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# ADDITIVE PROPERTIES OF CERTAIN CLASSES OF PATHOLOGICAL FUNCTIONS

**Abstract**

Some additive properties of the following three families of "pathological" functions are briefly discussed: continuous nowhere differentiable functions, Sierpiński-Zygmund functions, and absolutely nonmeasurable functions.

In this note we will be dealing with additive properties of some families of "pathological" functions (cf. [2]–[6], [8], [9], [13], [14]).

Let us consider the following three classes of functions acting from the real line $\mathbf{R}$ into itself:

1. Continuous nowhere differentiable functions;
2. Sierpiński-Zygmund functions, i.e., those functions whose restrictions to all subsets of $\mathbf{R}$ of cardinality continuum are discontinuous;
3. Absolutely nonmeasurable functions, i.e., those functions which are nonmeasurable with respect to all nonzero $\sigma$-finite continuous measures on $\mathbf{R}$.

The functions belonging to the first class are very bad from the differential point of view, the functions belonging to the second class are very bad from the topological point of view, and the functions belonging to the third class can be regarded as very bad from the measure-theoretical point of view.

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As is well known, there are concrete individual examples of continuous nowhere differentiable functions (recall that the first examples of such kind are due to Bolzano and Weierstrass). The next important step was made by Banach [1] and Mazurkiewicz [11]. They demonstrated (independently) that in the space $C[0,1]$ of all continuous real-valued functions defined on the unit segment $[0,1]$ the family of nowhere differentiable functions is co-meager, i.e., is the complement of a first category subset of $C[0,1]$.

A nontrivial consequence of their result is the following fact:

> Any continuous real-valued function on $\mathbf{R}$ can be represented as a sum (difference) of two continuous nowhere differentiable functions.

This fact does not follow from concrete individual constructions of a continuous nowhere differentiable function on $\mathbf{R}$, so needs an argument based on the above-mentioned result of Banach and Mazurkiewicz. Indeed, for any real numbers $a$ and $b$ such that $a < b$, consider the Banach space $C[a,b]$ of all continuous real-valued functions on $[a,b]$ and denote by $D \subset C[a,b]$ the set of all nowhere differentiable functions on $[a,b]$. Take any function $g$ from $C[a,b]$ and observe that both sets $D$ and $g + D$ are co-meager in $C[a,b]$. Consequently, $D \cap (g + D) \neq \emptyset$ which directly implies the existence of two functions $g_1 \in D$ and $g_2 \in D$ such that $g = g_1 - g_2$. Now, let $f : \mathbf{R} \to \mathbf{R}$ be an arbitrary continuous function and let $\mathbf{Z}$ denote, as usual, the set of all integers. For every integer $n$, let $f_n$ be the restriction of $f$ to the segment $[n, n+1]$. According to the said above, we may write

$$f_n = \phi_n - \psi_n \qquad (n \in \mathbf{Z}),$$

where $\phi_n$ and $\psi_n$ are some continuous and nowhere differentiable functions on $[n, n+1]$. Without loss of generality, we may assume that

$$\phi_n(n+1) = \phi_{n+1}(n+1) \qquad (n \in \mathbf{Z}).$$

Keeping in mind these "contact conditions" for the family of functions $\{\phi_n : n \in \mathbf{Z}\}$ and taking into account the trivial equalities

$$f_n(n+1) = f_{n+1}(n+1) \qquad (n \in \mathbf{Z}),$$

we get the analogous "contact conditions" for the family of functions $\{\psi_n : n \in \mathbf{Z}\}$, i.e.,

$$\psi_n(n+1) = \psi_{n+1}(n+1) \qquad (n \in \mathbf{Z}).$$

Therefore, denoting by $\phi$ (respectively, by $\psi$) the common extension of all $\phi_n$ $(n \in \mathbf{Z})$ (respectively, of all $\psi_n$ $(n \in \mathbf{Z})$), we obtain that both $\phi$ and $\psi$ are continuous nowhere differentiable functions on $\mathbf{R}$ and $f = \phi - \psi$.

