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A CONVERGENCE-THEORETIC VIEWPOINT ON THE ARZELÀ-ASCOLI THEOREM

Abstract

This is an expository note, hopefully accessible to students, on how continuous convergence and convergence-theroretic techniques can provide insight on the classical Arzelà-Ascoli theorem.

1 Introduction

In its original form, the Ascoli theorem provides conditions for a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous real-valued functions on a closed interval $[a,b]$ to have a *uniformly* convergent subsequence. To this end, Arzelà and Ascoli independently introduced the notion of an equicontinuous sequence: the sequence $(f_n)_{n\in\mathbb{N}}$ is *equicontinuous at* $x\in[a,b]$ if for every $\varepsilon>0$ there is $\delta>0$ such that

$$|x-y|<\delta \Longrightarrow |f_n(x)-f_n(y)|<\varepsilon$$

for every $n\in\mathbb{N}$, and $(f_n)_{n\in\mathbb{N}}$ is called equicontinuous if it is equicontinuous at every x in $[a,b]$. The sequence is *uniformly bounded* if there is an M such that $|f_n(x)|<M$ for all $x\in[a,b]$ and all $n\in\mathbb{N}$. The combination of both conditions yields the desired property:

Theorem 1 (Ascoli)**.** *[3] If a sequence of functions $f_n:[a,b]\to\mathbb{R}$ is equicontinuous and uniformly bounded, then it has a uniformly convergent subsequence.*

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In other words, under these conditions, the set $\{f_n : n \in \mathbb{N}\}$ has sequentially compact closure in the space $C([a,b],\mathbb{R})$ of real-valued continuous functions on $[a,b]$, endowed with the topology of uniform convergence. Since $C([a,b],\mathbb{R})$ is metrizable in this topology (by the uniform norm), sequential compactness and compactness are equivalent. Arzelà extended Ascoli's theorem to general sets of functions [2], obtaining what in modern terms would be called a criterion of compactness of a set of real-valued continuous functions. Therefore, the *abstract Arzelà-Ascoli quest* is to find sufficient (and hopefully, also necessary) conditions on subsets H of a space $C(X,Y)$ of continuous functions between two topological spaces X and Y to have compact closure, that is, to be *relatively compact*, for some appropriate analogue of the topology of uniform convergence. The literature is rich in results of that type. Theorems of the Arzelà-Ascoli type have become an ubiquitous and efficient tool in a variety of contexts. As such, the theorem, under one form or another, finds its place in standard Topology courses, as well as standard Functional Analysis courses. It is often stated only for continuous (sometimes only real-valued) functions over a compact metric space X, e.g., [16], or a compact topological space, e.g., [8], [12], [15]. While [9] provides a more comprehensive and far reaching treatement of Arzelà-Ascoli theorems, the following is probably the most general form that can be easily found from several textbooks (e.g., [11, Theorem 47.1]):

Theorem 2. *Let X be a topological space and Y be a metric space. If $H \subseteq C(X,Y)$ is equicontinuous and pointwise bounded then H is relatively compact in $C_k(X,Y)$, that is, in $C(X,Y)$ endowed with the topology of uniform convergence on compact subsets of X. Moreover, if X is locally compact, the converse is true.*

The setting of a metric range space allows a straightforward extension of the notion of equicontinuity ([1]). In absence of compactness, the topology of uniform convergence on compact subsets of X turns out to be the relevant analogue of the topology of uniform convergence, and pointwise boundedness (where H is pointwise bounded if for each $x \in X$ the set $\{h(x) : h \in H\}$ is bounded in Y) is the needed analogue of uniform boundedness. A full characterization is obtained among locally compact spaces. This is not too much of a restriction if your idea of a topological space is a manifold. However, if

[1]by declaring $H \subseteq C(X,Y)$ equicontinuous at x if for every positive ε there is a neighborhood U of x such that

$$y \in U, f \in H \Longrightarrow d(f(y), f(x)) < \varepsilon.$$

your favorite topological spaces are topological vector spaces, local compactness fails as soon as you leave the finite dimensional case (e.g., [15, Theorem 1.22]). Regardless of your preference, it should be clear that the natural context to treat the question of characterizing (relative) compactness of a set of continuous functions should be the most natural setting to consider continuity and compactness. That would lead most people to investigate this question in the realm of topological spaces. Kelley [9] provides the most complete, and in my opinion most lucid, exposition of results of the Arzelà-Ascoli type in this context.

It is however far more natural to define continuity as preservation of limits than in the usual topological way. As for compactness, the obscure definition in terms of open covers becomes more transparent when interpreted in terms of convergence: every ultrafilter converges. This viewpoint also proves often more efficient, as a quick comparison of proofs of Tychonoff's theorem with or without ultrafilters shows. The purpose of this expository note is to show that *convergence spaces,* in which the notion of limit is primal, offers an ideal context for such investigations, providing surprisingly easy proofs while delivering more general results than the classical ones.

