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UNIFORM LIMITS OF PREPONDERANTLY CONTINUOUS FUNCTIONS

Abstract

There are a few nonequivalent definitions of preponderant continuity [1, 2, 3, 4, 6]. In this paper we investigate uniform limits of preponderantly continuous functions. We show that preponderantly continuous functions in the O'Malley sense are closed under uniform limit, while the family of preponderantly continuous functions in Denjoy sense are not closed under uniform limit and find closure of this family in the topology of uniform convergence. Finally, we prove that the set of preponderantly continuous in Denjoy sense functions is a first category subset of its uniform closure.

Let $\mathbb{R}$ be the set of all reals, $I=(a,b)$ be an open interval and λ denote Lebesgue measure in $\mathbb{R}$. If $E\subset\mathbb{R}$ is a measurable set we define lower and upper density of E at $x_0\in\mathbb{R}$ by:

$$\underline{d}\,(E,x_0)=\liminf_{\substack{\lambda(J)\to 0,\\ x\in J}}\frac{\lambda(J\cap E)}{\lambda(J)}\quad\text{and}\quad \overline{d}\,(E,x_0)=\limsup_{\substack{\lambda(J)\to 0,\\ x\in J}}\frac{\lambda(J\cap E)}{\lambda(J)},$$

where J is a closed interval. If $\underline{d}\,(E,x_0)=\overline{d}\,(E,x_0)$ then we denote this common value by $d(E,x_0)$ and call it the density of E at x_0. In an obvious way we also define one-sided lower and upper density of the set E at the point x_0: $\underline{d}^+\,(E,x_0)$, $\underline{d}^-\,(E,x_0)$, $\overline{d}^+\,(E,x_0)$ and $\overline{d}^-\,(E,x_0)$. It is easy to verify that

$$\underline{d}\,(E,x_0)=\min\{\,\underline{d}^+\,(E,x_0)\,,\underline{d}^-\,(E,x_0)\},\ \text{and}$$
$$\overline{d}\,(E,x_0)=\max\{\,\overline{d}^+\,(E,x_0)\,,\overline{d}^-\,(E,x_0)\}.$$

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By $\mathcal{A}$ we denote approximately continuous functions (a function $f\colon I \to \mathbb{R}$ is approximately continuous at $x_0 \in I$ if there exists a measurable set $E \subset I$ such that $x_0 \in E$, $d(E, x_0) = 1$ and $f_{|E}$ is continuous at x_0). Symbol $D_{ap}(f)$ stands for a set of all points at which a function f is not approximately continuous.

Definition 1. *A point $x_0 \in \mathbb{R}$ is said to be the point of preponderant density in Denjoy sense of a measurable set $E \subset \mathbb{R}$ if $\underline{d}\,(E, x_0) > \frac{1}{2}$.*

In an obvious way we may define one-sided preponderant density in Denjoy sense and x_0 is the point of preponderant density in Denjoy sense of a measurable set $E \subset \mathbb{R}$ if and only if it is the point of preponderant density in Denjoy sense of the measurable set E from the right and from the left.

Definition 2. *[1, 2] A function $f\colon I \to \mathbb{R}$ is said to be preponderantly continuous in the Denjoy sense at $x_0 \in I$ if there exists a measurable set $E \subset I$ containing x_0 such that x_0 is a point of preponderant density of E in the Denjoy sense and $f_{|E}$ is continuous at x_0. A function f is said to be preponderantly continuous in Denjoy sense if it is preponderantly continuous at each point $x \in I$. The class of all preponderantly continuous functions in Denjoy sense will be denoted by $\mathcal{PD}$.*

In [6] O'Malley gives another definition of preponderant continuity.

Definition 3. *[6] A point $x_0 \in \mathbb{R}$ is said to be the point of preponderant density in the O'Malley sense of a measurable set $E \subset \mathbb{R}$ if there exists $\varepsilon > 0$ such that for every closed interval J satisfying conditions $x_0 \in J$ and $J \subset [x_0 - \varepsilon, x_0 + \varepsilon]$, the inequality $\frac{\lambda(E \cap J)}{\lambda(J)} > \frac{1}{2}$ holds.*

In an obvious way we may define one-sided preponderant density in the O'Malley sense. That is, x_0 is the point of preponderant density in the O'Malley sense of a measurable set $E \subset \mathbb{R}$ if and only if it is the point of preponderant density in the O'Malley sense of the measurable set E from the right and from the left.

Definition 4. *[6] A function $f\colon I \to \mathbb{R}$ is said to be preponderantly continuous in the O'Malley sense at $x_0 \in U$ if there exists a measurable set $E \subset I$ containing x_0 such that x_0 is the point of preponderant density in the O'Malley sense of the set E and $f_{|E}$ is continuous at x_0. A function $f\colon I \to \mathbb{R}$ is said to be preponderantly continuous in the O'Malley sense if it is preponderantly continuous in the O'Malley sense at each $x_0 \in I$. The class of all functions which are preponderantly continuous in the O'Malley sense will be denoted by $\mathcal{PO}$.*

In [3] Z. Grande defined a property of real functions, called A_1 property. Based on this we may define a similar property, which extends the notion of preponderant continuity.

Definition 5. *[3, 4] A function $f\colon I \to \mathbb{R}$ is said to have A_1 property in Denjoy sense at $x_0 \in I$ if there exist measurable sets $E_1 \subset I$ and $E_2 \subset I$ containing x_0 such that x_0 is the point of preponderant density in Denjoy sense of both sets E_1 and E_2, $f_{|E_1}$ is upper semicontinuous at x_0 and $f_{|E_2}$ is lower semicontinuous at x_0. A function $f\colon I \to \mathbb{R}$ has A_1 property in Denjoy sense if it has A_1 property in Denjoy sense at each $x_0 \in U$. The class of all functions which have A_1 property in Denjoy sense will be denoted by $\mathcal{GPD}$.*