On the other hand, it was shown that there are sufficiently large vector subspaces $U$ of $C[0,1]$ such that all members of $U \setminus \{0\}$ are nowhere differentiable functions (see, for instance, [5] and [6]). Moreover, it was proved in the article [14] that, for every separable Banach space $W$, there exists a closed vector subspace of $C[0,1]$ which is isometric to $W$ and all nonzero members of which are nowhere differentiable functions.

The second type of "pathological" functions are Sierpiński-Zygmund functions first introduced in [15]. They are usually constructed by the method of transfinite recursion and it is clear that their existence needs uncountable forms of the Axiom of Choice (since every such function turns out to be nonmeasurable with respect to the standard Lebesgue measure on $\mathbf{R}$).

Moreover, every Sierpiński-Zygmund function is nonmeasurable with respect to the completion of any nonzero continuous (diffused) $\sigma$-finite Borel measure on $\mathbf{R}$. However, it was demonstrated in [7] that there exists a translation invariant measure $\mu$ on $\mathbf{R}$ extending the standard Lebesgue measure and such that some Sierpiński-Zygmund functions become measurable with respect to $\mu$.

For various interesting properties of Sierpiński-Zygmund functions, see, e.g., [2] and references given therein.

It is not difficult to prove the following two statements (cf. [13], Proposition 1).

**Theorem 1.** *Any function from $\mathbf{R}^{\mathbf{R}}$ can be represented as a sum (difference) of two injective Sierpiński-Zygmund functions.*

**Theorem 2.** *Any additive function from $\mathbf{R}^{\mathbf{R}}$ can be represented as a sum (difference) of two injective additive Sierpiński-Zygmund functions.*

For the sake of completeness, we present the proof of Theorem 2 here. The proof of Theorem 1 can be done analogously and, in fact, is much easier.

Let $\mathbf{Q}$ denote the field of all rational numbers, $\omega$ denote the least infinite cardinal number, and let $\mathbf{c}$ denote the cardinality of the continuum. We may identify $\mathbf{c}$ with the initial ordinal number equinumerous with $\mathbf{c}$. Fix a Hamel basis $\{e_\xi : \xi < \mathbf{c}\}$ of $\mathbf{R}$. Let $\{h_\xi : \xi < \mathbf{c}\}$ be the family of all real-valued partial Borel functions whose domains are uncountable Borel subsets of $\mathbf{R}$.

Let $f : \mathbf{R} \to \mathbf{R}$ be any additive function. We are going to construct by transfinite recursion two injective additive functions

$$f_1 : \mathbf{R} \to \mathbf{R}, \quad f_2 : \mathbf{R} \to \mathbf{R}$$

such that $f = f_1 + f_2$. For this purpose, it suffices to define recursively the values

$$f_1(e_\xi), \quad f_2(e_\xi) \qquad (\xi < \mathbf{c}).$$

Suppose that, for an ordinal number $\xi < \mathbf{c}$, the two $\xi$-sequences of real numbers

$$\{f_1(e_\zeta) : \zeta < \xi\}, \quad \{f_2(e_\zeta) : \zeta < \xi\}$$

have already been defined so that the corresponding partial additive functions

$$f_1 : E_\xi \to \mathbf{R}, \quad f_2 : E_\xi \to \mathbf{R}$$

are injective, where $E_\xi$ denotes the vector space over $\mathbf{Q}$ generated by the family $\{e_\zeta : \zeta < \xi\}$. We may assert that there exist two real numbers $y_1$ and $y_2$ satisfying the relations:

$$f(e_\xi) = y_1 + y_2, \quad y_1 \notin f_1(E_\xi), \quad y_2 \notin f_2(E_\xi),$$

$$(f_1(E_\xi) + \mathbf{Q}y_1) \cap (\cup\{h_\zeta(E_\xi + \mathbf{Q}e_\xi) : \zeta < \xi\}) = \emptyset,$$

$$(f_2(E_\xi) + \mathbf{Q}y_2) \cap (\cup\{h_\zeta(E_\xi + \mathbf{Q}e_\xi) : \zeta < \xi\}) = \emptyset.$$