As a disclaimer, I should mention that the convergence-theoretic viewpoint on Arzelà-Ascoli theorems is far from a novel idea, and has been explored before, in more details than in this note, e.g., [13], [14]. The results stated here are known, but the proofs, in particular the use of (2.2) in this context, are original. In particular, with this technique, the usual need for Tychonoff's theorem is bypassed.

2 Preliminaries: convergence spaces

2.1 Convergent objects: filters

Most students, as well as many professional mathematicians, associate *convergence* with convergence of sequences. Convergent sequences, however, do not suffice to describe a topological space, as is shown in most topology textbooks. To recover all the information encoded in a topology in terms of the convergent objects, generalizations of sequences have to be considered: either nets or filters. A *filter* $\mathcal{F}$ *on a set* X is a family of non-empty subsets of X that is closed under supersets (that is, $B \in \mathcal{F}$ whenever $A \in \mathcal{F}$ and $A \subseteq B$) and under finite intersections (that is, $A \cap B \in \mathcal{F}$ whenever $A \in \mathcal{F}$ and $B \in \mathcal{F}$). The only family of subsets of X that is closed under supersets and contains the empty set is the power set of X, that we will also refer to as *the degenerate filter on* X. A family $\mathcal{B}$ of non-empty subsets of X with the property that there is $B \in \mathcal{B}$ with $B \subseteq B_1 \cap B_2$ whenever B_1 and B_2 are in $\mathcal{B}$ is called a *filter-base,* and $\mathcal{B}^{\uparrow} := \{A \subseteq X : \exists B \in \mathcal{B}, B \subseteq A\}$ is the filter *generated by* $\mathcal{B}$. Hence, filters will be usually described in terms of a filter-base, even though a filter may have many different filter-bases. In fact, you may think of filters as equivalence classes of filter-bases modulo $\sim$, where $\mathcal{A} \sim \mathcal{B}$ if $\mathcal{A}^{\uparrow} = \mathcal{B}^{\uparrow}$. Consider for instance the collection $\mathcal{N}(x)$ of neighborhoods of a given point x of a topological space X, which is a filter. In many cases, it is described by giving one of its filter-bases. For example, consider the neighborhood filter $\mathcal{N}(0)$ of the origin in the plane with its usual topology. Of course, there are many different metrics inducing this topology. For any one of these metrics, the collection of balls centered at the origin forms a filter-base for $\mathcal{N}(0)$. Hence, considering Euclidian balls or "square" balls centered at 0 does not change what characterizes the topology at 0 : the filter $\mathcal{N}(0)$.

As already mentioned, filters can also be thought of as generalized sequences. Indeed, to a sequence $(x_n)_{n\in\mathbb{N}}$ of points of X we can associate a filter

$$\mathcal{F}_{(x_n)} := \{\{x_n : n \geq k\} : k \in \mathbb{N}\}^{\uparrow}.$$

From the viewpoint of convergence in a topological space, it is not the sequence $(x_n)_{n\in\mathbb{N}}$ that matters, but the filter $\mathcal{F}_{(x_n)}$. For instance, changing a finite number of terms in a sequence yields a different sequence, yet this affects neither convergence nor the filter associated to the sequence. In fact, the usual definition of convergence of a sequence $(x_n)_{n\in\mathbb{N}}$ to a point x, that is, that for every neighborhood V of x there is $k \in \mathbb{N}$ such that $x_n \in V$ for every $n \geq k$, is a statement about the filters $\mathcal{N}(x)$ and $\mathcal{F}_{(x_n)}$: every element V of $\mathcal{N}(x)$ contains an element $\{x_n : n \geq k\}$ of $\mathcal{F}_{(x_n)}$. Since filters are closed under supersets, this means that

$$(x_n)_{n\in\mathbb{N}} \to x \Longleftrightarrow \mathcal{F}_{(x_n)} \supseteq \mathcal{N}(x).$$

More generally, a filter $\mathcal{F}$ is said to *converge to* x in a topological space X if $\mathcal{F} \supseteq \mathcal{N}(x)$ and this convergence completely determines the topology.

