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SETS OF DISCONTINUITIES OF LINEARLY CONTINUOUS FUNCTIONS

Abstract

The class of linearly continuous functions $f\colon \mathbb{R}^n \to \mathbb{R}$, that is, having continuous restrictions $f\restriction\ell$ to every straight line ℓ, have been studied since the dawn of the twentieth century. In this paper we refine a description of the form that the sets $D(f)$ of points of discontinuities of such functions can have. It has been proved by Slobodnik that $D(f)$ must be a countable union of isometric copies of the graphs of Lipschitz functions $h\colon K \to \mathbb{R}$, where K is a compact nowhere dense subset of $\mathbb{R}^{n-1}$. Since the class $\mathcal{D}^n$ of all sets $D(f)$, with $f\colon \mathbb{R}^n \to \mathbb{R}$ being linearly continuous, is evidently closed under countable unions as well as under isometric images, the structure of $\mathcal{D}^n$ will be fully discerned upon deciding precisely which graphs of the Lipschitz functions $h\colon K \to \mathbb{R}$, $K \subset \mathbb{R}^{n-1}$ being compact nowhere dense, belong to $\mathcal{D}^n$. Towards this goal, we prove that $\mathcal{D}^2$ contains the graph of any such $h\colon K \to \mathbb{R}$ whenever h can be extended to a C^2 function $\bar{h}\colon \mathbb{R} \to \mathbb{R}$. Moreover, for every $n > 1$, $\mathcal{D}^n$ contains the graph of any $h\colon K \to \mathbb{R}$, where K is closed nowhere dense in $\mathbb{R}^{n-1}$ and h is a restriction of a convex function $\bar{h}\colon \mathbb{R}^{n-1} \to \mathbb{R}$. In addition, we provide an example, showing that the above mentioned result on $\mathcal{C}^2$ functions need not hold when $\bar{h}$ is just differentiable with bounded derivative (so Lipschitz).

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1 Background

A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is: *separately continuous* if the restriction $f\restriction\ell$ is continuous for any line ℓ parallel to one of the coordinate axes; it is *linearly continuous* whenever $f\restriction\ell$ is continuous for every line ℓ in $\mathbb{R}^n$. Separate continuity is frequently described as a continuity in each variable separately.

Clearly, continuity implies linear continuity which, in turn, implies separate continuity. None of these implications can be reversed, although this may not be obvious at the first glance. Actually, the study of separately continuous functions seems to have been sparked by a mistake of Cauchy who, in his 1821 book *Cours d'analyse*, incorrectly claimed that separate continuity of $f\colon \mathbb{R}^2 \to \mathbb{R}$ implies its continuity. (See, e.g., [15] or [11].)

The study of discontinuous separately continuous functions has had a long and illustrious history, riddled with contributions from some of the giants of nineteenth and early twentieth century mathematics. Perhaps the simplest example of a discontinuous separately continuous function is the following

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} & \text{if } \langle x,y\rangle \neq \langle 0,0\rangle \\ 0 & \text{if } \langle x,y\rangle = \langle 0,0\rangle. \end{cases} \tag{1}$$

This example, attributed to Peano, first appeared in 1884 calculus text of Genocchi and Peano [10]. A somewhat more complicated example of such a function, apparently due to E. Heine (see [15]), appeared earlier in the 1870 calculus text of J. Thomae [22]. The early contributions to the theory of separately continuous functions came also from Volterra (cited by Baire, see [1]), Baire (1899, citation in [12] and [1]), and Hahn (1919, citation in [12]). A question on the regularity of separately continuous functions was settled in 1905 by Lebesgue (see [13] or [12]), who proved that every separately continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ is of $(n-1)^{\text{st}}$ Baire class and that the Baire class cannot be lowered in this result.

A full characterization of the sets $D(f)$ of discontinuity points of separately continuous functions $f\colon \mathbb{R}^n \to \mathbb{R}$ was given by Kershner [12] and reads as follows.

Theorem 1.1. *A set $D \subset \mathbb{R}^n$ is the set of discontinuities of a separately continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ if, and only if, D is an F_σ set and the projection of D onto each $(n-1)$-dimensional coordinate hyperplane is meager.*

In particular, a separately continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ can be discontinuous at every point of an $(n-2)$-dimensional linear manifold parallel to the coordinate axes (e.g., on $\mathbb{R}^{n-2} \times \{0\} \times \{0\}$). Actually, the same is also true for the class of linearly continuous function from $\mathbb{R}^n$ to $\mathbb{R}$, as immediately follows from our Corollary 3.5 (used with $K = \mathbb{R}^{n-2} \times \{0\}$).