Indeed, the existence of $y_1$ and $y_2$ easily follows from the inequalities

$$\mathrm{card}(E_\xi) \leq \mathrm{card}(\xi) + \omega < \mathbf{c},$$

$$\mathrm{card}(\cup\{h_\zeta(E_\xi + \mathbf{Q}e_\xi) : \zeta < \xi\}) \leq \mathrm{card}(\xi) + \omega < \mathbf{c}.$$

Now, we put $f_1(e_\xi) = y_1$ and $f_2(e_\xi) = y_2$. Proceeding in this manner, we obtain all the values

$$f_1(e_\xi), \quad f_2(e_\xi) \qquad (\xi < \mathbf{c})$$

and, consequently, the corresponding two injective additive functions

$$f_1 : \mathbf{R} \to \mathbf{R}, \quad f_2 : \mathbf{R} \to \mathbf{R}.$$

It is not difficult to verify that both $f_1$ and $f_2$ are Sierpiński-Zygmund functions.

**Remark 1.** Let $f : \mathbf{R} \to \mathbf{R}$ be a function. We shall say that $f$ is a Sierpiński-Zygmund function in the strong sense if, for every set $X \subset \mathbf{R}$ with $\mathrm{card}(X) = \mathbf{c}$, the restriction $f|X$ is not a Borel function. Both functions $f_1$ and $f_2$ constructed above are Sierpiński-Zygmund functions in the strong sense. Notice that, under Martin's Axiom, there exist Sierpiński-Zygmund functions which are not Sierpiński-Zygmund functions in the strong sense.

The family $\mathbf{R}^\mathbf{R}$ of all functions acting from $\mathbf{R}$ into itself carries the canonical structure of a vector space over the field $\mathbf{R}$. It was demonstrated in the article [3] that there exists a vector subspace $V$ of $\mathbf{R}^\mathbf{R}$ such that all members of $V \setminus \{0\}$ are Sierpiński-Zygmund functions and the cardinality of $V$ is strictly greater than the cardinality of the continuum.

The following statement strengthens the above-mentioned result of [3].

**Theorem 3.** *There exists a vector subspace $V$ of $\mathbf{R}^{\mathbf{R}}$ such that:*

(1) *all members of $V$ are additive functions;*

(2) *all members of $V \setminus \{0\}$ are Sierpiński-Zygmund functions;*

(3) *the cardinality of $V$ is strictly greater than the cardinality of the continuum.*

Proof. Let $\{e_\xi : \xi < \mathbf{c}\}$ be again a Hamel basis of $\mathbf{R}$.

Let $\{K_\xi : \xi < \mathbf{c}\}$ be an increasing (by inclusion) transfinite sequence of subfields of $\mathbf{R}$ such that

$$K_0 = \mathbf{Q}, \quad \cup\{K_\xi : \xi < \mathbf{c}\} = \mathbf{R}, \quad \mathrm{card}(K_\xi) \leq \mathrm{card}(\xi) + \omega$$

for each ordinal number $\xi < \mathbf{c}$.

Let $\{h_\xi : \xi < \mathbf{c}\}$ be again the family of all real-valued partial Borel functions whose domains are uncountable Borel subsets of $\mathbf{R}$.

Let $\mathbf{c}^+$ denote the successor of $\mathbf{c}$, i.e., the least cardinal number strictly greater than $\mathbf{c}$. We may assume that $\mathbf{c}^+$ coincides with the initial ordinal number of the same cardinality, i.e, for any ordinal number $\alpha < \mathbf{c}^+$, we have $\mathrm{card}(\alpha) < \mathbf{c}^+$.