Definition 6. *[3, 4] A function $f\colon I \to \mathbb{R}$ is said to have A_1 property in the O'Malley sense at $x_0 \in I$ if there exist measurable sets $E_1 \subset I$ and $E_2 \subset I$ containing x_0 such that x_0 is the point of preponderant density in the O'Malley sense of both sets E_1 and E_2, $f_{|E_1}$ is upper semicontinuous at x_0 and $f_{|E_2}$ is lower semicontinuous at x_0. A function $f\colon I \to \mathbb{R}$ has A_1 property in the O'Malley sense if it has A_1 property in the O'Malley sense at every x_0. The class of all functions which have A_1 property in the O'Malley sense will be denoted by $\mathcal{GPO}$.*

Corollary 1. *$\mathcal{PO} \subset \mathcal{GPO}$, $\mathcal{PD} \subset \mathcal{GPD}$, $\mathcal{GPD} \subset \mathcal{GPO}$ and $\mathcal{PD} \subset \mathcal{PO}$.*

In [3] it is proven that each function $f \in \mathcal{GPO}$ is Baire 1. Thus if $f \in \mathcal{PD} \cup \mathcal{PO} \cup \mathcal{GPD} \cup \mathcal{GPO}$ then f is a Baire 1 function.

In the sequel we will consider "interval sets", that is sets of the particular form $E = \bigcup_{n=1}^{\infty} [a_n, b_n]$, where $b_{n+1} < a_n$ for each n and $x_0 = \lim_{n\to\infty} a_n$. In [4] it is shown.

Corollary 2. *[4, Corollary 6] Let $E = \bigcup_{n=1}^{\infty} [a_n, b_n]$, where $b_{n+1} < a_n < b_n$ for every n and $x_0 = \lim_{n\to\infty} a_n$. Then*

1. $\underline{d}^{+}(E, x_0) = \liminf\limits_{n\to\infty} \frac{\lambda([x_0,a_n]\cap E)}{\lambda([x_0,a_n])}$ *and* $\overline{d}^{+}(E, x_0) = \limsup\limits_{n\to\infty} \frac{\lambda([x_0,b_n]\cap E)}{\lambda([x_0,b_n])}$.
2. *The point x_0 is the point of preponderant density in the O'Malley sense of the set E if and only if there exists $n_0 \in \mathbb{N}$ such that $\dfrac{\lambda(E \cap [x_0, a_n])}{\lambda([x_0, a_n])} > \dfrac{1}{2}$ for every $n \geq n_0$.*

We have the following conditions equivalent to preponderant continuity and property A_1.

Theorem 1. *[4, Theorem 2]*

(i) *A measurable function* $f\colon I \to \mathbb{R}$ *is preponderantly continuous in Denjoy sense at* $x_0 \in I$ *if and only if* $\lim_{n\to\infty} \underline{d}\Big(\{x \in I\colon |f(x)-f(x_0)| < \frac{1}{n}\}, x_0\Big) > \frac{1}{2}$,

(ii) *A measurable function* $f\colon I \to \mathbb{R}$ *has property* A_1 *in the O'Malley sense at* $x_0 \in I$ *if and only if* $\lim_{n\to\infty} \underline{d}\Big(\{x \in I\colon f(x) < f(x_0) + \frac{1}{n}\}, x_0\Big) > \frac{1}{2}$ *and* $\lim_{n\to\infty} \underline{d}\Big(\{x \in I\colon f(x) > f(x_0) - \frac{1}{n}\}, x_0\Big) > \frac{1}{2}$.

Theorem 2. *[4, Theorem 3]*

(i) *A measurable function* $f\colon I \to \mathbb{R}$ *is preponderantly continuous in the O'Malley sense at* $x_0 \in I$ *if and only if for each* $\varepsilon > 0$, x_0 *is the point of preponderant density in the O'Malley sense of the set* $\{x \in I\colon |f(x) - f(x_0)| < \frac{1}{n}\}$,

(ii) *A measurable function* $f\colon I \to \mathbb{R}$ *has* A_1 *property in the O'Malley sense at a point* $x_0 \in I$ *if and only if for each* $\varepsilon > 0$, x_0 *is a point of preponderant density in the O'Malley sense of both sets* $\{x \in I\colon f(x) < f(x_0) + \frac{1}{n}\}$ *and* $\{x \in I\colon f(x) > f(x_0) - \frac{1}{n}\}$.

First, we study uniform limits of functions from $\mathcal{PO}$ and $\mathcal{GPO}$.

Theorem 3. *i) Uniform limit of functions from* $\mathcal{PO}$ *belongs to* $\mathcal{PO}$.

ii) Uniform limit of functions from $\mathcal{GPO}$ *belongs to* $\mathcal{GPO}$.

PROOF. *i)* Let a sequence $(f_n)_{n\geq 1} \subset \mathcal{PO}$ be uniformly convergent to f. Certainly, f is measurable. Fix any $x_0 \in I$ and $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$ for all $n \geq n_0$ and $x \in I$. If $|f_{n_0}(x) - f_{n_0}(x_0)| < \frac{\varepsilon}{3}$ then

$$|f(x) - f(x_0)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| + |f_{n_0}(x_0) - f(x_0)| < \varepsilon.$$

Therefore $\{x \in I\colon |f_{n_0}(x) - f_{n_0}(x_0)| < \frac{\varepsilon}{3}\} \subset \{x \in I\colon |f(x) - f(x_0)| < \varepsilon\}$. Thus for each $\varepsilon > 0$, x_0 is a point of preponderant density in the O'Malley sense of $\{x \in I\colon |f(x) - f(x_0)| < \varepsilon\}$. By Theorem 2, f is preponderantly continuous in the O'Malley sense at x_0.

ii) The proof is similar. We show that $\{x \in I\colon f_{n_0}(x) < f_{n_0}(x_0) + \frac{\varepsilon}{3}\} \subset \{x \in I\colon f(x) < f(x_0) + \varepsilon\}$ and $\{x \in I\colon f_{n_0}(x) > f_{n_0}(x_0) - \frac{\varepsilon}{3}\} \subset \{x \in I\colon f(x) > f(x_0) - \varepsilon\}$ for all $\varepsilon > 0$. Again, by Theorem 2, f has property A_1 in the O'Malley sense at x_0. □

We will show that $\mathcal{PD}$ and $\mathcal{PGD}$ are not closed under uniform limit.