Recall that the necessary and sufficient condition for a subset of $\mathbb{R}^n$ to be the set of discontinuities of a function from $\mathbb{R}^n$ into $\mathbb{R}$ is precisely that the set be an F_σ set. (See e.g. [14].) So, Kershner's result tells us that a subset of $\mathbb{R}^n$ which is a set of discontinuity points of some function, is a set of discontinuity points of a separately continuous function if, and only if, its projection onto each of the $(n-1)$-dimensional coordinate hyperplanes is meager. Kershner's result has been the object of intense study over the past several years and has been generalized in many ways—see, for instance, [1] and the references therein.

The first examples of discontinuous linearly continuous functions were discovered at about the same time as those of discontinuous separately continuous functions. Genocchi and Peano in their text [10] give an easy example of a discontinuous linearly continuous function:

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{if } \langle x,y\rangle \neq \langle 0,0\rangle \\ 0 & \text{if } \langle x,y\rangle = \langle 0,0\rangle. \end{cases} \tag{2}$$

Notice that although the function from (1) is separately continuous, it is not linearly continuous. So the two classes of functions discussed above are indeed different.

The study of linearly continuous functions has been developing at a considerably slower rate than that of separately continuous functions. In 1910 Young and Young [23] (see also [12, 16]) constructed a linearly continuous function $f\colon [0,1]^2 \to \mathbb{R}$ for which the set of points of discontinuity is uncountable in any open set. However, to the authors' best knowledge, presently no characterization in the vein of Kershner's exists for the sets of discontinuities of linearly continuous functions.[1] The problem of finding such a characterization was posed by Kronrod (see [21]) and the only result toward this goal seems to be a 1976 result of Slobodnik, which we describe in detail in the next section.

There has also been work in examining functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ which are continuous when restricted to the curves in $\mathbb{R}^2$ more general than lines. In particular, Scheeffer (1890, see [20]) and Lebesgue (1905, see [13, pp. 199-200]) noticed, that the continuity along all analytic curves does not implies continuity. The most powerful result in this direction was proved in 1955 by Rosenthal [19]:

[1]We have recently found a characterization of the sets $D(f)$ of discontinuity point of linearly continuous functions $f\colon \mathbb{R}^2 \to \mathbb{R}$, see [5]. However, our characterization is in terms of a topology on the set of all lines in $\mathbb{R}^2$, which lacks the elegant simplicity of Kershner's result.

Theorem 1.2. *For any function $F: \mathbb{R}^2 \to \mathbb{R}$, if*

- *$F \restriction G$ is continuous whenever G is a graph of a continuously differentiable function $f: \mathbb{R} \to \mathbb{R}$ from x to y or from y to x,*

then G is continuous. However, there exists a discontinuous function $H: \mathbb{R}^2 \to \mathbb{R}$ with $H \restriction G$ continuous whenever G is a graph of twice differentiable function $f: \mathbb{R} \to \mathbb{R}$ from x to y or from y to x.

Notice, that every function H as in Theorem 1.2 is, in particular, linearly continuous. In this direction, the authors have recently constructed an example of a function H as in Theorem 1.2 for which the set of points of discontinuity has Hausdorff dimension 1. (See [4].) By the results discussed in the next section, this is the best possible result in this direction.

We use the standard terminology and notation as in [3], [14], or [9]. In particular, we often identify a function with its graph. We will write $D(f)$ for the set of discontinuity points of a function f. We use the notation

$$\mathcal{D}^n = \{D(f): f: \mathbb{R}^n \to \mathbb{R} \text{ is linearly continuous}\}.$$

Recall, that the *oscillation* of $f: \mathbb{R}^n \to \mathbb{R}$ at x_0 is defined as

$$\operatorname{osc}(f, x_0) = \lim_{\delta \to 0^+} \{|f(x) - f(y)| : x, y \in B(x_0, \delta)\},$$

where $B(x_0, \delta) = \{x \in \mathbb{R}^n : ||x - x_0|| < \delta\}$ is an open ball centered at x_0 with radius δ. (We use the euclidean norm $|| \cdot ||$.) Note (see e.g. [14]) that f is continuous at x_0 if, and only if, $\operatorname{osc}(f, x_0) = 0$. The support of a function $g: \mathbb{R}^n \to \mathbb{R}$, denoted as $\operatorname{supp}(g)$, will be defined, as usual, as the closure of the set $\{x \in \mathbb{R}^n : g(x) \neq 0\}$.

