INROADS

Real Analysis Exchange
Vol. 37(2), 2011/2012, pp. 489–492

Zafer Ercan, Department of Mathematics, Abant İzzet Baysal University, Gölköy Kampüsü, Bolu 14280, Turkey. email: zercan@ibu.edu.tr

A NEW PROOF OF THE SOBCZYK-HAMMER DECOMPOSITION THEOREM

Abstract

In this short note, we give a simple proof of the Sobczyk-Hammer Decomposition Theorem in terms of Dedekind complete Riesz spaces.

1 The Sobczyk-Hammer Decomposition Theorem.

Recall that an algebra $\mathcal{A}$ on a nonempty set X is a subset of the power set 2^X such that $X \in \mathcal{A}$ and $X \setminus A$ and $A \cap B$ belong to $\mathcal{R}$ whenever A and B do. A map $\mu : \mathcal{A} \to [-\infty, \infty]$ is called an *additive signed measure* (or, a *charge*) if $\mu(\emptyset) = 0$, and $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathcal{A}$ are disjoint. If $\mu(\mathcal{A})$ is bounded in $\mathcal{R}$ then μ is said to have *bounded variation*. The set of charges on $\mathcal{A}$ with finite bounded variation is denoted by $ba(\mathcal{A})$. If $\mu \in ba(\mathcal{A})$ and $\mu(A) \geq 0$ for each $A \in \mathcal{A}$ the we say μ is *positive*. Let $\mu \in ba(\mathcal{A})$ be positive; then μ is called *continuous* if for each $\epsilon > 0$ there exists a partition $\{A_1, A_2, ..., A_n\}$ of X in $\mathcal{A}$ such that

$$\sup_i \mu(A_i) < \epsilon.$$

A necessary and sufficient condition for the continuity of μ is

$$\inf \left\{ \sup_{i \leq r} \mu(A_i) : \mathcal{P} = \{A_1, ..., A_r\} \text{ is a partition of } X \right\} = 0.$$

The Sobczyk-Hammer Decomposition Theorem can be stated as follows.

Mathematical Reviews subject classification: Primary: 28A10
Key words: charge, finitely additive measure, Riesz space
Received by the editors October 19, 2011
Communicated by: Brian S. Thomson

Theorem 1 (Sobczyk-Hammer)**.** *Let* $\mu \in ba(\mathcal{R})$ *be positive. Then there exist a unique positive continuous* $\nu_0 \in ba(\mathcal{R})$ *and a subset* $\{\mu_n : n \in \mathbb{N}\}$ *of two-valued positive additive measure with* $\sum_n \mu_n(X) < \infty$ *such that*

$$\mu = \nu + \sum_n \mu_n.$$

The above theorem is given in [4] by Sobczyk and Hammer. Another proof of it can be found in [3].

2 Proof of the Theorem.

We will give a proof of Theorem 1 using Riesz space (vector lattice) theory. For all unexplained notation and terminology concerning Riesz spaces, we refer to [1].

It is well-known that $ba(\mathcal{A})$ is a Dedekind complete Riesz space under the pointwise algebraic operations and pointwise order. In this case, the absolute value $\mu \in \mathcal{A}$ is given by

$$|\mu|(A) = \sup\left\{\sum_{i=1}^{n} |\mu(A_i)| : \{A_1, A_2, ..., A_n\} \text{ is a partition of } X\right\}.$$

Moreover, it can be verified directly or consult on [1, Theorem 8.70] that the Dedekind complete Riesz space $ba(\mathcal{A})$ is an AL-space under the norm

$$||\mu|| = |\mu|(X).$$

Let

$$cba(\mathcal{A}) := \{\mu \in ba(\mathcal{R}) : |\mu| \text{ is continuous}\}.$$

Lemma 2. *The set* $cba(\mathcal{A})$ *is a band in the Dedekind complete Riesz space* $ba(\mathcal{A})$*.*