The set $\mathbb{F}X$ of filters on X is partially ordered by inclusion. Every family $(\mathcal{F}_\alpha)_{\alpha\in I}$ in $\mathbb{F}X$ has a greatest lower bound, which is $\bigcap_{\alpha\in I} \mathcal{F}_\alpha$ ([2]). In contrast, if two filters $\mathcal{F}$ and $\mathcal{G}$ have a least upper bound $\mathcal{F} \vee \mathcal{G}$ in $\mathbb{F}X$, then $F \cap G \in \mathcal{F} \vee \mathcal{G}$ is necessarily non-empty, for each $F \in \mathcal{F}$ and each $G \in \mathcal{G}$, because $\mathcal{F} \vee \mathcal{G}$ is a filter. Moreover, if $F \cap G \neq \varnothing$ whenever $F \in \mathcal{F}$ and $G \in \mathcal{G}$–a

[2]and $\{\bigcup_{\alpha\in I} F_\alpha : \forall\alpha \in I, F_\alpha \in \mathcal{F}_\alpha\}$ is a filter-base for $\bigcap_{\alpha\in I} \mathcal{F}_\alpha$. Indeed $F \in \mathcal{F}_\alpha$ for each $\alpha \in I$ if and only if, for each $\alpha \in I$, there is $F_\alpha \in \mathcal{F}_\alpha$ such that $F_\alpha \subseteq F$, because each $\mathcal{F}_\alpha$ is closed under supersets. Hence $F \in \bigcap_{\alpha\in I} \mathcal{F}_\alpha$ if $\bigcup_{\alpha\in I} F_\alpha \subseteq F$ for a selection of sets $F_\alpha \in \mathcal{F}_\alpha$, for each $\alpha \in I$.

condition that we will refer to as "$\mathcal{F}$ and $\mathcal{G}$ *mesh*", or in symbols, $\mathcal{F}\#\mathcal{G}$–then $\{F \cap G : F \in \mathcal{F},\, G \in \mathcal{G}\}$ is a filter-base for a filter that contains both $\mathcal{F}$ and $\mathcal{G}$. Hence it generates $\mathcal{F} \vee \mathcal{G}$.

If $(\mathcal{F}_\alpha)_{\alpha\in I}$ is a chain in $\mathbb{F}X$ then $\bigcup_{\alpha\in I} \mathcal{F}_\alpha$ is a filter (hence the least upper bound of $(\mathcal{F}_\alpha)_{\alpha\in I}$), so that, by Zorn's Lemma, $\mathbb{F}X$ has maximal elements called *ultrafilters*. In fact, every filter is contained in an ultrafilter. Let us denote by $\mathbb{U}X$ the set of ultrafilters on X and $\mathbb{U}(\mathcal{F})$ the set of ultrafilters containing $\mathcal{F}$.

2.2 Calculus of relations

One advantage (among many) of describing convergence in terms of filters rather than nets ([3]) is that a notion of product filter is readily available, and well behaved in the sense of (2.2) below. If $\mathcal{F} \in \mathbb{F}X$ and $\mathcal{G} \in \mathbb{F}Y$ then

$$\mathcal{F} \times \mathcal{G} := \{F \times G :\ F \in \mathcal{F},\, G \in \mathcal{G}\}^{\uparrow}$$

is a filter on $X \times Y$. A subset R of $X \times Y$ can be seen as a relation R from X to Y or R^- from Y to X. Defining as usual $Rx := \{y \in Y : (x,y) \in R\}$ and $RF := \bigcup_{x\in F} Rx$, it is immediate that if $F \subseteq X$, $G \subseteq Y$ and $R \subseteq X \times Y$ then

$$(F \times G)\,\#R \Longleftrightarrow RF\#G \Longleftrightarrow F\#R^-G,$$

where $\#$ means that the two sets have non-empty intersection. Therefore if $\mathcal{F} \in \mathbb{F}X$ and $\mathcal{G} \in \mathbb{F}Y$ and $R \subseteq X \times Y$ then

$$(\mathcal{F} \times \mathcal{G})\,\#R \Longleftrightarrow R\,[\mathcal{F}]\,\#\mathcal{G} \Longleftrightarrow \mathcal{F}\#R^-\,[\mathcal{G}]\,, \tag{2.1}$$

where $R\,[\mathcal{F}]$ is the (possibly degenerate) filter generated on Y by $\{RF : F \in \mathcal{F}\}$ and $R^-\,[\mathcal{G}]$ is defined similarly. If now $\mathcal{R} \in \mathbb{F}(X \times Y)$, each of its elements

[3] A *net on* X is a map $N : D \to X$ where D is a *directed set*, that is, a partially ordered set in which for each d_1, d_2 in D, there is $d \in D$ such that $d \geq d_1$ and $d \geq d_2$. A net N on X *converges to* $x \in X$ if for every neighborhood V of x there is $d_0 \in D$ such that $\{N(d) : d \geq d_0\} \subseteq V$.