2 Sets of points of discontinuity of linear continuous functions

We say that two subsets of $\mathbb{R}^n$ are *isometric* provided there is an isometry (i.e., distance preserving map from $\mathbb{R}^n$ onto $\mathbb{R}^n$) which maps one into another. Consider the following theorem of Slobodnik [21].

Theorem 2.1. *If $D \subset \mathbb{R}^n$ is the set of discontinuity points of some linearly continuous function $f: \mathbb{R}^n \to \mathbb{R}$, then D admits a representation $D = \bigcup_{i=1}^{\infty} D_i$, where each D_i is isometric to the graph of a Lipschitz function $\phi_i: K_i \to \mathbb{R}$ with K_i being a compact nowhere dense subset of $\mathbb{R}^{n-1}$.*

Let $\mathcal{D}^n_s$ be the collection of all sets $D(f)$, where $f\colon \mathbb{R}^n \to \mathbb{R}$ is separately continuous. Clearly $\mathcal{D}^n \subset \mathcal{D}^n_s$.

Theorems 1.1 and 2.1 show a striking difference between the sizes of the sets belonging to families $\mathcal{D}^n$ and $\mathcal{D}^n_s$. The family $\mathcal{D}^n_s$ contains sets of full n-dimensional Lebesgue measure (e.g., M^n, where $M \subset \mathbb{R}$ is a meager and of full measure). At the same time, every $D \in \mathcal{D}^n$ is of n-dimensional Lebesgue measure 0, since this is is true for a graph of any continuous function. Actually, a graph of a Lipschitz function $\phi\colon K \to \mathbb{R}$ with $K \subset \mathbb{R}^{n-1}$ (so, also any $D \in \mathcal{D}^n_s$) can have Hausdorff dimension at most $n-1$, since Lipschitz maps cannot raise Hausdorff dimension (see, for instance [9]). At the same time, it follows from Corollary 3.5 that there are sets in $\mathcal{D}^n$ of infinite Hausdorff $(n-1)$-dimensional measure. In fact, $\mathcal{D}^n$ contains any meager F_σ subset of an $(n-1)$-dimensional hyperplane H, including one of full measure in H.

Notice also that, by Kershner's result, Theorem 1.1, the projection $\pi_n[D]$ along the nth axis of any $D \in \mathcal{D}^n_s$ must be meager in $\mathbb{R}^{n-1}$. In particular, if $\phi \restriction K \in \mathcal{D}^n_s$ for some function $\phi\colon \mathbb{R}^{n-1} \to \mathbb{R}$, then K must be meager, that is, a countable union of compact nowhere dense. This explains the restriction condition on the sets K_i in Theorem 2.1.

For a class $\mathcal{F}$ of (possibly partial) functions from $\mathbb{R}^{n-1}$ into $\mathbb{R}$, let $\mathcal{E}(\mathcal{F})$ be a collection of all sets $D = \bigcup_{i=1}^\infty D_i$, where each D_i is isometric to $\phi \restriction K$ for some $\phi \in \mathcal{F}$ and a compact nowhere dense $K \subset \mathbb{R}^{n-1}$. In this notation, Theorem 2.1 reduces to

$$\mathcal{D}^n \subset \mathcal{E}(\mathrm{Lip}(\mathbb{R}^{n-1})),$$

where $\mathrm{Lip}(\mathbb{R}^{n-1})$ is the class of all Lipschitz functions $\phi\colon \mathbb{R}^{n-1} \to \mathbb{R}$. (Recall that every partial Lipschitz function on a subset of $\mathbb{R}^n$ can be extended to a Lipschitz function on the entire space, see e.g. [8, p. 80].) Moreover, since the family $\mathcal{D}^n$ is clearly closed under isometric images (i.e., if $D \in \mathcal{D}^n$, then $i[D] \in \mathcal{D}^n$ for every isometry i of $\mathbb{R}^n$) and under countable unions,[2] we have the following result, where $R(\mathcal{F})$ represents the property:

$R(\mathcal{F})$**:** $\phi \restriction K \in \mathcal{D}^n$ for every $\phi \in \mathcal{F}$ and a compact nowhere dense $K \subset \mathbb{R}^{n-1}$.

