RESEARCH

Real Analysis Exchange
Vol. 37(2), 2011/2012, pp. 291–304

Dan Dobrovolschi, National Meteorological Administration, Sos. Bucuresti–Ploiesti 97, 013 686 Bucharest, Romania. email: danidbrv@yahoo.com

STRICTLY MONOTONE FUNCTIONS ON PREIMAGES OF OPEN SETS LEADING TO LYAPUNOV FUNCTIONS

Abstract

We consider a real function f of a real variable such that, for every point x of the preimage $f^{-1}(D)$ of a set $D \subseteq \mathbb{R}$, f is strictly monotone at x, and give sufficient conditions of strict monotonicity of f on $f^{-1}(D)$. In particular, we prove that a differentiable function f on an open interval, whose derivative is strictly negative on $f^{-1}(D)$, where $D \subseteq \mathbb{R}$ is an open set, is strictly decreasing on $f^{-1}(D)$.

The latter result has applications in stability theory of differential equations on $\mathbb{R}^N$. The first application provides Lyapunov functions V for preimages under V of closed sets. The second application is a generalization of the Lyapunov stability theorem, in which the role of the asymptotically equilibrium point is played by $V^{-1}(-\infty, c_0]$, where V is a Lyapunov function for $V^{-1}(-\infty, c_0]$, and all sublevel sets of V are assumed to be compact. Moreover, due to compactness, all solutions of the differential equation are global to the right.

The second application is also a generalization of a boundedness result from Geophysical Fluid Dynamics; in particular, it proves rigorously that all trajectories of the famous Lorenz system eventually enter a compact set.

Mathematical Reviews subject classification: Primary: 26A24, 26A48, 26A15; Secondary: 34D20

Key words: monotone function, preimage of an open set, differentiable function, Darboux function, Lyapunov function

Received by the editors October 12, 2010

Communicated by: Brian S. Thomson

1 Introduction.

The purpose of the paper is to present two new monotonicity theorems for real functions of a real variable, Theorems 3.1, and its corollary, Theorem 3.2, and some applications of Theorem 3.2 to stability theory of *ordinary differential equations* (ODEs) on $\mathbb{R}^N$ (Theorems 4.2 (4), and 4.3, and Corollary 4.4). The more general version of Theorem 4.3, Theorem 4.1, is also presented, because its proof is easier to follow, and Theorem 4.3 is an immediate consequence of Theorem 4.1 and Theorem 4.2 (4).

The following monotonicity result is well–known from elementary calculus:

Theorem 1.1. *Let $I \subseteq \mathbb{R}$ be an open interval and $f\colon I \to \mathbb{R}$ a differentiable function on I. If $f' < 0$ on I, then f is strictly decreasing on I.*

A direct consequence of Theorem 1.1, well–known in stability theory of ODEs, is that a strict Lyapunov function V for an equilibrium point $\bar{\mathbf{x}}$ of a given ODE is strictly decreasing along the trajectories of the ODE. The monotonicity of V along the trajectories is further used to prove the asymptotic stability of $\bar{\mathbf{x}}$, i.e. the strong part of the famous Lyapunov's Stability Theorem (LST) (for definitions and more details, see Section 2).

In this paper, we present two generalizations of Theorem 1.1: Theorem 3.1 and Theorem 3.2. Theorem 3.2 states that if a differentiable function f is strictly negative on the preimage $f^{-1}(D)$ of an open set $D \subseteq \mathbb{R}$, then f is strictly decreasing on $f^{-1}(D)$. In Theorem 3.1 we prove that f remains strictly decreasing on $f^{-1}(D)$, even if we weaken the regularity of f such that f is Darboux, right–continuous, and strictly decreasing at x for all $x \in f^{-1}(D)$. Since Theorem 3.2 follows immediately from Theorem 3.1, it is sufficient to prove only the latter one.