Note that a directed partial order $\geq$ on D defines a filter $\mathcal{F}_D$ on D generated by the filter-base $\{\{d \in D : d \geq d_0\} : d_0 \in D\}$ and that conversely, a filter-base $\mathcal{F} := \{F_d : d \in D\}$ indexed by D defines a directed partial order on D via

$$d_1 \leq d_2 \Longleftrightarrow F_{d_2} \subseteq F_{d_1}.$$

The convergence of the net N is nothing but the convergence of the image filter $N(\mathcal{F}_D)$. Hence nets can be thought of as images of filters. Yet, from the viewpoint of convergence, what matters is the filter and not the many nets for which $N(\mathcal{F}_D)$ coincide with this filter.

can be seen as a relation, and we can consider the (possibly degenerate) filter $\mathcal{R}[\mathcal{F}] := \{RF : R \in \mathcal{R}, F \in \mathcal{F}\}^{\uparrow}$. Clearly,

$$(\mathcal{F} \times \mathcal{G}) \# \mathcal{R} \Longleftrightarrow \mathcal{R}[\mathcal{F}] \# \mathcal{G} \Longleftrightarrow \mathcal{F} \# \mathcal{R}^{-}[\mathcal{G}]. \tag{2.2}$$

We will see that we need little more than this very simple set-theoretic observation to prove one of the directions in our characterization of relatively compact subsets of function spaces below, which generalizes Theorem 2.

2.3 Convergence and pseudotopology

A *convergence* ξ on a set X is a relation between X and the set $\mathbb{F}X$ of filters on X, denoted $x \in \lim_{\xi} \mathcal{F}$ if $(x, \mathcal{F}) \in \xi$, such that $\lim \mathcal{F} \subseteq \lim \mathcal{G}$ whenever $\mathcal{F} \subseteq \mathcal{G}$. A convergence is *centered* if $x \in \lim\{x\}^{\uparrow}$ for every $x \in X$. A centered convergence such that

$$\lim \mathcal{F} = \bigcap_{\mathcal{U} \in \mathbb{U}(\mathcal{F})} \lim \mathcal{U}$$

for every $\mathcal{F} \in \mathbb{F}X$ is a *pseudotopology*, as introduced by Gustave Choquet [5]. Unless stated otherwise, all convergences will be assumed centered ([4]). A pair (X, ξ) where ξ is a convergence is called a *convergence space*. We may denote a convergence space simply by X when no ambiguity can arise.

Note that a topology on X induces a convergence via

$$x \in \lim \mathcal{F} \Longleftrightarrow \mathcal{F} \supseteq \mathcal{N}(x),$$

so that topologies can be considered as convergences. In fact, each topology is a pseudotopology. On the other hand, non topological convergences arise naturally in a variety of context. For instance, the convergence of ultrafilters on a topological space, say on the real line, is clearly non-topological because if $(\mathcal{U}_{\alpha})_{\alpha \in I}$ is a collection of ultrafilters converging to 0, the filter $\bigcap_{\alpha \in I} \mathcal{U}_{\alpha}$ is not an ultrafilter and therefore does not converge. Hence, for this convergence, there is no minimal filter converging to 0, and the convergence is not topological.

A function $f : (X, \xi) \to (Y, \sigma)$ is *continuous* if

$$x \in \lim_{\xi} \mathcal{F} \Longrightarrow f(x) \in \lim_{\sigma} f[\mathcal{F}],$$

where $f[\mathcal{F}] := \{f(F) : F \in \mathcal{F}\}^{\uparrow}$. Note that if the convergences ξ and σ happen to be topologies, continuity coincides with the usual topological notion ([5]).

[4] In fact, the condition that $x \in \lim\{x\}^{\uparrow}$ for every $x \in X$ is usually included in the definition of a convergence.

[5] because continuity of f at x in the topological sense translates into $f[\mathcal{N}_{\xi}(x)] \supseteq \mathcal{N}_{\sigma}(f(x))$, that is, $f(x) \in \lim_{\sigma} f[\mathcal{N}_{\xi}(x)]$.