We are going to construct by transfinite recursion a family $\{f_\alpha : \alpha < \mathbf{c}^+\}$ of additive functions acting from $\mathbf{R}$ into $\mathbf{R}$.

Suppose that, for an ordinal $\beta < \mathbf{c}^+$, the partial family $\{f_\alpha : \alpha < \beta\}$ has already been constructed.

In order to define the additive function $f_\beta$, it suffices to define the values

$$f_\beta(e_\xi) \quad (\xi < \mathbf{c}).$$

Since $\mathrm{card}(\beta) \leq \mathbf{c}$, we may represent $\{f_\alpha : \alpha < \beta\}$ in the form of a $\mathbf{c}$-sequence $\{g_\xi : \xi < \mathbf{c}\}$. Here we do not assume that $\{g_\xi : \xi < \mathbf{c}\}$ is necessarily injective (if $\mathrm{card}(\beta) < \mathbf{c}$, then it is clear that the corresponding $\{g_\xi : \xi < \mathbf{c}\}$ cannot be injective).

For every ordinal $\xi < \mathbf{c}$, let $V_\xi$ denote the vector space over $K_\xi$ generated by the family of functions $\{g_\zeta : \zeta < \xi\}$.

Also, for every ordinal $\xi < \mathbf{c}$, introduce the notation:

$E_\xi$ = the vector space over $\mathbf{Q}$ generated by the family $\{e_\zeta : \zeta < \xi\}$;

$E'_\xi$ = the vector space over $\mathbf{Q}$ generated by the family $\{e_\zeta : \zeta \leq \xi\}$.

Now, we define the values $f_\beta(e_\xi)$ $(\xi < \mathbf{c})$ by transfinite recursion over $\xi$.

Suppose that, for $\xi < \mathbf{c}$, the partial family $\{f_\beta(e_\zeta) : \zeta < \xi\}$ has already been constructed. Then we may consider the corresponding additive functional $f_\beta$ on the vector space $E_\xi$. As usual, denote

$$V_\xi(E'_\xi) = \{g(x) : g \in V_\xi,\ x \in E'_\xi\}$$

and choose the value $f_\beta(e_\xi)$ so that the relation

$$(V_\xi(E'_\xi) + K_\xi f_\beta(E_\xi) + (K_\xi \setminus \{0\}) f_\beta(e_\xi)) \cap (\cup\{h_\zeta(E'_\xi) : \zeta < \xi\}) = \emptyset$$

would be satisfied. It is not difficult to check that such a choice of $f_\beta(e_\xi)$ is always possible because of the inequalities

$$\mathrm{card}(V_\xi) \leq \mathrm{card}(\xi) + \omega < \mathbf{c},$$

$$\mathrm{card}(E_\xi) \leq \mathrm{card}(E'_\xi) \leq \mathrm{card}(\xi) + \omega < \mathbf{c}.$$

Proceeding in this manner, we obtain the family of real numbers $f_\beta(e_\xi)$ ($\xi < \mathbf{c}$) and, consequently, the associated additive function $f_\beta : \mathbf{R} \to \mathbf{R}$.

So we come to the transfinite sequence of additive functions $\{f_\alpha : \alpha < \mathbf{c}^+\}$ each of which acts from $\mathbf{R}$ into itself.

Let $V$ denote the vector space over $\mathbf{R}$ generated by $\{f_\alpha : \alpha < \mathbf{c}^+\}$. We claim that $V$ is as required, i.e., $V$ satisfies conditions (1) - (3) of the theorem.

Indeed, condition (1) is trivially valid.

Let us show that condition (2) holds true, too. For this purpose, take any nonzero function $f$ from $V$ and any function $h$ from the family $\{h_\xi : \xi < \mathbf{c}\}$.