Fact 2.2. *For every family $\mathcal{F}$ of functions from $H \subset \mathbb{R}^{n-1}$ into $\mathbb{R}$, if $R(\mathcal{F})$ holds then then $\mathcal{E}(\mathcal{F}) \subset \mathcal{D}^n$.*

The main result of this paper is as follows, where symbols $\mathrm{Conv}(\mathbb{R}^n)$ and $\mathcal{C}^2(\mathbb{R}^n)$ stand for all functions $\phi\colon \mathbb{R}^n \to \mathbb{R}$ that are, respectively, convex and continuously twice differentiable.

[2]We may fix the oscillation of f on any set $D \in \mathcal{D}^n$ to be at most 1 and use a procedure similar to what is found in [14, pp. 31–32].

Theorem 2.3. *The property $R(\mathcal{F})$ holds for $\mathcal{F} = \mathrm{Conv}(\mathbb{R}^{n-1})$ for any $n \geq 2$ and for $\mathcal{F} = \mathcal{C}^2(\mathbb{R}^1)$.*

We will prove this theorem in the latter sections. Below, we discuss its consequences.

Corollary 2.4. *For every $n \geq 2$*

(a) $\mathcal{E}(\mathrm{Conv}(\mathbb{R}^{n-1})) \subsetneq \mathcal{D}^n \subsetneq \mathcal{E}(\mathrm{Lip}(\mathbb{R}^{n-1}))$,

(b) $\mathcal{E}(\mathcal{C}^2(\mathbb{R})) \subsetneq \mathcal{D}^2 \subsetneq \mathcal{E}(\mathrm{Lip}(\mathbb{R}))$.

Proof. The inclusions follow from Theorem 2.3, Fact 2.2, and Theorem 2.1.

To see that they are all strict, first notice that, clearly, neither of the classes $\mathcal{C}^2(\mathbb{R})$ and $\mathrm{Conv}(\mathbb{R})$ contains the other. Actually, if $\Phi\colon \mathbb{R} \to \mathbb{R}$ is a convex function with a (countable) dense sets A of points at which Φ'' does not exist (see e.g. [18]) and K is a nowhere dense perfect set for which the set $A_0 = \{a \in A\colon a \text{ is a bilateral limit point of } K\}$ is dense in K, then $f \restriction K \in \mathcal{E}(\mathrm{Conv}(\mathbb{R})) \setminus \mathcal{E}(\mathcal{C}^2(\mathbb{R}))$. It is also possible to find a $\mathcal{C}^\infty$ function $\Phi\colon \mathbb{R} \to \mathbb{R}$ and a nowhere dense perfect set $K \subset \mathbb{R}$ for which $f \restriction K \notin \mathcal{E}(\mathrm{Conv}(\mathbb{R}))$. Therefore, the first inclusions in (a) and (b) are indeed strict (at least, for $n = 2$).

The fact that the remaining inclusions are strict follows from Proposition 2.5. □

Now, let $bD^1(\mathbb{R}^n)$ stand for the class of all differentiable functions $\phi\colon \mathbb{R}^n \to \mathbb{R}$ with bounded derivative. Notice that $bD^1(\mathbb{R}^n) \subset \mathrm{Lip}(\mathbb{R}^n)$.

Proposition 2.5. *There exists a differentiable function $\phi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ and a nowhere dense perfect set $K \subset \mathbb{R}^{n-1}$ such that $\phi \restriction K \notin \mathcal{D}^n_s$. In particular, $\mathcal{E}(bD^1(\mathbb{R}^{n-1}))$ is contained in none of the classes $\mathcal{E}(\mathcal{F})$ from Corollary 2.4.*

Proof. First, we prove this for $n = 2$. Let $K \subset [0,1]$ be nowhere dense of positive measure. By Kershner result, Theorem 1.1, it is enough to find a function $\phi\colon \mathbb{R} \to \mathbb{R}$ from $bD^1(\mathbb{R})$ for which $\phi[K]$ has non-empty interior (as $\phi[K]$ is a projection of $\phi \restriction K$ along x-axis). The easiest example of such a function is given by $\phi(x) = \int_0^x g(t)\,dt$ for an appropriate function $g\colon \mathbb{R} \to [0,1]$. Notice, that $g = \chi_K$, the characteristic function of K, has almost all these properties: it maps K onto a non-empty interval and resulting ϕ has bounded derivative almost everywhere. To insure that ϕ is actually everywhere differentiable, it is enough to take as $g\colon \mathbb{R} \to [0,1]$ a nonnegative approximately continuous function with $\{x \in \mathbb{R}\colon g(x) > 0\}$ being a subset of K of positive measure. Then $\phi[K]$ still has a non-empty interior and $\phi'(x) = g(x)$ for all $x \in \mathbb{R}$. (For more on approximately continuous functions, see e.g. [2] or [6].)