If ξ and τ are two convergences on the same set, we say that ξ is *finer than* τ (or that τ is *coarser than* ξ), in symbols $\xi \geq \tau$, if $\lim_\xi \mathcal{F} \subseteq \lim_\tau \mathcal{F}$ for every filter $\mathcal{F}$. Of course, if ξ and τ are topologies, this partial order coincides with the usual partial order on topologies. Induced convergence, quotient convergence and product convergence are defined as usual via continuity. More precisely, if A is a subset of a convergence space (X, ξ), the *induced convergence* $\xi_{|A}$ is the coarsest convergence on A making the inclusion map $i : A \to (X, \xi)$ continuous. In other words, $a \in \lim_{\xi_{|A}} \mathcal{F}$ if $a \in \lim_\xi i[\mathcal{F}]$. If $f : (X, \xi) \to Y$ is onto, the *quotient convergence* $f\xi$ on Y is the finest convergence on Y making f continuous. In other words, $y \in \lim_{f\xi} \mathcal{F}$ if there is a filter $\mathcal{G}$ on X such that $x \in \lim_\xi \mathcal{G}$, $f(x) = y$ and $\mathcal{F} \supseteq f[\mathcal{G}]$. If (X, ξ) and (Y, σ) are two convergence spaces, then the *product convergence* $\xi \times \sigma$ on $X \times Y$ is the coarsest convergence on $X \times Y$ making each projection continuous. In other words, $(x, y) \in \lim_{\xi \times \sigma} \mathcal{F}$ if $x \in \lim_\xi p_X[\mathcal{F}]$ and $y \in \lim_\sigma p_Y[\mathcal{F}]$, where p_X and p_Y are the projection maps. The reader versed in the use of filters in topology will note that if ξ and σ are topologies then $\xi_{|A}$ is nothing but the induced topology, $\xi \times \sigma$ is the product topology, but $f\xi$ is **not** the quotient topology.

2.4 Separation

Recall that a topological space is regular if at each point, it admits a base of neighborhood composed of closed sets. An analogue for convergences of the topological closure is that of adherence. If A is a subset of a convergence space X, let $\mathrm{adh}A := \bigcup_{\mathcal{U} \in \mathbb{U}(A)} \lim \mathcal{U}$. More generally, the adherence of a filter $\mathcal{F}$ on X is

$$\mathrm{adh}\mathcal{F} := \bigcup_{\mathcal{U} \in \mathbb{U}(\mathcal{F})} \lim \mathcal{U}.$$

Note that if the convergence is topological $\mathrm{adh}A$ is the usual topological closure of A and $\mathrm{adh}\mathcal{F}$ is the usual set of cluster points of $\mathcal{F}$. In general however, $\mathrm{adh}A$ may not be closed, that is, adh may not be idempotent. To extend the definition of regularity from topological to convergence spaces, consider for each filter $\mathcal{F}$ the filter

$$\mathrm{adh}^\natural \mathcal{F} := \{\mathrm{adh}F : F \in \mathcal{F}\}^\uparrow.$$

A convergence space is *regular* if $\lim \mathcal{F} \subseteq \lim(\mathrm{adh}^\natural \mathcal{F})$ for each filter $\mathcal{F}$. It should be a good exercise for the reader to verify that a topology is regular in that sense exactly if it is regular in the usual topological sense.

On the other hand, we extend the definition of a *Hausdorff space* from topological to convergence spaces by observing that a topological space is Hausdorff if each filter has at most one limit point.

2.5 Function space structures

The most important reason to consider convergence structures is the lack of a well behaved canonical topology on function spaces. More precisely, if X and Y are two topological spaces, $C(X,Y)$ denotes the set of continuous maps from X to Y. Consider now the evaluation map

$$e = \langle \cdot, \cdot \rangle : X \times C(X,Y) \to Y$$

defined by $e(x,f) = \langle x, f \rangle = f(x)$. Of course, the question of continuity of this coupling depends on the structure on $C(X,Y)$. Several classically used function space topologies can be defined in terms of $\langle \cdot, \cdot \rangle$. Indeed, the *topology of pointwise convergence* is the coarsest convergence (and also the coarsest topology) on $C(X,Y)$ that makes the point-evaluation map $\langle x, \cdot \rangle$ continuous, for each $x \in X$. Similarly, the *compact-open topology*, also called topology of uniform convergence on compact subsets, is the coarsest convergence (or topology) on $C(X,Y)$ that makes the restriction of $\langle \cdot, \cdot \rangle$ to $K \times C(X,Y)$ continuous, for each compact subset K of X. The *continuous convergence* is the coarsest convergence on $C(X,Y)$ making $\langle \cdot, \cdot \rangle$ (jointly) continuous.

There is, in general, no coarsest *topology* on $C(X,Y)$ making $\langle \cdot, \cdot \rangle$ continuous, which is the reason for the use of a range of ad-hoc function space topologies depending on the context. In contrast, continuous convergence provides the canonical function space structure in the realm of convergences. I hope to convince the reader that it should also be considered the canonical structure when X and Y are topological spaces. We will denote by $C_p(X,Y)$, $C_k(X,Y)$ and $C_c(X,Y)$ respectively the set $C(X,Y)$ endowed with the topology of pointwise convergence, the compact-open topology and the continuous convergence respectively and by $\lim_p$, $\lim_k$ and $\lim_c$ the corresponding limit operators. By definition $C_p(X,Y) \leq C_k(X,Y) \leq C_c(X,Y)$ and

$$f \in \lim_c \mathcal{F} \Longleftrightarrow \forall x \in X, (x \in \lim_X \mathcal{G} \Longrightarrow f(x) \in \lim_Y < \mathcal{G}, \mathcal{F} >) .$$