The function $f$ can be written as

$$f = t_1 f_{\alpha_1} + t_2 f_{\alpha_2} + \cdots + t_n f_{\alpha_n} + t f_\beta,$$

where $n$ is a natural number, $t_1, t_2, \ldots, t_n, t$ are some nonzero real numbers, and $\alpha_1, \alpha_2, \ldots, \alpha_n, \beta$ are ordinals such that

$$\alpha_1 < \alpha_2 < \cdots < \alpha_n < \beta < \mathbf{c}^+.$$

Obviously, we can find an ordinal number $\xi_0 < \mathbf{c}$ such that

$$\{t_1, t_2, \ldots, t_n, t\} \subset K_\xi,$$

$$\{f_{\alpha_1}, f_{\alpha_2}, \ldots, f_{\alpha_n}\} \subset \{g_\zeta : \zeta < \xi\},$$

$$h \in \{h_\zeta : \zeta < \xi\}$$

for every ordinal number $\xi$ satisfying the inequalities $\xi_0 < \xi < \mathbf{c}$.

Now, let us consider an element $z \in \mathbf{R}$ whose representation via our Hamel basis $\{e_\xi : \xi < \mathbf{c}\}$ looks as follows:

$$z = q_1 e_{\zeta_1} + q_2 e_{\zeta_2} + \cdots + q_m e_{\zeta_m} + q e_\xi,$$

where $m$ is a natural number, $q_1, q_2, \ldots, q_m, q$ are some nonzero rational numbers, and $\zeta_1, \zeta_2, \ldots, \zeta_m, \xi$ are some ordinal numbers such that

$$\zeta_1 < \zeta_2 < \cdots < \zeta_m < \xi, \quad \xi_0 < \xi.$$

Then it is not difficult to see from the definition of $f_\beta(e_\xi)$ that $f(z) \neq h(z)$. Consequently, we have

$$\mathrm{card}(\{x \in \mathbf{R} : f(x) = h(x)\}) \leq \mathrm{card}(\xi_0) + \omega < \mathbf{c},$$

which yields that $f$ is a Sierpiński-Zygmund function. Moreover, $f$ has the following much stronger property: for every set $X \subset \mathbf{R}$ with $\mathrm{card}(X) = \mathbf{c}$, the restriction of $f$ to $X$ is not a Borel function, i.e., $f$ is a Sierpiński-Zygmund function in the strong sense.

It remains to check the validity of condition (3). The preceding argument shows, in particular, that if ordinals $\alpha$ and $\beta$ are such that $\alpha < \beta < \mathbf{c}^+$, then the difference $f_\alpha - f_\beta$ is a Sierpiński-Zygmund function and, consequently, $f_\alpha \neq f_\beta$. It immediately follows from this observation that

$$\mathrm{card}(V) = \mathbf{c}^+ > \mathbf{c},$$

i.e., condition (3) is satisfied. Notice also that if $f$ and $f^*$ are any two distinct functions from $V$, then the difference $f - f^*$ also belongs to $V$ and is a nonzero function. According to the said above, $f - f^*$ is a Sierpiński-Zygmund function, so

$$\mathrm{card}(\{x \in \mathbf{R} : (f - f^*)(x) = 0\}) < \mathbf{c}$$

or, equivalently,

$$\mathrm{card}(\{x \in \mathbf{R} : f(x) = f^*(x)\}) < \mathbf{c}.$$

In other words, the graphs of $f$ and $f^*$ are almost disjoint subsets of the plane $\mathbf{R}^2 = \mathbf{R} \times \mathbf{R}$. Theorem 3 has thus been proved. □