Other generalizations of Theorem 1.1 can be found e.g. in [2, Chapter 11], [12, Chapter 5], [4], [1, Chapter 5] and [13, Chapter 7]. To our knowledge, all these generalizations conclude that f is monotone (strictly or not) on the whole domain of f, and none of them involves preimages of open sets.

As we shall see in Section 4, Theorem 3.2 has applications to stability theory of ODEs (Theorem 4.2 (4) and Theorem 4.3) which remind us the previously mentioned applications of Theorem 1.1. However, this time, Lyapunov functions correspond to closed sets instead of equilibrium points. Theorem 4.2 (4) says that if a given ODE of vectorfield f has a strict Lyapunov function V for the set $V^{-1}(F)$, where $F \subset \mathbb{R}$ is a closed set, then V is strictly decreasing along the trajectories of the ODE (it is assumed that the solutions of (1) are uniquely determined by initial conditions, and that the domains of f and V are equal). If, in addition, we assume that F is of the form $V^{-1}(-\infty, c_0]$ and the sublevel sets of V, i.e. the sets of the form $V^{-1}(-\infty, c)$, are all compact, then all trajectories of the ODE eventually enter every open neighborhood of $V^{-1}(F)$ of the form $V^{-1}(-\infty, c)$ where $c > c_0$ (Theorem 4.3). Moreover, they will ultimately enter and remain permanently in a compact neighborhood of $V^{-1}(F)$ of the form $V^{-1}(-\infty, c]$, with $c > c_0$, proving that all solutions are global to the right (Corollary 4.4). Theorem 4.3 is a generalization of the strong part of LST, in which the compact set $V^{-1}(-\infty, c_0]$ of Theorem 4.3 plays the role of the asymptotically stable equilibrium point of LST. Notice that in LST, before proving asymptotic stability, one proves stability of the equilibrium point. In Theorems 4.1 and 4.3, since we lack the stability part, we need to assume the boundedness (compactness) of the sublevel sets of V (hypothesis (2)).

A differential system (1) (see § 2.3) for which every component of its vectorfield is a polynomial in its variables, of degree less than or equal to two, and at least one component of the vectorfield is a polynomial of degree two is called a *quadratic system*. Theorem 4.3 was previously known in Geophysical Fluid Dynamics ([9, 5]) for the particular class of quadratic systems (3), called *forced dissipative systems* (see the first two examples of Section for more details). The corresponding strict Lyapunov function was also quadratic. The novelty of Theorem 4.3 is not only its more general hypothesis, but also the rigor of its proof (previously [9, 5], it was considered as "evident" that the strict Lyapunov function V is strictly decreasing along trajectories if its derivative along them, $\frac{d}{dt}V\phi(t, \mathbf{x})$, is strictly negative outside the region bounded by an ellipsoid; see also Remark 2.2).

Theorems 4.1 and 4.3 give sufficient conditions for the ω–limit set (see § 2.5 for the definition) of the solutions of a differential system to be included in a certain set. A related result is the LaSalle invariance principle (LSIP) [8], [6, Lemma 11.1]. There are, however, some differences between the above mentioned Theorems and LSIP. Thus, regarding the assumptions:

a) The inequality defining the Lyapunov function is non–strict in LSIP, and strict in our theorems.

b) The set G on which this inequality holds is arbitrary in LSIP, while in our theorems G is the complement of a certain closed set,

$$G = \mathbb{R}^N \setminus V^{-1}(-\infty, c_0].$$

c) In LSIP, the solutions of the differential system must be contained by G for all positive times, while our theorems do not have such a restriction.

The differences between the conclusions: in LSIP the solutions $\mathbf{x(t)}$ of the differential system would tend, as $t \to \infty$, to the union of all solutions that remain in the set $\{\mathbf{x} \in \operatorname{cl} G : \nabla V \cdot f(\mathbf{x}) = 0\}$ on their maximal interval of definition, while in our theorems they tend to the closed (and bounded) set $V^{-1}(-\infty, c_0]$. Unlike in [6, Lemma 11.1] we do not need to assume that the solutions are defined for all $t \geq 0$, see Corollary 4.4.

