_0$. If $n > n_0$ then $I_n^{k_n} \cap D_n \ne \emptyset$ and therefore for every $n > n_0$ we can choose points $x_n, y_n \in K_n \cap I_n^{k_n}, \ x_n < x_0 < y_n$. We may demand $\operatorname{dist}(x_n, y_n) < \frac{2}{n}\delta_n$. Evidently $x_n \uparrow x_0 \wedge y_n \downarrow x_0$ and

$$g(x_n) = g_{n_0}(x_n) = g_{n_0}(x_0) = g(x_0),$$
$$g(y_n) = g_{n_0}(y_n) = g_{n_0}(x_0) = g(x_0)$$

and thus the function $g$ satisfies the condition $(v)$ from Lemma 7.

Now let $x_0$ be an arbitrary point of discontinuity of the function $h$. Again, because $D_h \subset D_f$, there exists $n_0$ such that $x_0 \in D_{n_0}$ with $x_0 \notin D_n$ for $n < n_0$ and there exists a sequence of contiguous intervals

$$I_{n_0} \supset I_{n_0+1} \supset I_{n_0+2} \supset \cdots \supset \{x_0\},$$

and of perfect sets

$$P^*_{n_0} \subset_c P^*_{n_0+1} \subset_c P^*_{n_0+2} \subset \dots .$$

Let $I_{n_0+j} = (x_j, y_j)$, $j = 1, 2, \dots$ . Because

$$I_{n_0+j} \cap D_{n_0+j} \supset I_{n_0+j} \cap D_{n_0} \supset \{x_0\} \neq \emptyset$$

the points $x_j, y_j \in F_{n_0+j}$. Using the same arguments as in the the paragraph above, $x_j \uparrow x_0 \wedge y_j \downarrow x_0$. According to Remark 3, from $(7), (8)$ it follows that the function $h_{n_0+j} \in CK\left(P^*_{n_0+j}\right)$. Then the function $h_{n_0+j}$ is constant on the interval $[x_j, y_j]$ and therefore

$$h_{n_0+j}(x_j) = h_{n_0+j}(x_0) = h_{n_0+j}(y_j).$$

The point $x_j \in F_{n_0+j} \subset F_{n_0+j+1} \subset F_{n_0+j+2} \subset \dots$ . Then according to (iii.) we have

$$h_{n_0+j}(x_j) = h_{n_0+j+1}(x_j) = h_{n_0+j+2}(x_j) = \dots = h(x_j)$$

and

$$h(x_j) = h_{n_0+j}(x_j) = h_{n_0+j}(x_0) \le h(x_0).$$

Based on the same reasoning

$$h(y_j) = h_{n_0+j}(y_j) = h_{n_0+j}(x_0) \le h(x_0)$$

holds, too. The function $h$ also satisfies the condition $(v)$ from Lemma 7 and then by Lemma 7 the functions $g, h \in S^*lsc$. □

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