**Remark 2.** Let $\mathcal{F}$ be a family of functions acting from $\mathbf{R}$ into $\mathbf{R}$. The cardinal number $\mathrm{A}(\mathcal{F})$ is usually defined as the smallest cardinality of a family $\mathcal{H} \subset \mathbf{R}^{\mathbf{R}}$ for which there exists no $h \in \mathbf{R}^{\mathbf{R}}$ such that $h + \mathcal{H} \subset \mathcal{F}$. This cardinal number was investigated for many concrete classes of real-valued functions on $\mathbf{R}$ (see e.g. [13] and references therein). A more general concept was also introduced in [13]. Namely, let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two subfamilies of $\mathbf{R}^{\mathbf{R}}$. Define the cardinal number $\mathrm{Add}(\mathcal{F}_1, \mathcal{F}_2)$ as the smallest cardinality of a family $\mathcal{H} \subset \mathbf{R}^{\mathbf{R}}$ for which there exists no $h \in \mathcal{F}_1$ such that $h + \mathcal{H} \subset \mathcal{F}_2$. By using an argument somewhat similar to the proof of Theorem 3, it can be demonstrated that if $\mathcal{F}_1$ is the family of all additive real-valued functions on $\mathbf{R}$ and $\mathcal{F}_2$ is the family of all Sierpiński-Zygmund functions on $\mathbf{R}$, then

$$\mathrm{Add}(\mathcal{F}_1, \mathcal{F}_2) > \mathbf{c}.$$

For details, see Theorem 10 (iv) from [13] and its proof. In this context, it is natural to consider the cardinal number $\mathrm{A}(\mathcal{G})$, where $\mathcal{G}$ denotes the family of all those real-valued functions on $\mathbf{R}$ which simultaneously are additive and Sierpiński-Zygmund functions. It is not difficult to check that

$$\mathrm{A}(\mathcal{G}) = \mathrm{Add}(\mathcal{G}, \mathcal{G}) + 1 = 2.$$

Notice also that the equality $\mathrm{A}(\mathcal{F}) = \mathrm{Add}(\mathcal{F}, \mathcal{F}) + 1$ holds true for any family $\mathcal{F} \subset \mathbf{R}^{\mathbf{R}}$ (see again [13], Proposition 1).

Certain analogues of Theorems 1–3 are valid for absolutely nonmeasurable functions on $\mathbf{R}$. To formulate them, let us recall some notions.

For a given nonempty set $A$, we shall denote by $\mathcal{M}(A)$ the class of all nonzero $\sigma$-finite continuous measures on $A$ (their domains are, in general, various $\sigma$-algebras of subsets of $A$).

A function $f : A \to \mathbf{R}$ which is nonmeasurable with respect to any measure from $\mathcal{M}(A)$ can be regarded as an extremely nonmeasurable real-valued function on $A$. Such an $f$ will be called an absolutely nonmeasurable function on $A$.

In order to describe such functions, we need the classical notion of a universal measure zero subset of $\mathbf{R}$.

Let $Z \subset \mathbf{R}$. We recall that $Z$ is a universal measure zero set if, for any $\sigma$-finite continuous Borel measure $\mu$ on $\mathbf{R}$, we have $\mu^*(Z) = 0$ where $\mu^*$ denotes the outer measure associated with $\mu$.

Equivalently, we may say that $Z \subset \mathbf{R}$ is a universal measure zero set if there exists no nonzero $\sigma$-finite continuous Borel measure on $Z$ (where $Z$ is assumed to be endowed with the induced topology).

The following statement yields a characterization of absolutely nonmeasurable functions with respect to the class $\mathcal{M}(A)$.

**Theorem 4.** *For any function $f : A \to \mathbf{R}$, these two assertions are equivalent:*

(1) *$f$ is absolutely nonmeasurable with respect to $\mathcal{M}(A)$;*

(2) *the range of $f$ is a universal measure zero subset of $\mathbf{R}$ and, for each point $t \in \mathbf{R}$, the set $f^{-1}(t)$ is at most countable.*

The proof of this theorem is not difficult and can be found in [10].

It directly follows from Theorem 4 that:

(i) if a function $f : A \to \mathbf{R}$ is injective and the range of $f$ is a universal measure zero set, then $f$ is absolutely nonmeasurable with respect to $\mathcal{M}(A)$;

(ii) the composition of any two functions which are absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$ is absolutely nonmeasurable with respect to the same class.