Classical (and relatively easy) facts about continuous convergence that I will leave without proofs (which can be found for instance in [4], [6]) include:

Theorem 3. *1. The map $i : X \to C_c(C_c(X,Y),Y)$ defined by $i(x) = \langle x, \cdot \rangle$ is continuous;*

2. If Y is a pseudotopological space then so is $C_c(X,Y)$;

3. If Y is regular (respectively Hausdorff) then so is $C_c(X,Y)$;

4. *If* Y *is a convergence group or convergence vector space* ([6]), *so is* $C_c(X, Y)$;

5.
$$C_c(X \times Y, Z) = C_c(Y, C_c(X, Z))$$
where equality stands for homeomorphism via the map associating to $f \in C(X \times Y, Z)$ *its companion map* ${}^t f \in C(Y, C_c(X, Z))$ *defined by* ${}^t f(y)(x) = f(x, y)$.

Less straightforward yet classical is the following important fact:

Theorem 4. *[1] Let* X *be a completely regular topological space. The following are equivalent:*

1. $C_c(X, \mathbb{R})$ *is topological;*
2. $C_c(X, \mathbb{R}) = C_k(X, \mathbb{R})$*;*
3. X *is locally compact.*

Moreover, Theorem 4 above extends to $C(X, Y)$ where Y is also completely regular.

A simple fact that belongs to folklore is that "continuous limits are continuous"– a useful feature shared with uniform convergence. More precisely, we can consider continuous convergence on the set Y^X of all functions from X to Y, even though it is then centered only at continuous functions. Nevertheless, we have:

Lemma 5. *If* Y *is a regular convergence space,* $f \in Y^X$*,* $\mathcal{F}$ *is a filter on* Y^X*, and* $f \in \lim_c \mathcal{F}$ *then* f *is continuous.*

PROOF. Let $x_0 \in \lim_X \mathcal{G}$. We want to show that $f(x_0) \in \lim_Y f[\mathcal{G}]$. Since $\mathcal{F}$ converges continuously to f, we have $f(x_0) \in \lim_Y \langle \mathcal{G}, \mathcal{F} \rangle$. By regularity, $f(x_0) \in \lim_Y \operatorname{adh}^\natural \langle \mathcal{G}, \mathcal{F} \rangle$ and $f[\mathcal{G}] \supseteq \operatorname{adh}^\natural \langle \mathcal{G}, \mathcal{F} \rangle$ because $f(x) \in \lim_Y \langle x, \mathcal{F} \rangle$ for each $x \in G \in \mathcal{G}$ so that $f(G) \subseteq \operatorname{adh} \langle G, F \rangle$. Therefore, $f(x_0) \in \lim_Y f[\mathcal{G}]$ and f is continuous. □

3 Arzelà-Ascoli

3.1 Interpreting equicontinuity and compactness

A classical interpretation of compactness of a topological space is that every ultrafilter converges. This readily extends to convergence spaces. More precisely a subset K of a convergence space X is *compact* if $\lim \mathcal{U} \cap K \neq \varnothing$ for

[6] A group (vector space) equipped with a convergence structure is a *convergence group* (*convergence vector space*) if the convergence makes the group operation and inversion continuous (if it is a convergence group and the convergence makes scalar multiplication continuous).

every ultrafilter $\mathcal{U}$ on X that contains K. Note that this viewpoint makes pseudotopologies the natural setting to study compactness, because compactness and pseudotopologies are both defined modulo the convergence of ultrafilters.

As for equicontinuity, in its classical formulation, it is a metric concept. It is easily extended to the context of uniform spaces. However, since we are aiming at a characterization of (relatively) compact collections of continuous maps, we do not want to restrict ourselves to the case where Y is a uniform space. This brings us to the concept of even continuity (e.g., [9]). If X and Y are topological spaces, a subset H of $C(X,Y)$ is called *evenly continuous at* x if for each $y \in Y$ and each neighborhood U of y there is a neighborhood V of x and a neighborhood W of y such that $f(V) \subseteq U$ whenever $f(x) \in W$ and $f \in H$. The collection H is *evenly continuous* if it is evenly continuous at every x in X. It is a simple exercise in ε-cutting to see that when Y is a metric (or a uniform) space, even continuity for the induced topology on Y is implied by equicontinuity (e.g., [9, Theorem 22, p.237]). If the definition of even continuity does not seem very palatable, it is because it is a statement about filters. It becomes more transparent when rephrased in those terms:

Lemma 6. *$H \subseteq C(X,Y)$ is evenly continuous at x if and only if for every filter $\mathcal{F}$ on $C(X,Y)$ such that $H \in \mathcal{F}$*

$$y \in \lim_Y \langle x, \mathcal{F} \rangle, x \in \lim_X \mathcal{G} \Longrightarrow y \in \lim_Y \langle \mathcal{G}, \mathcal{F} \rangle . \tag{3.1}$$