For $n > 2$, let K be as above and put $\hat{K} = K \times [0,1]^{n-2}$. Define $\phi_n(x_1, x_2, \ldots, x_{n-1}) = \phi(x_1)$. Then the projection $\pi_{x_1}[\phi \restriction K]$ has nonempty interior. □

3 The case of convex functions

Our construction will be based on the following simple general observation.

Lemma 3.1. *Let $P \subset \mathbb{R}^n$ be closed nowhere dense and let*

(∗) $\{B_i : i < \omega\}$ be a family of pairwise disjoint closed balls in $\mathbb{R}^n$ disjoint with P, with non-empty interiors, and such that the closure of $\bigcup_{i<\omega} B_i$ is equal to $P \cup \bigcup_{i<\omega} B_i$.

For every $i < \omega$, let f_i be a continuous function from $\mathbb{R}^n$ onto $[0,1]$ with the support contained in B_i. Then $f = \sum_{i<\omega} f_i$ is from $\mathbb{R}^n$ onto $[0,1]$ and

(a) $D(f) = P = \{x \in \mathbb{R}^n : \operatorname{osc}(f,x) = 1\}$.

Moreover, if

(b) for every $N > 0$ and line ℓ in $\mathbb{R}^n$ the set $\{i < \omega : \ell \cap B_i \cap [-N,N]^n \neq \emptyset\}$ is finite,

then f is linearly continuous.

PROOF. Condition (b) implies that the family $\{\operatorname{supp}(f_i) \cap \ell : i < \omega\}$ is locally finite for every line ℓ. So, indeed, $f \restriction \ell$ is continuous.

Property (a) is obvious, after one notices that every open set intersecting P fully contains one of the balls B_i. (This can be additionally imposed in the assumptions, but in $\mathbb{R}^n$ it actually is already ensured by the current assumptions.) □

All linearly continuous functions we construct in this paper will be of the form of a function f from Lemma 3.1. Clearly, we will use this with P being $\phi \restriction K$ for appropriate ϕ and K. The construction of the balls satisfying (∗) is an easy exercise. It is the property (b) that will require care.

We will also need the following simple result.

Lemma 3.2. *Let $\hat{B}_i$'s and $\hat{P} \subset \mathbb{R}^{n-1}$ be as in Lemma 3.1. Then there exists a $\mathcal{C}^2$ function $h : \mathbb{R}^{n-1} \to [0,1]$ such that $h(x) > 0$ if, and only if, x belongs to the interior of one of the balls $\hat{B}_i$.*

Proof. Actually, such a function can be even $\mathcal{C}^\infty$. Simply, for any $i < \omega$ choose a C^∞ function $h_i: \mathbb{R}^{n-1} \to [0,1]$ for which $\{x \in \mathbb{R}^{n-1}: h_i(x) > 0\}$ is the interior of the ball $\hat{B}_i$. Then, for appropriately chosen numbers $a_i > 0$ (with the sequence $\langle a_i \rangle_i$ converging quickly to 0), the function $h = \sum_{i<\omega} a_i h_i$ is $\mathcal{C}^\infty$ and as required. (For similar constructions of $\mathcal{C}^\infty$ functions, see also [7].) $\square$

We will also use the following lemma, in which the term "a line ℓ in $\mathbb{R}^n$ is non-vertical" is understood as "ℓ is not parallel to the last axis of $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$."

Lemma 3.3. *Let $\phi: \mathbb{R}^{n-1} \to \mathbb{R}$ be an arbitrary continuous function, $\hat{P} \subset \mathbb{R}^{n-1}$ be closed nowhere dense, and let h and the $\hat{B}_i$'s be as in Lemma 3.2. For every $i < \omega$ choose an arbitrary closed ball B_i inside the set*

$$T_i = \{\langle x, y \rangle : x \in \hat{B}_i \ \&\ y \in (\phi(x), (\phi + h)(x))\}.$$

In particular, each B_i is strictly above the graph of ϕ. Then, for every $N > 0$ and line ℓ in $\mathbb{R}^n$ if the set $\{i < \omega : \ell \cap B_i \cap [-N, N]^n \neq \emptyset\}$ is infinite, then