2 Preliminaries.

2.1 By $I(x,\delta)$, where $x \in \mathbb{R}$ and $\delta > 0$ we denote the interval $(x-\delta, x+\delta)$. Given a set $A \subseteq \mathbb{R}$, and a point $x \in A$, the function $f\colon A \to \mathbb{R}$ is called *strictly decreasing at* x if there is a $\delta > 0$ such that $f \geq f(x)$ on $(x-\delta, x) \cap A$ and $f \leq f(x)$ on $(x, x+\delta) \cap A$. When we need to specify also the set on which the above inequalities hold, we shall say that f is strictly decreasing at x *on* $I(x,\delta)$. Similarly, we define f strictly increasing at x.

2.2 If A and B are open subsets of $\mathbb{R}$ or $\mathbb{R}^N$, then $C^1(A,B)$ is the set of continuously differentiable functions from A to B; $C^0(A,B)$ is the set of continuous functions from A to B. For V in $C^1(\mathbb{R}^N,\mathbb{R})$, ∇V is the gradient of function V. If u and v are vectors in $\mathbb{R}^N$, then $u \cdot v$ denotes their canonical scalar product. The closure of a set $U \subset \mathbb{R}^N$ is denoted by $\operatorname{cl} U$.

2.3 In the paper we consider the autonomous ordinary differential equation on $\mathbb{R}^N$ (also called differential system on $\mathbb{R}^N$)

$$\dot{\mathbf{x}} = f(\mathbf{x}) \tag{1}$$

defined by a function $f \in C^1(W, \mathbb{R}^N)$, called the *vectorfield* of (1), where W is an open subset of $\mathbb{R}^N$. Under these conditions, the solutions of (1) are uniquely determined by initial conditions. For every $\mathbf{x} \in W$ there exists a unique nonextendible solution $\phi(\cdot, \mathbf{x})\colon I_{\mathbf{x}} \subseteq \mathbb{R} \to W$ of equation (1) such that $\phi(0,\mathbf{x}) = \mathbf{x}$, defined on a maximal open interval $I_{\mathbf{x}}$. Since (1) is autonomous, one may assume without loss of generality that $0 \in I_{\mathbf{x}} = (t^-(\mathbf{x}), t^+(\mathbf{x}))$. The image of the function $\phi(\cdot,\mathbf{x})$ is called the *trajectory* of (1) through $\mathbf{x}$ at $t = 0$. The independent variable t is called time due to physical applications.

The function $\phi\colon \Omega \to W$, where $\Omega = \{(t,\mathbf{x}) \in \mathbb{R} \times W | t \in I_{\mathbf{x}}\}$, is called the *flow* of (1), and has the following basic properties [7]:

(ϕ 1) $\phi(0,\mathbf{x}) = \mathbf{x}$, for all $\mathbf{x} \in W$;

(ϕ 2) $\phi(t, \phi(s,\mathbf{x})) = \phi(t+s, \mathbf{x})$, for all $\mathbf{x} \in W, s \in I_{\mathbf{x}}$, and $t \in I_{\phi(s,\mathbf{x})}$;

(ϕ 3) ϕ is C^1.

Regarding property (ϕ 2), one can show, by using the uniqueness of solutions of (1), that if one side of equality (ϕ 2) is defined, then so is the other, and they are equal (see e.g. [7]).

If $t^+(\mathbf{x}) = +\infty$ for all $\mathbf{x} \in W$, then the flow ϕ is called *global to the right*; if $I_{\mathbf{x}} = \mathbb{R}$ for all $\mathbf{x} \in W$, then the flow ϕ is called *global*.

2.4 (The Theorem on extending solutions [7]) Consider system (1). If $\mathbf{x}(\cdot)$ is a solution on a maximal open interval $I = (t^-, t^+)$ with $t^+ < +\infty$, then given any compact set $K \subset W$, there exists $\tau \in I$ with $\mathbf{x}(\tau) \notin K$.