Proof. If H is evenly continuous, $H \in \mathcal{F}$, $y \in \lim_Y \langle x, \mathcal{F} \rangle$ and $x \in \lim_X \mathcal{G}$ then for each $U \in \mathcal{N}_Y(y)$, consider $W \in \mathcal{N}_Y(y)$ as in the definition of even continuity. There is $F \in \mathcal{F}$ such that $F \subseteq H$ and $\langle x, F \rangle \subseteq W$. By even continuity, there is $V \in \mathcal{N}_X(x) \subseteq \mathcal{G}$ such that $\langle V, F \rangle \subseteq U$. Hence $y \in \lim_Y \langle \mathcal{G}, \mathcal{F} \rangle$.

Conversely, assume by contrapositive that there is $U_0 \in \mathcal{N}_Y(y)$ for some $y \in Y$ such that for every neighborhood V of x and every neighborhood W of y there is a map $f_{V,W}$ in H such that $f_{V,W}(x) \in W$ but $f_{V,W}(V) \nsubseteq U_0$. For each $W \in \mathcal{N}_Y(y)$, consider $N_W := \{f_{V,P} : V \in \mathcal{N}_X(x), P \in \mathcal{N}_Y(y), P \subseteq W\}$. By definition, the collection $\{N_W : W \in \mathcal{N}_Y(y)\}$ is a filter-base, and the generated filter $\mathcal{F}$ satisfies $y \in \lim_Y \langle x, \mathcal{F} \rangle$ but $y \notin \lim_Y \langle \mathcal{N}_X(x), \mathcal{F} \rangle$. Hence there is a filter $\mathcal{F}$ containing H for which (3.1) fails. □

Therefore, we extend the definition of even continuity to the case where X and Y are convergence spaces by declaring H evenly continuous at x if (3.1) is true for every filter $\mathcal{F}$ on $C(X,Y)$ such that $H \in \mathcal{F}$. As an immediate consequence, we have:

Corollary 7. *If X and Y are convergence spaces, $H \subseteq C(X,Y)$ is evenly continuous and $H \in \mathcal{F}$ then*

$$f \in \lim_p \mathcal{F} \Longrightarrow f \in \lim_c \mathcal{F}.$$

In other words, the topology of pointwise convergence and the continuous convergence coincide on H.

3.2 Arzelà-Ascoli for the continuous convergence

Theorem 8. *Let X be a convergence space and Y be a regular convergence space. If $H \subseteq C(X,Y)$ is evenly continuous and $\langle x,H\rangle$ is relatively compact in Y for each x in X, then H is relatively compact in $C_c(X,Y)$.*

PROOF. Let $\mathcal{U}$ be an ultrafilter on $C(X,Y)$ containing H. We need to show that $\lim_c \mathcal{U} \neq \varnothing$. Since $\langle x,H\rangle$ belongs to the ultrafilter $\langle x,\mathcal{U}\rangle$ and $\langle x,H\rangle$ is relatively compact, there is $y(x) \in \lim_Y \langle x,\mathcal{U}\rangle$, for each $x \in X$. Therefore, $y(\cdot) \in \lim_{Y^X} \mathcal{U}$ and $H \in \mathcal{U}$ so that $y(\cdot) \in \lim_c \mathcal{U}$ because of Corollary 7. In view of Lemma 5, $y(\cdot)$ is continuous, which concludes the proof. □

As for the converse of Theorem 8, we will obtain it using almost exclusively the calculus of relations (2.2).

Theorem 9. *Let X be a convergence space and Y be a Hausdorff pseudotopological space. If H is relatively compact in $C_c(X,Y)$ then $\langle x,H\rangle$ is relatively compact in Y for each x in X and H is evenly continuous.*

PROOF. If H is (relatively) compact in $C_c(X,Y)$ then $\langle x,H\rangle$ is (relatively) compact in Y for each x in X because each map $\langle x,\cdot\rangle$ is continuous and therefore preserves (relative) compactness.