Theorem 4 also implies that if $\mathrm{card}(A) > \mathbf{c}$, then there exist no functions on $A$ which are absolutely nonmeasurable with respect to $\mathcal{M}(A)$. More precisely, the existence of an absolutely nonmeasurable function with respect to $\mathcal{M}(A)$ is equivalent to the existence of a universal measure zero set $Z \subset \mathbf{R}$ with $\mathrm{card}(Z) = \mathrm{card}(A)$. Consequently, the following two assertions are equivalent:

(a) there exists a function $f : \mathbf{R} \to \mathbf{R}$ absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$;

(b) there exists a universal measure zero set $Z \subset \mathbf{R}$ with $\mathrm{card}(Z) = \mathbf{c}$.

**Remark 3.** Several classical constructions (within **ZFC** theory) of uncountable universal measure zero subsets of $\mathbf{R}$ are known. Those constructions belong to Hausdorff, Luzin, Sierpiński, Marczewski, and others. According to them, every nonempty perfect set $P \subset \mathbf{R}$ contains an uncountable universal measure zero subset. It was also shown that there exists a model of **ZFC** set theory in which the Continuum Hypothesis fails to be true and every universal measure zero subset of $\mathbf{R}$ has cardinality less than or equal to $\omega_1$, where $\omega_1$ stands for the least uncountable cardinal number. Therefore, the existence of absolutely nonmeasurable functions acting from $\mathbf{R}$ into $\mathbf{R}$ cannot be established within **ZFC** set theory.

Recall that $L \subset \mathbf{R}$ is a Luzin set if $L$ is uncountable and the intersection of $L$ with any first category subset of $\mathbf{R}$ is at most countable.

Various properties of Luzin sets are presented in widely known text-book by Oxtoby [12].

A set $L' \subset \mathbf{R}$ is a generalized Luzin set if $\mathrm{card}(L') = \mathbf{c}$ and the intersection of $L'$ with any first category subset of $\mathbf{R}$ has cardinality strictly less than $\mathbf{c}$.

Notice that every Luzin subset of $\mathbf{R}$ and, under Martin's Axiom, every generalized Luzin subset of $\mathbf{R}$ are universal measure zero sets in $\mathbf{R}$. These two facts are easy to prove (see, e.g., [12]).

**Theorem 5.** *Assuming Martin's Axiom, for any function $f : \mathbf{R} \to \mathbf{R}$, there exist two injective functions $f_1 : \mathbf{R} \to \mathbf{R}$ and $f_2 : \mathbf{R} \to \mathbf{R}$ which are absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$ and for which the equality $f = f_1 + f_2$ holds true.*

**Theorem 6.** *Assume Martin's Axiom. Let $f : \mathbf{R} \to \mathbf{R}$ be any additive function. Then there exist two injective additive functions $f_1 : \mathbf{R} \to \mathbf{R}$ and $f_2 : \mathbf{R} \to \mathbf{R}$ which are absolutely nonmeasurable with respect to $\mathcal{M}(\mathbf{R})$ and satisfy the equality $f = f_1 + f_2$.*

The proofs of Theorems 5 and 6 are given in [9].

**Theorem 7.** *Under the Continuum Hypothesis, there exists a group* $G \subset \mathbf{R}^{\mathbf{R}}$ *satisfying the relations:*

(1) $\mathrm{card}(G) = \mathbf{c}^+$;

(2) *every* $g \in G$ *is an additive function;*

(3) *every* $g \in G \setminus \{0\}$ *is a function absolutely nonmeasurable with respect to* $\mathcal{M}(\mathbf{R})$.

*In particular, assuming* $2^{\mathbf{c}} = \mathbf{c}^+$, *we obtain that* $\mathrm{card}(G) = \mathrm{card}(\mathbf{R}^{\mathbf{R}})$.

The proof of Theorem 7 can be found in [8].