Example 1. We shall construct a sequence of functions preponderantly continuous in Denjoy sense uniformly convergent to a function which does not have property A_1 in Denjoy sense. Let $I_n = [\frac{1}{3^n}, \frac{2}{3^n}]$ and $x_n = \frac{1}{2}\left(\frac{1}{3^n} + \frac{2}{3^{n+1}}\right) = \frac{1}{2} \cdot \frac{5}{3^{n+1}}$ for $n \geq 1$. Define $f\colon \mathbb{R} \to \mathbb{R}$ letting

$$f(x) = \begin{cases} 0 \quad for \quad x \in (-\infty, 0] \cup [\frac{2}{3}, \infty] \cup \bigcup\limits_{n=1}^{\infty} I_n, \\ 1 \quad for \quad x = x_n, \quad n = 1, 2, \dots, \\ \text{linear in intervals } [\frac{2}{3^{n+1}}, x_n], [x_n, \frac{1}{3^n}],\ n = 1, 2, \dots. \end{cases}$$

Clearly, f is continuous at each point except at 0. Moreover, f is continuous from the right at 0, $f(x) \leq f(0)$ for all x and

$$\begin{aligned} &\{x \in \mathbb{R}\colon f(x) < f(0) + \tfrac{1}{k}\} \cap [0, x_1] = \\ &= \bigcup_{n=2}^{\infty} \left(\tfrac{1}{3^n} - \tfrac{1}{k}(\tfrac{1}{3^n} - x_n), \tfrac{2}{3^n} + \tfrac{1}{k}(x_{n-1} - \tfrac{2}{3^n})\right) = \\ &= \bigcup_{n=2}^{\infty} (\tfrac{1}{3^n} - \tfrac{1}{2k \cdot 3^{n+1}}, \tfrac{2}{3^n} + \tfrac{1}{2k \cdot 3^n}) \end{aligned}$$

for each $k \geq 1$. Hence, by Corollary 2, we obtain

$$\begin{aligned} &\underline{d}^+ \left(\{x \in \mathbb{R}\colon f(x) < f(0) + \tfrac{1}{k}\}, 0\right) = \\ &= \liminf_{n\to\infty} \frac{\lambda\left(\bigcup\limits_{i=1}^{\infty} (\frac{1}{3^i} - \frac{1}{2k \cdot 3^{i+1}}, \frac{2}{3^i} + \frac{1}{2k \cdot 3^i}) \cap [0, \frac{1}{3^n} - \frac{1}{2k \cdot 3^{n+1}}]\right)}{\lambda\left([0, \frac{1}{3^n} - \frac{1}{2k \cdot 3^{n+1}}]\right)} = \\ &= \liminf_{n\to\infty} \frac{\lambda\left(\bigcup\limits_{i=n+1}^{\infty} (\frac{1}{3^i} - \frac{1}{2k \cdot 3^{i+1}}, \frac{2}{3^i} + \frac{1}{2k \cdot 3^i})\right)}{\lambda([0, \frac{1}{3^n} - \frac{1}{2k \cdot 3^{n+1}}])} = \liminf_{n\to\infty} \frac{\sum\limits_{i=n+1}^{\infty} (\frac{1}{3^i} + \frac{2}{3k \cdot 3^i})}{\frac{1}{3^n} - \frac{1}{2k \cdot 3^{n+1}}} = \\ &= \liminf_{n\to\infty} \frac{\frac{1}{2 \cdot 3^n} + \frac{1}{3k \cdot 3^n}}{\frac{1}{3^n} - \frac{1}{2k \cdot 3^{n+1}}} = \frac{\frac{1}{2} + \frac{1}{3k}}{1 - \frac{1}{6k}} \end{aligned}$$

for all $k \geq 1$. Therefore

$$\lim_{k\to\infty} \underline{d}^+\left(\{x \in \mathbb{R}\colon f(x) < f(0) + \tfrac{1}{k}\}, 0\right) = \lim_{k\to\infty} \frac{\frac{1}{2} + \frac{1}{3k}}{1 - \frac{1}{6k}} = \frac{1}{2}$$

and, by Theorem 1, f does not have property A_1 in Denjoy sense at 0. Thus $f \notin \mathcal{GPD}$.

Let $f_n\colon \mathbb{R} \to \mathbb{R}$, $f_n = \max\{f, \frac{1}{n}\}$ for $n \geq 1$. Each function f_n is continuous at each point except at 0 and it is continuous from the right at 0. Moreover,

$$\underline{d}^+\Big(\{x\colon f_n(x) = f_n(0)\}, 0\Big) = \underline{d}^+\Big(\{x\colon |f(x) - f(0)| \leq \tfrac{1}{n}\}, 0\Big) = \frac{\frac{1}{2} + \frac{1}{3n}}{1 - \frac{1}{6n}} > \frac{1}{2}.$$

Again, by Theorem 1, f_n is preponderantly continuous at 0. Hence $f_n \in \mathcal{PD}$ for $n \geq 1$. Obviously, the sequence $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent to f.

Definition 7. *Let $\mathcal{P}_1$ be the family of all measurable functions $f\colon I \to \mathbb{R}$ satisfying the condition*

$$\underline{d}\Big(\{y \in I\colon |f(x) - f(y)| < \varepsilon\}, x\Big) > \frac{1}{2}$$

for each $x \in I$ and $\varepsilon > 0$.

Definition 8. *Let $\mathcal{P}_2$ be the family of all measurable functions $f\colon I \to \mathbb{R}$ satisfying the conditions*

$$\underline{d}\Big(\{y \in I\colon f(x) < f(y) + \varepsilon\}, x\Big) > \frac{1}{2}$$

and

$$\underline{d}\Big(\{y \in I\colon f(x) > f(y) - \varepsilon\}, x\Big) > \frac{1}{2}$$

for each $x \in I$ and $\varepsilon > 0$.

It follows from Theorem 1 that $\mathcal{PD} \subset \mathcal{P}_1$ and $\mathcal{GPD} \subset \mathcal{P}_2$.