Proof. Since $|\mu(A)| \leq |\mu|(A)$ for each $A \in ba(\mathcal{A})$, it is easy to see that $ba(\mathcal{A})$ is an order ideal. Let $\mu_\alpha \uparrow \mu$ in $ba(\mathcal{A})$ with $\mu_\alpha \in cba(\mathcal{A})$. Let $\epsilon > 0$ be given. Since $\mu_\alpha(X) \uparrow \mu(X)$ in $\mathbb{R}$, there exists an α_0 such that

$$0 \leq \mu(A) - \mu_{\alpha_0}(A) \leq \mu(X) - \mu_{\alpha_0}(X) < \frac{\epsilon}{2}$$

for each $A \in ba(\mathcal{A})$. As μ_{α_0} is continuous, there exists a partition

$$\{P_1, P_2, ..., P_n\}$$

of X in $ba(\mathcal{A})$ such that

$$\mu_{\alpha_0}(A_i) < \frac{\epsilon}{2}$$

for each $i = 1, 2, ..., n$. Now it is clear that

$$\mu(A_i) < \epsilon$$

for each $i = 1, 2, ..., n$, completing the proof. □

For a subset A of a Riesz space E, the *disjoint complement* of A is defined by

$$A^d := \{x \in E : |x| \wedge |y| = 0 \text{ for each } y \in A\}.$$

If $0 \leq \mu \in ba(\mathcal{A})$, then set

$$m_\mu := \inf\{\sup_{i \leq r} \mu(A_i) : \mathcal{P} = \{A_1, ..., A_r\} \text{ is a partition of } X\}.$$

Lemma 3. *One has*

$$cba(\mathcal{A})^d \subset \left\{ \sum_{n=1}^{\infty} \mu_n : \mu_n \in ba(\mathcal{A}) \textit{ is two-valued and } \sum_{n=1}^{\infty} |\mu_n|(X) < \infty \right\}.$$

Proof. First we note that if $0 < \mu \in ba(\mathcal{A})$ is not continuous, then there exists a two-valued $0 < \lambda \in ba(\mathcal{A})$ with $\lambda \leq \mu$ (see Lemma in [3]). Moreover we can arrange λ so that it is $0 - m_\mu$ valued. Suppose that there exists $\mu \in cba(\mathcal{A})^d$ which is not of the form $\sum_{n=1}^{\infty} \mu_n$. Choose a $0 - m_\mu$ valued $\mu_1 \in ba(\mathcal{A})$. Then $0 < \mu - \mu_1$ holds. Suppose now that $\mu_1, \mu_2, ..., \mu_n$ is two-valued and non-zero in $ba(\mathcal{A})$, and that

$$0 < \lambda_n = \mu - \sum_{i=1}^{n} \mu_i.$$

Now define μ_{n+1} as being $0 - m_{\lambda_n}$ valued in $ba(\mathcal{A})$ with $\mu_{n+1} \leq \lambda_n$. So, by the induction step, a sequence (μ_n) in $ba(\mathcal{A})$ consisting of positive and two-valued term and satisfying $\mu_{n+1} \leq \lambda_n$ for all n is defined. Since μ is bounded $\sum_{n=1}^{\infty} \mu_n(X) < \infty$ holds, whence $\mu_n(X) \to 0$. Let

$$\mu_0 := \mu - \sum_{n=1}^{\infty} \mu_n.$$

By the assumption, μ_0 is not continuous, as $0 < \mu_0 \leq \mu$. On the other hand,

$$0 < m_{\mu_0} \leq m_{\lambda_n} = m_{\mu_n} = \mu_n(X) \to 0$$

is satisfied, and this contradiction proves the lemma. □

We are now in a position to give the proof of the Sobczyk-Hammer Decomposition Theorem.

Proof of Theorem 1. Since $ba(\mathcal{A})$ is Dedekind complete and $cba(\mathcal{A})$ is a band in $ba(\mathcal{A})$, we have

$$ba(\mathcal{A}) = cba(\mathcal{A}) \oplus cba(\mathcal{A})^d.$$

Then there exist unique $\mu_s \in cba(\mathcal{A})$ and $\mu_d \in cba(\mathcal{A})^d$ such that

$$\mu = \mu_s + \mu_d.$$

By Lemma 3, there exists a sequence (μ_n) of nonzero, two-valued, finitely additive measures such that

$$\mu_d = \sum_{n=1}^{\infty} \mu_n,$$

and the proof is complete. □

The referee has informed me that in [2] a technique based on the theory of Riesz spaces has been used also. In [2] the measures have values in l-groups and are defined on a structure (D-lattice) more general than Boolean algebras.

References

[1] C. D. Aliprantis and K. C. Border, *Infinite Dimensional Analysis. A Hitchhiker's Guide*, Springer, Berlin, 2006.

[2] G. Barbieri, A. Valente and H. Weber, *Decomposition of l-group valued measures*, Czechoslovak Math. Journal, to appear.

[3] W. Siebe, *On the Sobczyk-Hammer decomposition of additive set functions*, Proc. Amer. Math. Soc. **86**(3) (1982) 447–450.

[4] A. Sobczyk and P. C. Hammer, *A decomposition of additive set functions*, Duke Math. J. **11** (1944) 839–846.