(i) ℓ is non-vertical, so it can be identified with a function L from a line ℓ_0 in $\mathbb{R}^{n-1}$ into $\mathbb{R}$,

(ii) there is a sequence $\langle b_{i_k} \in \ell_0 \cap \hat{B}_{i_k} : k < \omega \rangle$ converging monotonically on ℓ_0 to a $b \in \ell_0 \cap \hat{P}$ and such that: $\phi(b) = L(b)$ and $\lim_{k\to\infty} \frac{\phi(b_{i_k}) - \phi(b)}{\|b_{i_k} - b\|}$ exists and is equal $\lim_{k\to\infty} \frac{L(b_{i_k}) - L(b)}{\|b_{i_k} - b\|}$, the slope of the line $L: \ell_0 \to \mathbb{R}$ when ℓ_0 is oriented in such a way, that points b_{i_k} are to the right of b.

Proof. Clearly ℓ cannot be vertical, since each B_i is a subset of the set $T_i = \{\langle x, y \rangle : x \in \hat{B}_i \ \&\ y \in (\phi(x), (\phi + h)(x))\}$ and $\hat{B}_i$'s are pairwise disjoint. So, there is an ℓ_0 as in (i) and we can choose a sequence $\langle b_{i_k} \rangle_{k<\omega}$ of points in ℓ_0, each from a different ball $\hat{B}_{i_k}$, with $\langle b_{i_k}, L(b_{i_k}) \rangle \in B_{i_k} \cap [-N, N]^n$ for every $k < \omega$. By $(*)$, choosing a subsequence, if necessary, we can assume that $\langle b_{i_k} \rangle_{k<\omega}$ converges monotonically on ℓ_0 to some $b \in \hat{P} \cap \ell_0$. So, $L(b) = \lim_{k\to\infty} L(b_{i_k})$ and $\phi(b) = \lim_{k\to\infty} \phi(b_{i_k})$.

The rest is a consequence of the squeeze theorem. Indeed, for every $k < \omega$ we have $\phi(b_{i_k}) \leq L(b_{i_k}) \leq (\phi + h)(b_{i_k}) = \phi(b_{i_k}) + h(b_{i_k})$. Hence, taking limit over $k \to \infty$, we get $\phi(b) \leq L(b) \leq \phi(b) + h(b) = \phi(b)$, where $h(b) = 0$ follows from the fact that $b \in \hat{P}$ does not belong to any interior of a ball $\hat{B}_i$. Similarly,

$$\lim_{k\to\infty} \frac{h(b_{i_k}) - h(b)}{\|b_{i_k} - b\|} = 0,$$

since the limit equals the directional derivative $D_{\vec{u}}h(b)$ of h at b in the direction of the line ℓ_0 and $D_{\vec{u}}h(b)=0$, since between any b_{i+k} and b there is a $c_{i_k}\in\ell_0$ (from the boundary of $\hat{B}_{i_k}$) with $h(c_{i_k})=0$.

Finally, for every $k<\omega$ we have $L(b_{i_k})-h(b_{i_k})\le\phi(b_{i_k})\le L(b_{i_k})$, so

$$\begin{aligned}\frac{L(b_{i_k})-L(b)}{\|b_{i_k}-b\|}-\frac{h(b_{i_k})-h(b)}{\|b_{i_k}-b\|} &= \frac{(L(b_{i_k})-h(b_{i_k}))-\phi(b)}{\|b_{i_k}-b\|}\\ &\le \frac{\phi(b_{i_k})-\phi(b)}{\|b_{i_k}-b\|}\le\frac{L(b_{i_k})-L(b)}{\|b_{i_k}-b\|}\end{aligned}$$

and, taking limit over $k\to\infty$, we get

$$\lim_{k\to\infty}\frac{L(b_{i_k})-L(b)}{\|b_{i_k}-b\|}-0\le\lim_{k\to\infty}\frac{\phi(b_{i_k})-\phi(b)}{\|b_{i_k}-b\|}\le\lim_{k\to\infty}\frac{L(b_{i_k})-L(b)}{\|b_{i_k}-b\|}$$

and the desired equation $\lim_{k\to\infty}\frac{\phi(b_{i_k})-\phi(b)}{\|b_{i_k}-b\|}=\lim_{k\to\infty}\frac{L(b_{i_k})-L(b)}{\|b_{i_k}-b\|}$ holds. □

Theorem 3.4. *If $\phi\colon\mathbb{R}^{n-1}\to\mathbb{R}$ is convex and $\hat{P}$ is a closed nowhere dense subset of $\mathbb{R}^{n-1}$, then $\phi\restriction\hat{P}\in\mathcal{D}^n$.*

Proof. Notice, that ϕ is continuous, as it is convex. (See e.g. [17].)