Lemma 1. *If $f \in \mathcal{P}_1$ $(f \in \mathcal{P}_2)$ and $g \in \mathcal{A}$ then $f + g \in \mathcal{P}_1$ $(f + g \in \mathcal{P}_2)$.*

PROOF. First, assume that $f \in \mathcal{P}_1$. Fix any $x \in I$ and $\varepsilon > 0$. Then $\underline{d}\left(\{y \in I\colon |g(y) - g(x)| < \frac{\varepsilon}{2}\}, x\right) = 1$ and $\underline{d}\left(\{y \in I\colon |f(y) - f(x)| < \frac{\varepsilon}{2}\}, x\right) > \frac{1}{2}$. Hence

$$\overline{d}\Big(\{y\colon |f(y) - f(x)| \geq \tfrac{\varepsilon}{2}\}, x\Big) = 1 - \underline{d}\left(\{y \in I\colon |f(y) - f(x)| < \tfrac{\varepsilon}{2}\}, x\right) < \frac{1}{2}.$$

Since

$$\{y \in I\colon |f(y) - f(x)| < \tfrac{\varepsilon}{2}\} \cap \{y \in I\colon |g(y) - g(x)| < \tfrac{\varepsilon}{2}\} \subset \\ \subset \{y \in I\colon |(f+g)(y) - (f+g)(x)| < \varepsilon\},$$

we have

$$\underline{d}\Big(\{y\in I\colon |(f+g)(y)-(f+g)(x)|<\varepsilon\},x\Big)\geq$$
$$\geq \underline{d}\Big(\{y\colon |g(y)-g(x)|<\tfrac{\varepsilon}{2}\},x\Big)-\overline{d}\Big(\{y\colon |f(y)-f(x)|\geq\tfrac{\varepsilon}{2}\},x\Big)>\frac{1}{2}.$$

(We used the well known formula $\underline{d}(E\cap F,x)\geq\underline{d}(E,x)-\overline{d}(\mathbb{R}\setminus F,x)$ which holds for each measurable sets E,F and each point $x\in\mathbb{R}$.) Thus we have proven that

$$\underline{d}\Big(\{y\in I\colon |(f+g)(y)-(f+g)(x)|<\varepsilon\},x\Big)>\frac{1}{2}$$

for all $x\in I$ and $\varepsilon>0$. Therefore $f+g\in\mathcal{P}_1$.

Proof in the case $f\in\mathcal{P}_2$ is analogous. □

Theorem 4. *1. The family $\mathcal{P}_1$ is closed under uniform convergence.*

2. The family $\mathcal{P}_2$ is closed under uniform convergence.

Proof. Let a sequence $(f_n)_{n\in\mathbb{N}}\subset\mathcal{P}_1$ be uniformly convergent to $f\colon I\to\mathbb{R}$. We shall show that $f\in\mathcal{P}_1$. Obviously, f is measurable. Fix any $x\in I$ and $\varepsilon>0$. There exists $k\in\mathbb{N}$ such that $\sup\limits_{y\in I}\{|f(y)-f_k(y)|\}<\frac{\varepsilon}{3}$. As it was shown earlier

$$\{y\in I\colon |f_k(x)-f_k(y)|<\tfrac{\varepsilon}{3}\}\subset\{y\in I\colon |f(x)-f(y)|<\varepsilon\}.$$

Hence

$$\underline{d}\Big(\{y\in I\colon |f(x)-f(y)|<\varepsilon\},x\Big)\geq\underline{d}\Big(\{y\in I\colon |f_k(x)-f_k(y)|<\tfrac{\varepsilon}{3}\},x\Big)>\frac{1}{2},$$

because $f_k\in\mathcal{P}_1$. Thus $f\in\mathcal{P}_1$.

2) The proof is analogous. □

Let ϱ be a metric in the space of all functions $f\colon I\to\mathbb{R}$ defined in the following way $\varrho(f,g)=\max\{1,\sup_{x\in I}|f(x)-g(x)|\}$. It is well known that convergence in this metric is equivalent to uniform convergence. We shall find closure of $\mathcal{PD}$ and $\mathcal{GPD}$ in the metric ϱ. In the next proof we will use result from [7].

Lemma 2. *[7] Let $f\colon I\to\mathbb{R}$ be a Baire 1 function. If $E\subset I$ and $\lambda(E)=0$ then there exists an approximately continuous function $g\colon I\to\mathbb{R}$ such that $f_{|E}=g_{|E}$.*

(Actually, the above Lemma is an obvious corollary from [7, Theorem 3.2].)

Theorem 5. *1. $\mathcal{P}_1$ is a closure of $\mathcal{PD}$ in the metric ϱ.*

2. $\mathcal{P}_2$ is a closure of $\mathcal{GPD}$ in the metric ϱ.

PROOF. 1) We know that $\mathcal{PD} \subset \mathcal{P}_1$ and, by Theorem 4, the family $\mathcal{P}_1$ is closed under uniform limit. It remains to prove that each function from $\mathcal{P}_1$ is a uniform limit of a sequence from $\mathcal{PD}$.

Let $f \in \mathcal{P}_1$. Since f is Baire 1 function (because $\mathcal{P}_1 \subset \mathcal{PO}$), we have $\lambda(D_{ap}(f)) = 0$. By Lemma 2, there exists an approximately continuous function $g\colon I \to \mathbb{R}$ such that $f(x) = g(x)$ for all $x \in D_{ap}(f)$. Let $h = f - g$. By Lemma 1, $h \in \mathcal{P}_1$. Besides, $D_{ap}(h) = D_{ap}(f)$ and $h(x) = 0$ for all $x \in D_{ap}(h)$. Define a sequence of function $(h_n)_{n\in\mathbb{N}}$ from I to $\mathbb{R}$ letting

$$h_n(x) = \begin{cases} h(x) - \frac{1}{n} & if \quad h(x) > \frac{1}{n}, \\ 0 & if \quad |h(x)| \le \frac{1}{n}, \\ h(x) + \frac{1}{n} & if \quad h(x) < -\frac{1}{n}. \end{cases}$$

Since $\varrho(h_n, h) \le \frac{1}{n}$ for all $n \ge 1$, it is obvious that $(h_n)_{n\in\mathbb{N}}$ is convergent to f in the metric ϱ. By definition of h_n, we have $D_{ap}(h_n) \subset D_{ap}(h)$. Hence if $x \in D_{ap}(h_n)$ then $h_n(x) = h(x) = 0$. Let $n \in \mathbb{N}$ and $x \in D_{ap}(h_n)$. Then

$$\underline{d}\Big(\{y \in I\colon h_n(y) = h_n(x) = 0\}, x\Big) = \underline{d}\Big(\{y \in I\colon |h(y)| \le \tfrac{1}{n}\}, x\Big) > \frac{1}{2},$$

because $h_n \in \mathcal{P}_1$. It follows that each h_n is preponderantly continuous in Denjoy sense at x. Thus $h_n \in \mathcal{PD}$ for all $n \in \mathbb{N}$. Define $f_n = h_n + g$ for $n \in \mathbb{N}$. Then each f_n is preponderantly continuous in Denjoy sense. Finally, $\varrho(f_n, f) = \varrho(h_n, h) \le \frac{1}{n}$ for $n \in \mathbb{N}$ and $(f_n)_{n\in\mathbb{N}} \subset \mathcal{PD}$ is uniformly convergent to f.