**Remark 4.** The usage of the Continuum Hypothesis in the formulation of Theorem 7 is in some sense necessary. Indeed, suppose that Martin's Axiom and the negation of the Continuum Hypothesis hold. Then, as is well known, we have the equalities

$$2^{\omega} = 2^{\omega_1} = \mathbf{c},$$

where $\omega_1$ stands, as usual, for the least uncountable ordinal number. Let $G \subset \mathbf{R}^{\mathbf{R}}$ be a group satisfying condition (3) of Theorem 7. Fix a subset $X$ of $\mathbf{R}$ with $\mathrm{card}(X) = \omega_1$. For any function $g \in G$, consider the restriction of $g$ to $X$ and let $Gr(g|X)$ denote the graph of this restriction. Thus, we have the mapping

$$F(g) = Gr(g|X) \qquad (g \in G)$$

acting from $G$ into the family of all those subsets of $\mathbf{R} \times \mathbf{R}$ whose cardinalities are equal to $\omega_1$. By virtue of Theorem 4, the introduced mapping $F$ is injective, which yields

$$\mathrm{card}(G) \leq \mathrm{card}((\mathbf{R} \times \mathbf{R})^{\omega_1}) = 2^{\omega_1} = \mathbf{c}.$$

So we conclude that, under Martin's Axiom and the negation of the Continuum Hypothesis, there is no large subgroup of $\mathbf{R}^{\mathbf{R}}$ all nonzero members of which are absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R})$.

**Remark 5.** As has been already mentioned, in some models of set theory there are no absolutely nonmeasurable functions with respect to $\mathcal{M}(\mathbf{R})$. On the other hand, if we assume Martin's Axiom, then the class of Sierpiński-Zygmund functions on $\mathbf{R}$ and the class of absolutely nonmeasurable functions with respect to $\mathcal{M}(\mathbf{R})$ are in general position, i.e., they have nonempty intersection and none of them contains another one.

Finishing this note, let us introduce a version of Sierpiński-Zygmund functions which is closely connected with the notion of absolute nonmeasurability.

Let $T$ be a non-universal measure zero topological space (all singletons in which are Borel).

We shall say that a function $f : T \to \mathbf{R}$ is a Sierpiński-Zygmund type function (in the measure-theoretical sense) if, for any non-universal measure zero set $X \subset T$, the restriction $f|X$ is not a Borel function.

**Theorem 8.** *Let $T$ be a non-universal measure zero topological space (all singletons in which are Borel) and let $f : T \to \mathbf{R}$ be an absolutely nonmeasurable function. Then there exists a non-universal measure zero subset $Y$ of $T$ such that the restriction $f|Y$ is an injective Sierpiński-Zygmund type function on $Y$.*

Proof. According to Theorem 4, the set $\mathrm{ran}(f)$ is universal measure zero and the set $f^{-1}(x)$ is at most countable for every point $x \in \mathbf{R}$. Consequently, there exists a countable disjoint family $\{Y_i : i \in I\}$ such that $\cup\{Y_i : i \in I\} = T$ and the restriction $f|Y_i$ is injective for any $i \in I$. Since $T$ is not universal measure zero, at least one set $Y_i$ is not universal measure zero either. Let $Y$ denote one of such sets. We assert that the restriction $f|Y$ is an injective Sierpiński-Zygmund type function on $Y$. Suppose otherwise, i.e., suppose that there exists a non-universal measure zero set $Z \subset Y$ for which the corresponding restriction $f|Z$ is Borel. Let $\mu$ be some probability continuous Borel measure on $Z$. For every Borel subset $B$ of $\mathrm{ran}(f)$, define

$$\nu(B) = \mu((f|Z)^{-1}(B)).$$

In this way, we come to a probability continuous Borel measure $\nu$ on $\mathrm{ran}(f)$, which contradicts the fact that $\mathrm{ran}(f)$ is a universal measure zero set. The obtained contradiction finishes the proof. □

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