Choose balls $\hat{B}_i$'s for $\hat{P}$ satisfying $(*)$. Use Lemma 3.3 to find balls B_i, each with non-empty interior. Clearly these balls satisfy $(*)$ from Lemma 3.1 used with $P=\phi\restriction\hat{P}$. Therefore, to finish the proof, it is enough to show that its property (b) is satisfied.

So, choose $N>0$ and a line ℓ in $\mathbb{R}^n$ and, by way of contradiction, assume that the set $\{i<\omega\colon\ell\cap B_i\cap[-N,N]^n\neq\emptyset\}$ is infinite. Then, there is a sequence satisfying (ii) from Lemma 3.3. In particular, the line ℓ is tangent to ϕ (more precisely, tangent to $\phi\restriction\ell_0$ from the side of points b_{i_k}) and so, by the convexity of ϕ, ℓ is below the graph of ϕ. This gives us the desired contradiction, since no line below the graph of ϕ can intersect any ball B_i, each ball B_i being chosen, according to Lemma 3.3, strictly above the graph of ϕ. □

Clearly, Theorem 2.3 for $\mathcal{F}=\mathrm{Conv}(\mathbb{R}^{n-1})$ immediately follows from Theorem 3.4. Moreover, we can conclude the following.

Corollary 3.5. *If K is an F_σ meager subset of $\mathbb{R}^{n-1}$, then $K\times\{0\}\in\mathcal{D}^n$. Moreover, if K is closed in $\mathbb{R}^{n-1}$, then there exists a linearly continuous function $f\colon\mathbb{R}^n\to\mathbb{R}$ such that $D(f)=\{x\in\mathbb{R}^n\colon\mathrm{osc}(f,x)=1\}=K\times\{0\}$.*

Proof. If K is closed in $\mathbb{R}^{n-1}$, then $K\times\{0\}=\Phi\restriction K$, where $\phi\colon\mathbb{R}^{n-1}\to\{0\}$ is constant (so convex) function. Thus, Theorem 3.4 implies the second part of the corollary. Then, the first part of the corollary follows, as $\mathcal{D}^n$ is closed under countable unions. □

4 The case of $\mathcal{C}^2$ functions

Throughout this section we assume that $n = 2$. Our goal is show, using the machinery developed in the previous section, that $\phi \restriction K \in \mathcal{D}^2$ for every $\mathcal{C}^2$ function $\phi\colon \mathbb{R} \to \mathbb{R}$ and closed nowhere dense $K \subset \mathbb{R}$.

We start with the following lemma which, in particular, shows that the function constructed in Proposition 2.5 (i.e., with an image of a nowhere dense set having non-empty interior), cannot be continuously differentiable.

Lemma 4.1. *If $\phi\colon \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^1$ and P is compact nowhere dense subset of $\mathbb{R}$, then $\phi[P]$ is also nowhere dense in $\mathbb{R}$.*

Proof. Let $Z = \{x \in P\colon \phi'(x) = 0\}$. Then, by Sard's theorem, $\phi[Z]$ has measure zero. Since Z is compact, as ϕ' is continuous, $\phi[Z]$ is also compact. Therefore, $\phi[Z]$ is nowhere dense in $\mathbb{R}$.

Next, let J_k be the closures the component intervals of the complement of Z. Since $\phi \restriction J_k$ is a homeomorphism, each set $M_k = \phi[P \cap J_k]$ is nowhere dense, as a homeomorphic image of a nowhere dense set. So, $\phi[P]$ is meager, being equal to a meager set $Z \cup \bigcup_k M_k$. Being compact, it must be nowhere dense. $\square$

Lemma 4.2. *Let $\phi\colon \mathbb{R} \to \mathbb{R}$ be $\mathcal{C}^2$ and P be compact nowhere dense subset of $\mathbb{R}$. For every $x \in \mathbb{R}$ let ℓ_x be the line tangent to ϕ at $\langle x, \phi(x)\rangle$. Then the set $K_P = \bigcup_{x\in P} \ell_x$ is nowhere dense in the plane.*

Proof. First, notice that K_P is closed in $\mathbb{R}^2$. Indeed, let $\langle a, b\rangle$ be in the closure of K_P. Choose points $\langle a_k, b_k\rangle \in K_P$ converging to $\langle a, b\rangle$. For every $k < \omega$ choose $x_k \in P$ for which $\langle a_k, b_k\rangle \in \ell_{x_k}$. By compactness of P, choosing a subsequence, if necessary, we can assume that points x_k converge to an $\hat{x} \in P$. Since $\phi'(x_k)$ converge to $\phi'(\hat{x})$, it is easy to see that $\langle a, b\rangle = \langle a, \ell_{\hat{x}}(a)\rangle \in \ell_{\hat{x}} \subset K_P$. So, indeed K_P is closed in $\mathbb{R}^2$.