2.5 If there exists a set $U \subset W \subseteq \mathbb{R}^N$ such that for every $\mathbf{x} \in W$ there is a $T > 0$ such that $\phi(t, \mathbf{x}) \in U$ for all $t \in (T, t^+(\mathbf{x}))$, we say that the solutions of (1) are *uniformly eventually enclosed* by U. If, in addition, U is compact, then system (1) is called *dissipative* [10], and its solutions are called *uniformly eventually bounded*.

Given $\mathbf{x} \in W$, the *ω–limit set* of $\phi(\cdot, \mathbf{x})$ is the set $\omega(\phi(\cdot, \mathbf{x}))$ of the points $\mathbf{y} \in W$ for which there exists a strictly increasing sequence $\{t_n\}$ in $(t^-(\mathbf{x}), t^+(\mathbf{x}))$ such that $t_n \to t^+(\mathbf{x})$ and $\phi(t_n, \mathbf{x}) \to \mathbf{y}$ when $n \to \infty$ ([7, 3]).

2.6 A point $\bar{\mathbf{x}} \in W$ is called an *equilibrium point* of (1) if $f(\bar{\mathbf{x}}) = 0$. The equilibrium point $\bar{\mathbf{x}}$ of (1) is *stable* if for every neighborhood V of $\bar{\mathbf{x}}$ in W there is a neighborhood U of $\bar{\mathbf{x}}$ in V such that if $\mathbf{x} \in U$ then $\phi(t, \mathbf{x}) \in V$ for all $t \in (0, t^+(\mathbf{x}))$ and $t^+(\mathbf{x}) = +\infty$. If U of the previous definition can be chosen such that, in addition, $\lim_{t \to +\infty} \phi(t, \mathbf{x}) = \bar{\mathbf{x}}$, then $\bar{\mathbf{x}}$ is called *asymptotically stable*.

2.7 Given (1) of flow ϕ, and a function $V \in C^1(W, \mathbb{R})$, it makes sense to consider for every $\mathbf{x} \in W$ the composition function $V \circ \phi(\cdot, \mathbf{x})\colon I_{\mathbf{x}} \to \mathbb{R}$. For convenience, we shall write $V\phi(t, \mathbf{x})$ instead of $V(\phi(t, \mathbf{x}))$.

2.8 A function $V \in C^1(W \setminus \{\bar{\mathbf{x}}\}, \mathbb{R}) \cap C^0(W, \mathbb{R})$ such that

(1) $V(\bar{\mathbf{x}}) = 0$, and $V(\mathbf{x}) > 0$ for $\mathbf{x} \neq \bar{\mathbf{x}}$;

(2) $\nabla V \cdot f \leq 0$ on $W \setminus \{\bar{\mathbf{x}}\}$,

is called a *Lyapunov function* for $\bar{\mathbf{x}}$. If in (2) above the inequality is strict, then V is called a *strict Lyapunov function* for $\bar{\mathbf{x}}$.

It is well–known that if $\bar{\mathbf{x}}$ has a (strict) Lyapunov function, then $\bar{\mathbf{x}}$ is (asymptotically) stable (LST, [7]).

Remark 2.1. *If V is a strict Lyapunov function for $\bar{\mathbf{x}}$, then*

$$\frac{d}{dt} V\phi(t, \mathbf{x}) = (\nabla V \cdot f)(\phi(t, \mathbf{x})) < 0,$$

for all $t \in \phi(\cdot, \mathbf{x})^{-1}(W \setminus \{\bar{\mathbf{x}}\})$. Since the solutions of system (1) *are uniquely determined by initial conditions, and the domains of V and f are equal, the set $\phi(\cdot, \mathbf{x})^{-1}(W \setminus \{\bar{\mathbf{x}}\})$ is equal to $I_{\mathbf{x}}$ for all $\mathbf{x} \in W \setminus \{\bar{\mathbf{x}}\}$. Since I_x is an interval, Theorem 1.1 can be applied to $V\phi(\cdot, \mathbf{x})$; hence, $V\phi(\cdot, \mathbf{x})$ is strictly decreasing on $\phi(\cdot, \mathbf{x})^{-1}(W \setminus \{\bar{\mathbf{x}}\}) = I_{\mathbf{x}}$ for all $\mathbf{x} \in W \setminus \{\bar{\mathbf{x}}\}$.*