2) The proof of this case is very similar to the proof the first part. We take $f \in \mathcal{P}_2$. Then f is Baire 1 function (because $\mathcal{P}_2 \subset \mathcal{GPO}$) and $\lambda(D_{ap}(f)) = 0$. There exists an approximately continuous function $g\colon I \to \mathbb{R}$ such that $f(x) = g(x)$ for all $x \in D_{ap}(f)$. Let $h = f - g$. Then $h \in \mathcal{P}_2$. Define a sequence of function $(h_n)_{n\in\mathbb{N}}$ from I to $\mathbb{R}$ letting

$$h_n(x) = \begin{cases} h(x) - \frac{1}{n} & if \quad h(x) > \frac{1}{n}, \\ 0 & if \quad |h(x)| \le \frac{1}{n}, \\ h(x) + \frac{1}{n} & if \quad h(x) < -\frac{1}{n}. \end{cases}$$

Then $(h_n)_{n\in\mathbb{N}}$ is convergent to f in ϱ. Moreover,

$$\underline{d}\Big(\{y \in I\colon h_n(y) \ge h_n(x) = 0\}, x\Big) = \underline{d}\Big(\{y \in I\colon h(y) \ge -\tfrac{1}{n}\}, x\Big) > \frac{1}{2}$$

and

$$\underline{d}\Big(\{y \in I\colon h_n(y) \le h_n(x) = 0\}, x\Big) = \underline{d}\Big(\{y \in I\colon h(y) \le \tfrac{1}{n}\}, x\Big) > \frac{1}{2},$$

for each $x \in D_{ap}(h_n)$, because $h \in \mathcal{P}_2$. It follows that each h_n has property A_1 in Denjoy sense at x. Thus $h_n \in \mathcal{GPD}$ for all $n \in \mathbb{N}$. Define $f_n = h_n + g$ for $n \in \mathbb{N}$. Then each f_n is belongs to $\mathcal{GPD}$ and $(f_n)_{n\in\mathbb{N}} \subset \mathcal{GPD}$ is uniformly convergent to f. □

In the sequel we will need two technical lemmas. First may be found, for example, in [5].

Lemma 3. *[5, Lemma 2.1] Let F be a measurable subset of $\mathbb{R}$ and let $x_0 \in \mathbb{R}$. There exist two sequences of closed intervals $\{I_n = [a_n, b_n]\colon a \le \dots b_n < a_{n+1} < \dots < x_0\}$ and $\{J_k = [c_k, d_k]\colon x_0 < \dots d_{k+1} < c_k < \dots \le b\}$ such that*

$$\overline{d}\left(F \setminus \left(\bigcup_{n=1}^{\infty} I_n \cup \bigcup_{k=1}^{\infty} J_k\right), x_0\right) = \overline{d}\left(\left(\bigcup_{n=1}^{\infty} I_n \cup \bigcup_{k=1}^{\infty} J_k\right) \setminus F, x_0\right) = 0.$$

The second lemma is probably known too. But since we were unable to find any reference, we give, for the sake of completeness, a direct proof.

Lemma 4. *Let $x_0 \in \mathbb{R}$ and E, F be measurable subsets of $\mathbb{R}$ such that $F \subset E$, $\underline{d}^+(E, x_0) = \alpha > 0$ and $\underline{d}^+(F, x_0) = \beta > 0$. Then for each $0 < \gamma < \beta$ there exists a measurable subset $H \subset F$ for which $\underline{d}^+(E \setminus H, x_0) = \alpha - \gamma$.*

Proof. Let $(x_n)_{n\ge1}$ be a decreasing sequence converging to x_0 such that $\lim\limits_{n\to\infty} \frac{\lambda(E\cap[x_0,x_n])}{x_n - x_0} = \beta$. We may assume that $\lim\limits_{n\to\infty} \frac{x_n - x_{n+1}}{x_{n+1} - x_0} = \infty$. Fix any $c \in (0, \min\{\frac{\beta-\gamma}{2}, \gamma\})$. Let $a_n \in [x_{n+1}, x_n]$ for $n = 1, 2, \dots$ be such that $a_n - x_{n+1} = \frac{c}{3}(x_n - x_{n+1})$. Then $\lim\limits_{n\to\infty} \frac{a_n - x_{n+1}}{x_{n+1} - x_0} = \infty$. Without loss of generality we may assume that $\frac{\lambda(E\cap[x_0,y])}{y - x_0} > \alpha - \frac{c}{2}$, $\frac{\lambda(F\cap[x_0,y])}{y - x_0} > \beta - \frac{c}{3}$ for each $y \in (x_0, x_1]$ and $\frac{\lambda(E\cap[x_0,x_n])}{x_n - x_0} < \alpha + \frac{c}{2}$, $\frac{x_{n+1} - x_0}{a_n - x_{n+1}} < \frac{c}{3}$ for each $n \in \mathbb{N}$. Then

$$\begin{aligned}\lambda(F \cap [a_n, x_n]) &\ge \lambda(F \cap [x_0, x_n]) - (a_n - x_{n+1}) - (x_{n+1} - x_0) > \\ &> (\beta - \tfrac{c}{3})(x_n - x_0) - (1 + \tfrac{c}{3})(a_n - x_{n+1}) > (\beta - \tfrac{c}{3})(x_n - x_{n+1}) - \\ &- (1 + \tfrac{c}{3})\tfrac{c}{3}(x_n - x_{n+1}) > (\beta - c)(x_n - x_{n+1}) > (\gamma + c)(x_n - x_{n+1}) \qquad (1)\end{aligned}$$