Next, we prove that for every vertical line ℓ, the intersection $K_P \cap \ell$ is nowhere dense in ℓ. By the Kuratowski-Ulam theorem, this implies that K_P is meager, so nowhere dense.

So, let $\ell = \{a\} \times \mathbb{R}$. Then $\ell \cap \ell_x = \langle a, \phi(x) + (a - x)\phi'(x)\rangle$. In particular, $\ell \cap K_P = \{a\} \times \hat{\phi}[P]$, where $\hat{\phi}\colon \mathbb{R} \to \mathbb{R}$ is defined as $\hat{\phi}(x) = \phi(x) + (a-x)\phi'(x)$. Since $\hat{\phi}$ is $\mathcal{C}^1$, as ϕ is $\mathcal{C}^2$, Lemma 4.1 implied that $\hat{\phi}[P]$ is nowhere dense in $\mathbb{R}$. So, indeed K_P is nowhere dense in $\mathbb{R}^2$. $\square$

Proof of Theorem 2.3 for $\mathcal{F} = \mathcal{C}^2(\mathbb{R})$. Let $\hat{P} \subset \mathbb{R}$ be compact nowhere dense and $\phi\colon \mathbb{R} \to \mathbb{R}$ be $\mathcal{C}^2$. We need to show that $\phi \restriction \hat{P} \in \mathcal{D}^2$. We proceed as in the previous section.

Choose balls (intervals) $\hat{B}_i$'s for $\hat{P}$ satisfying $(*)$. Use Lemma 3.3 to find balls B_i, each with non-empty interior. Since, by Lemma 4.2, the set $K_{\hat{P}}$ is nowhere dense in $\mathbb{R}^2$, we can additionally assume (this is the key trick) that the balls B_i are disjoint with $K_{\hat{P}}$. Once again, the balls B_i satisfy $(*)$ from Lemma 3.1 used with $P = \phi \restriction \hat{P}$.

To finish the proof, it is enough to show that the property (b) from Lemma 3.1 is satisfied. For this, choose a line ℓ in $\mathbb{R}^2$ and, by way of contradiction, assume that ℓ intersects infinitely many balls B_i. Then, there is a sequence satisfying (ii) from Lemma 3.3. This means, that ℓ is equal to the tangent line $\ell_b \subset K_{\hat{P}}$. So, ℓ could not have intersected any balls B_i, since all these balls are disjoint with $K_{\hat{P}}$. □

5 Final remarks

Notice, that we have actually proved a bit stronger result than perviously claimed:

Theorem 5.1. *Let $\phi\colon \mathbb{R}^{n-1} \to \mathbb{R}$ and let K be a closed nowhere dense subset of $\mathbb{R}^{n-1}$. Assume that either*

- *ϕ is convex, or*
- *K is compact and ϕ is continuously twice differentiable, where we allow the second derivative to be infinite.*

Then, there exists a linearly continuous function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $D(f) = \phi \restriction K = \{x \in \mathbb{R}^n\colon \operatorname{osc}(f,x) = 1\}$.

Proof. The oscillation strengthening comes directly from the formulation of Lemma 3.1. The fact that we can allow the second derivative to be infinite, requires noticing that we never used in our proof the assumption that the derivative is finite. □

Finally, let point out some natural open questions.

Problem 5.2. Does $\mathcal{E}(\mathcal{C}^2(\mathbb{R}^{n-1})) \subset \mathcal{D}^n$ for $n > 2$?

Notice that analogue of Lemma 4.2 for $n > 2$ is false.

Problem 5.3. Does $\mathcal{E}(\mathcal{C}^1(\mathbb{R})) \subset \mathcal{D}^2$? What about $\mathcal{E}(D^2(\mathbb{R})) \subset \mathcal{D}^2$?

Here, $D^2(\mathbb{R})$ stands for the class of twice differentiable functions.

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