To show that H is evenly continuous, assume that $H \in \mathcal{F}$, that $y \in \lim_Y \langle x,\mathcal{F}\rangle$ and that $x \in \lim_X \mathcal{G}$. Since Y is pseudotopological, to show that $y \in \lim_Y \langle \mathcal{G},\mathcal{F}\rangle$ we only need to show that $y \in \lim_Y \mathcal{W}$ for each ultrafilter $\mathcal{W}$ that contains $\langle \mathcal{G},\mathcal{F}\rangle$. Note that $\mathcal{W}\#e\,[\mathcal{G}\times\mathcal{F}]$. By (2.1)

$$e^{-}\,[\mathcal{W}]\,\#\,(\mathcal{G}\times\mathcal{F})\,,$$

which, in view of (2.2) amounts to

$$\left(e^{-}\,[\mathcal{W}]\right)[\mathcal{G}]\,\#\mathcal{F}.$$

In particular, the filters $(e^{-}\,[\mathcal{W}])\,[\mathcal{G}]$ and $\mathcal{F}$ have a least upper bound $\mathcal{H}$ that contains H. By relative compactness of H, there is an ultrafilter $\mathcal{U}$ finer than $\mathcal{H}$ and a function $f \in \lim_c \mathcal{U}$. In particular, $f(x) \in \lim_Y \langle x,\mathcal{U}\rangle$ and

$$\langle x,\mathcal{U}\rangle \supseteq \langle x,\mathcal{H}\rangle \supseteq \langle x,\mathcal{F}\rangle\,,$$

and $y \in \lim_Y \langle x,\mathcal{F}\rangle$ so that $y \in \lim_Y \langle x,\mathcal{U}\rangle$. As Y is Hausdorff, $y = f(x)$. Now, $y = f(x) \in \lim_Y \langle \mathcal{G},\mathcal{U}\rangle$ because $f \in \lim_c \mathcal{U}$. Moreover,

$$\mathcal{U}\#\left(e^{-}\,[\mathcal{W}]\right)[\mathcal{G}]\,,$$

so that, in view of (2.2),

$$e\left[\mathcal{U}\times\mathcal{G}\right]\#\mathcal{W}.$$

As $\mathcal{W}$ is an ultrafilter, $\mathcal{W}\supseteq\langle\mathcal{G},\mathcal{U}\rangle$ and $y\in\lim_Y\mathcal{W}$, which completes the proof. □

Corollary 10. *If X is a convergence space and Y is a Hausdorff regular pseudotopological space then a subset H of $C_c(X,Y)$ is relatively compact if and only if $\langle x,H\rangle$ is relatively compact in Y for each x in X and H is evenly continuous.*

3.3 Topological corollaries

Note first that Theorem 8, while immediate once proper definitions are introduced, significantly extends the first part of Theorem 2:

Corollary 11. *Let X and Y be topological spaces. If Y is regular, $H\subseteq C(X,Y)$ is evenly continuous and $\langle x,H\rangle$ is relatively compact in Y for each $x\in X$, then H is relatively compact in $C_c(X,Y)$ and therefore in $C_k(X,Y)$.*

As an immediate consequence of Theorem 4 and Corollary 10, we obtain:

Theorem 12. *Let X be a completely regular locally compact topological space and let Y be completely regular. Then a subset H of $C_k(X,Y)$ is relatively compact if and only if $\langle x,H\rangle$ is relatively compact in Y for each x in X and H is evenly continuous.*

The conclusion of Theorem 12 can be extended from locally compact to k-spaces, that is, spaces in which a subset whose intersection with each compact subset is closed is automatically closed. The class of k-spaces include both first-countable and locally compact spaces. Note that a map $f:X\to Y$ where Y is topological and X is a k-space is continuous if and only if $f_{|K}$ is continuous for every compact subset K of X (e.g., [7, Theorem 3.3.21]). A product of two k-spaces does not need to be a k-space, but a product of a Hausdorff locally compact space and a k-space is a k-space ([10], [7, Theorem 3.3.27]).

Theorem 13. *Let X be a k-space and let Y be a (Hausdorff) completely regular topological space. Then a subset H of $C_k(X,Y)$ is relatively compact if and only if $\langle x,H\rangle$ is relatively compact in Y for each x in X and H is evenly continuous.*

Proof. In view of Corollary 10, we only need to show that continuous convergence and compact-open topology coincide on H under our assumptions.

Because $\mathrm{cl}_k H$ is a compact Hausdorff topological space, it is locally compact. Thus, $X \times \mathrm{cl}_k H$ is a k-space, so that the continuity of the evaluation map $ev : X \times \mathrm{cl}_k H \to Y$ depends only on the continuity of restrictions of ev to compact subsets K of $X \times \mathrm{cl}_k H$. Let K be such a compact subset. Then $ev : p_X(K) \times \mathrm{cl}_k H \to Y$ is continuous, because $\mathrm{cl}_k H$ carries the compact-open topology and the projection $p_X(K)$ is a compact subset of X. But $K \subseteq p_X(K) \times \mathrm{cl}_k H$ so that $ev_{|K}$ is continuous. As a result, $ev : X \times \mathrm{cl}_k H \to Y$ is continuous, so that the compact-open topology and continuous convergence coincide on $\mathrm{cl}_k H$, hence on H. □

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