and

$$\lambda(E \cap [x_{n+1}, x_n]) < (\alpha + \tfrac{c}{2})(x_n - x_0) < (\alpha + c)(x_n - x_{n+1}) \qquad (2)$$

for all $n \in \mathbb{N}$. Similarly, for $y \in [a_n, x_n]$ and $n \in \mathbb{N}$ we have

$$\lambda(E \cap [x_{n+1}, y]) \geq \lambda(E \cap [x_0, y]) - (x_{n+1} - x_0) > (\alpha - \tfrac{c}{2})(y - x_0) - \\ - \tfrac{c}{3}(a_n - x_{n+1}) > (\alpha - c)(y - x_{n+1}) > (\alpha - \gamma)(y - x_{n+1}). \quad (3)$$

For each $n \geq 1$ define $f\colon [a_n, x_n] \to \mathbb{R}$ letting

$$f_n(x) = \inf \left\{ \lambda\big((E \setminus (F \cap [x, x_n])) \cap [x_{n+1}, y]\big) - (\alpha - \gamma)(y - x_{n+1}) \colon y \in [a_n, x_n] \right\}.$$

Obviously, all f_n are continuous. By 1) and 2), $f_n(a_n) < 0$ and, by 3), we have $f_n(x_n) \geq 0$. Therefore for each $n \in \mathbb{N}$ there exists $c_n \in [a_n, x_n]$ such that $\lambda\big((E \setminus (F \cap [c_n, x_n])) \cap [x_{n+1}, y]\big) \geq (\alpha - \gamma)(y - x_{n+1})$ for all $y \in [a_n, x_n]$ and $\lambda\big((E \setminus (F \cap [c_n, x_n])) \cap [x_{n+1}, y_n]\big) = (\alpha - \gamma)(y_n - x_{n+1})$ for some $y_n \in [a_n, x_n]$. Let $H = \bigcup_{n=1}^{\infty} F \cap [c_n, x_n]$. Then

$$\liminf_{n \to \infty} \frac{\lambda((E \setminus H) \cap [x_0, y_n])}{\lambda([x_0, y_n])} \leq \liminf_{n \to \infty} \left(\tfrac{(\alpha - \gamma)(y_n - x_{n+1})}{y_n - x_0} + \tfrac{x_{n+1} - x_0}{y_n - x_0} \right) = \alpha - \gamma.$$

Hence $\underline{d}^+(E \setminus H, x_0) \leq \alpha - \gamma$. Let $x \in [x_{n+1}, x_n]$. If $x \in [a_n, x_n]$ then

$$\frac{\lambda((E \setminus H) \cap [x_0, x])}{x - x_0} \geq \frac{(\alpha - \gamma)(x - x_{n+1})}{x - x_0} \geq \alpha - \gamma - \frac{x_{n+1} - x_0}{x - x_0}. \quad (4)$$

If $x \in [x_{n+1}, a_n]$ then

$$\lambda((E \setminus H) \cap [x_0, x]) = \lambda((E \setminus H) \cap [x_0, x_{n+1}]) + \lambda(E \cap [x_{n+1}, x]) = \\ = \sum_{k > n} \lambda((E \setminus H) \cap [x_{k+1}, x_k]) + \lambda(E \cap [x_0, x]) - \lambda(E \cap [x_0, x_{n+1}]) \geq \\ \geq (\alpha - \gamma)(x_{n+1} - x_0)\big) + \lambda(E \cap [x_0, x]) - \lambda(E \cap [x_0, x_{n+1}]).$$

and

$$\frac{\lambda((E \setminus H) \cap [x_0, x])}{x - x_0} \geq (\alpha - \gamma)\frac{x_{n+1} - x_0}{x - x_0} - \frac{\lambda(E \cap [x_0, x_{n+1}])}{x - x_0} + \\ + \frac{\lambda(E \cap [x_0, x])}{x - x_0} \geq \frac{\lambda(E \cap [x_0, x])}{x - x_0} - \gamma + \left(\alpha - \frac{\lambda(E \cap [x_0, x_{n+1}])}{x_{n+1} - x_0}\right)\frac{x_{n+1} - x_0}{x - x_0}. \quad (5)$$

By (4) and (5), we get $\underline{d}^+(E \setminus H, x_0) \geq \alpha - \gamma$. Finally, $\underline{d}^+(E \setminus F, x_0) = \alpha - \gamma$. $\square$

Corollary 3. *Let E be a measurable subset of $\mathbb{R}$, $x_0 \in \mathbb{R}$ and $\underline{d}^+(E, x_0) = \alpha > 0$. For every $0 < \beta < \alpha$ there exists a measurable subset $F \subset E$ such that $\underline{d}^+(E \setminus F, x_0) = \beta$.*

Theorem 6. *The family $\mathcal{PD}$ is a first category subset of $\mathcal{P}_1$.*

Proof. Fix any $x_0 \in I$. Let us define the sets

$$G_n = \Big\{ f \in \mathcal{P}_1 \colon \exists_{\eta > 0}\ \underline{d}\Big(\{x \in I \colon |f(x) - f(x_0)| < \eta\}, x_0 \Big) < \tfrac{1}{2} + \tfrac{1}{n} \Big\}$$

for each $n \geq 1$. By Theorem 1, we have $\mathcal{PD} \cap \bigcap_{n=1}^{\infty} G_n = \emptyset$. We shall prove that every G_n is open and dense in $\mathcal{P}_1$.

Let $n \in \mathbb{N}$ and $f \in G_n$. Then $\underline{d}\big(\{x \in I \colon |f(x) - f(x_0)| < \eta\}, x_0\big) < \frac{1}{2} + \frac{1}{n}$ for some $\eta > 0$. Let $g \in \mathcal{P}_1$ be any function for which $\varrho(f, g) < \frac{\eta}{3}$. If $|g(x) - g(x_0)| < \frac{\eta}{3}$ then

$$|f(x) - f(x_0)| < |f(x) - g(x)| + |g(x) - g(x_0)| + |g(x_0) - f(x_0)| < \eta.$$

Hence

$$\big\{x \in I \colon |g(x) - g(x_0)| < \tfrac{\eta}{3}\big\} \subset \{x \in I \colon |f(x) - f(x_0)| < \eta\}$$

and $\underline{d}\left(\{x \in I \colon |g(x) - g(x_0)| < \frac{\eta}{3}\}, x_0\right) < \frac{1}{2} + \frac{1}{n}$. Thus $g \in G_n$ and we have proven that every G_n is an open subset of $\mathcal{P}_1$.

Take any $n \in \mathbb{N}$, $f \in \mathcal{PD} \setminus G_n$ and $\varepsilon > 0$. Let E be a measurable set witnessing that f is preponderantly continuous at x_0. Without loss of generality we may assume that $|f(x) - f(x_0)| < \frac{\varepsilon}{3}$ for all $x \in E$. Choose any $p \in \left(\frac{1}{2}, \min\{\underline{d}(E, x_0), \frac{1}{2} + \frac{1}{n}\}\right)$. By Corollary 3, there exists a measurable set $F \subset E$ such that $\underline{d}^+(E \setminus F, x_0) = p$. Let $H = F \cup \big(\{x \in I \colon |f(x) - f(x_0)| < \frac{\varepsilon}{3}\} \setminus E\big)$. By Lemma 3, we can find a sequence of pairwise disjoint intervals $(I_n = [a_n, b_n])_{n \geq 1} \subset I$ such that $x_0 < \ldots b_{n+1} < a_n < b_n < \ldots < b_1 < b$ and $\overline{d}^+\left(\bigcup_{n=1}^{\infty} I_n \setminus H, x_0\right) = \overline{d}^+\left(H \setminus \bigcup_{n=1}^{\infty} I_n, x_0\right) = 0$. Let $(J_n = [c_n, d_n])_{n \geq 1} \subset I$ be a sequence of pairwise disjoint intervals satisfying conditions $\overline{d}^+\left(\bigcup_{n=1}^{\infty} (J_n \setminus I_n), x_0\right) = 0$ and $[a_n, b_n] \subset (c_n, d_n)$ for $n \geq 1$. Define two function $g, h \colon I \to \mathbb{R}$ letting

$$g(x) = \begin{cases} 0 \quad if \quad x \in (a, x_0] \cup [d_1, b) \cup \bigcup_{n=1}^{\infty} [d_{n+1}, c_n], \\ \frac{2\varepsilon}{3} \quad if \quad x \in \bigcup_{n=1}^{\infty} I_n, \\ \text{linear on all intervals } [c_n, a_n] \text{ and } [b_n, d_n], n = 1, 2, \ldots \end{cases}$$

and $h = f + g$. Let $E' = E \setminus \bigcup_{n=1}^{\infty} J_n$. We have $g(x) = 0$ for each $x \in E'$. Hence $h_{|E'}$ is continuous at x_0. Moreover,

$$\underline{d}^+(E', x_0) \geq \underline{d}^+(E \setminus F, x_0) - \overline{d}\left(\bigcup_{n=1}^{\infty} I_n \setminus F, x_0\right) - \\ - \overline{d}\left(\bigcup_{n=1}^{\infty} (J_n \setminus I_n), x_0\right) = \underline{d}^+(E \setminus F, x_0) = p$$

and $\underline{d}^-(E', x_0) = \underline{d}^-(E, x_0)$. Therefore $\underline{d}(E', x_0) = \min\{\underline{d}^-(E', x_0), p\} > \frac{1}{2}$. Hence h is preponderantly continuous at x_0. Since g has only one discontinuity point x_0, we deduce that $h = f + g \in \mathcal{PD}$.

Let $D = \{x \in [x_0, b) \colon |h(x) - h(x_0)| < \frac{\varepsilon}{3}\}$. Then $D \cap H \cap \bigcup_{n=1}^{\infty} I_n = \emptyset$. Hence

$$D \subset \left(\bigcup_{n=1}^{\infty} I_n \setminus H\right) \cup \left(D \cap (I \setminus \bigcup_{n=1}^{\infty} J_n)\right) \cup \bigcup_{n=1}^{\infty} (J_n \setminus I_n).$$

Obviously, $\overline{d}^+\left(\bigcup_{n=1}^{\infty} (J_n \setminus I_n), x_0\right) = 0$ and $\overline{d}^+\left(\bigcup_{n=1}^{\infty} I_n \setminus H, x_0\right) = 0$. Moreover, $h(x) = f(x)$ for $x \in I \setminus \bigcup_{n=1}^{\infty} J_n$. Therefore

$$\underline{d}^+(D, x_0) = \underline{d}^+\left(\{x \colon |f(x) - f(x_0)| < \tfrac{\varepsilon}{3}\} \cap \left(I \setminus \bigcup_{n=1}^{\infty} J_n\right), x_0\right) = \\ = \underline{d}^+\left(\{x \colon |f(x) - f(x_0)| < \tfrac{\varepsilon}{3}\} \cap (I \setminus H), x_0\right) = \underline{d}^+(E \setminus F, x_0) = p < \frac{1}{2} + \frac{1}{n}.$$

It follows that $h \in G_n$. Obviously, $\varrho(f, h) = \frac{2\varepsilon}{3} < \varepsilon$. Thus closure of each G_n contains $\mathcal{PD}$. Since $\mathcal{PD}$ is dense in $\mathcal{P}_1$, each G_n is dense in $\mathcal{P}_1$. Finally, $\mathcal{PD} \subset \bigcup_{n=1}^{\infty} (\mathcal{P}_1 \setminus G_n)$ is a first category subset of $\mathcal{P}_1$. □

Theorem 7. *The family $\mathcal{GPD}$ is a first category subset of $\mathcal{P}_2$.*

Proof. Fix any $x_0 \in I$. Let us define the sets

$$G_n = \left\{f \in \mathcal{P}_2 \colon \exists_{\eta > 0}\ \underline{d}\left(\{x \in I \colon f(x) < f(x_0) + \eta\}, x_0\right) < \tfrac{1}{2} + \tfrac{1}{n}\right\}$$

for each $n \geq 1$. By Theorem 1, we have $\mathcal{GPD} \cap \bigcap_{n=1}^{\infty} G_n = \emptyset$. It remains to prove that every G_n is open and dense in $\mathcal{P}_2$.

Let $n \in \mathbb{N}$ and $f \in G_n$. Then $\underline{d}\big(\{x \in I \colon f(x) < f(x_0) + \eta\}, x_0\big) < \frac{1}{2} + \frac{1}{n}$ for some $\eta > 0$. Let $g \in \mathcal{P}_2$ is any function for which $\varrho(f,g) < \frac{\eta}{3}$. Then

$$\big\{x \in I \colon g(x) < g(x_0) + \tfrac{\eta}{3}\big\} \subset \{x \in I \colon f(x) < f(x_0) + \eta\}$$

and $\underline{d}\left(\{x \in I \colon g(x) < g(x_0) + \frac{\eta}{3}\}, x_0\right) < \frac{1}{2} + \frac{1}{n}$. Thus $g \in G_n$ and we have proven that every G_n is an open subset of $\mathcal{P}_2$.

Take any $n \in \mathbb{N}$, $f \in \mathcal{PD} \setminus G_n$ and $\varepsilon > 0$. Let E_1 and E_2 be measurable sets witnessing that f has property A_1 at x_0. Without loss of generality we may assume that $f(x) < f(x_0) + \frac{\varepsilon}{3}$ for all $x \in E_1$ and $f(x) > f(x_0) - \frac{\varepsilon}{3}$ for all $x \in E_2$. Let $E = E_1 \cup \{x \in I \colon f(x) \leq f(x_0)\}$ and $F = E \cap E_2$. Then $f_{|E}$ is upper semicontinuous at x_0 and

$$\underline{d}^+(F, x_0) = \underline{d}^+(E \cap E_2, x_0) \geq \underline{d}^+(E, x_0) + \underline{d}^+(E_2, x_0) - 1 > \underline{d}^+(E, x_0) - \tfrac{1}{2}.$$

By Lemma 4, there exists a measurable set $H \subset F$ such that

$$\underline{d}^+(E \setminus H, x_0) = \tfrac{1}{2} + \tfrac{1}{2n}.$$

Let $G = H \cup \big(\{x \in I \colon f(x) < f(x_0) + \frac{\varepsilon}{3}\} \setminus E\big)$. By Lemma 3, we can find a sequence of pairwise disjoint intervals $(I_n = [a_n, b_n])_{n \geq 1} \subset I$ such that $x_0 < \ldots b_{n+1} < a_n < b_n < \ldots < b_1 < b$ and $\overline{d}^+\left(\bigcup_{n=1}^{\infty} I_n \setminus G, x_0\right) = \overline{d}^+\left(G \setminus \bigcup_{n=1}^{\infty} I_n, x_0\right) = 0$. Let $(J_n = [c_n, d_n])_{n \geq 1} \subset I$ be a sequence of pairwise disjoint intervals for which $\overline{d}^+\left(\bigcup_{n=1}^{\infty} (J_n \setminus I_n), x_0\right) = 0$ and $[a_n, b_n] \subset (c_n, d_n)$ for $n \geq 1$. Define two function $g, h \colon I \to \mathbb{R}$ letting

$$g(x) = \begin{cases} 0 & if \quad x \in (a, x_0] \cup [d_1, b) \cup \bigcup_{n=1}^{\infty} [d_{n+1}, c_n], \\ \frac{2\varepsilon}{3} & if \quad x \in \bigcup_{n=1}^{\infty} I_n, \\ \text{linear in all intervals } [c_n, a_n] \text{ and } [b_n, d_n], n = 1, 2, \ldots \end{cases}$$

and $h = f + g$. Let $E' = E \setminus \bigcup_{n=1}^{\infty} J_n$. We have $g(x) = 0$ for each $x \in E'$.

Moreover,

$$\underline{d}^+(E',x_0) \geq \underline{d}^+(E\setminus G,x_0) - \overline{d}\left(\bigcup_{n=1}^{\infty} I_n\setminus G,x_0\right) - \\ - \overline{d}\left(\bigcup_{n=1}^{\infty}(J_n\setminus I_n),x_0\right) = \underline{d}^+(E\setminus H,x_0) = \tfrac{1}{2}+\tfrac{1}{2n}$$

and $\underline{d}^-(E',x_0) = \underline{d}^-(E,x_0)$. Therefore $\underline{d}(E',x_0) = \min\{\underline{d}^-(E',x_0), \frac{1}{2}+\frac{1}{2n}\} > \frac{1}{2}$. Since $h_{|E'}$ is upper semicontinuous at x_0 and $h_{|E_2}$ is lower semicontinuous at x_0, h has property A_1 at x_0. Since g has only one discontinuity point x_0, we obtain that $h = f+g \in \mathcal{GPD}$.

Let $D = \{x \in [x_0,b)\colon h(x) < h(x_0) + \frac{\eta}{3}\}$. Then $D \cap G \cap \bigcup_{n=1}^{\infty} I_n = \emptyset$. Hence

$$D \subset \left(\bigcup_{n=1}^{\infty} J_n \setminus G\right) \cup (E\setminus G) \cup \bigcup_{n=1}^{\infty}(G\setminus J_n).$$

Therefore,

$$\underline{d}^+(D,x_0) \leq \underline{d}^+(E\setminus G,x_0) + \overline{d}^+\Big(\bigcup_{n=1}^{\infty} J_n\setminus G,x_0\Big) + \overline{d}^+\Big(G\setminus \bigcup_{n=1}^{\infty} I_n,x_0\Big) \\ = \underline{d}^+(E\setminus H,x_0) = \frac{1}{2}+\frac{1}{2n}.$$

It follows that $h \in G_n$. Obviously, $\varrho(f,h) = \frac{2\varepsilon}{3} < \varepsilon$. Thus closure of each G_n contains $\mathcal{P}_2$. Since $\mathcal{GPD}$ is dense in $\mathcal{P}_2$, each G_n is dense in $\mathcal{P}_2$. Finally, $\mathcal{GPD} \subset \bigcup_{n=1}^{\infty}(\mathcal{P}_2\setminus G_n)$ is a first category subset of $\mathcal{P}_2$. □

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