2.9 More generally, given a set $U \subset W$, we call $V \in C^1(W, \mathbb{R})$ a *strict Lyapunov function for the set* U, if for every $\mathbf{x} \in W \setminus U$ the function $V\phi(\cdot, \mathbf{x})$ is strictly decreasing on $\phi(\cdot, \mathbf{x})^{-1}(W \setminus U)$.

Remark 2.2. *In practice, it is easier to check whether* $\nabla V \cdot f < 0$ *on* $W \setminus U$, *i.e. whether* $\frac{d}{dt} V\phi(t, \mathbf{x}) < 0$ *on* $\phi(\cdot, \mathbf{x})^{-1}(W \setminus U)$ *for* $\mathbf{x} \in W \setminus U$. *However, in general,* $\phi(\cdot, \mathbf{x})^{-1}(W \setminus U)$ *is no more an interval (as happened when* U *was an equilibrium point of* (1)*, see Remark 2.1), which makes Theorem 1.1 inapplicable. The strict monotonicity of* $V\phi(\cdot, \mathbf{x})$ *on* $\phi(\cdot, \mathbf{x})^{-1}(W \setminus U)$ *is now ensured by theorems of the type of Theorem 3.2.*

2.10 Given a function $V : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$, and an interval $(a, b) \subset \mathbb{R}$ with $a = -\infty$ or $b = +\infty$, we shall write the preimage of (a, b) under V as $V^{-1}(a, b)$ instead of $V^{-1}((a, b))$.

3 The monotonicity results.

Let $I \subseteq \mathbb{R}$ be an open interval, $f : I \to \mathbb{R}$ a function, and $D \subseteq \mathbb{R}$ a set. Recall that f is *right continuous* on I if it is right continuous at every $x \in I$, i.e. the limit from the right of f at x exists, and it is equal to $f(x)$.

Theorem 3.1. *If* f *is*

(1) Darboux on I,

(2) right continuous on I, *and*

(3) strictly decreasing at x, *for all* $x \in f^{-1}(D)$,

then f *is strictly decreasing on* $f^{-1}(D)$.

Theorem 3.2. *If* $f' < 0$ *on* $f^{-1}(D)$, *and* D *is an open set, then* f *is strictly decreasing on* $f^{-1}(D)$.

The theorems 3.1 and 3.2 remain valid after replacing everywhere "decreasing" with "increasing".

4 The applications.

Consider the differential system (1) defined as in § 2.3, and let ϕ be its flow.

Theorem 4.1. *Let* $W = \mathbb{R}^N$, $F \subset \mathbb{R}^N$ *be a closed set, and let* M *denote* $\mathbb{R}^N$ *or* $\mathbb{R}^N \setminus F$. *If* $V \in C^1(\mathbb{R}^N, \mathbb{R})$ *is such that*

(1) for all $\mathbf{x} \in M$, $V\phi(\cdot, \mathbf{x})$ *is strictly decreasing on* $\phi(\cdot, \mathbf{x})^{-1}(\mathbb{R}^N \setminus F)$*;*

(2) for all $\mathbf{x} \in \mathbb{R}^N$, $V^{-1}(-\infty, V(\mathbf{x})]$ *is compact,*

then for every $\mathbf{x} \in M$*, and for every open set* U *of* F *such that* $U \supset F$*, there is a* $T \in (0, t^+(\mathbf{x}))$ *such that* $\phi(t, \mathbf{x}) \in U$ *for all* $t > T$*.*

An immediate consequence is that for every $\mathbf{x} \in M$, and for every open set U of F such that $U \supset F$, $\omega(\phi(\cdot, \mathbf{x})) \subseteq \mathrm{cl}(U)$.

Theorem 4.2. *If* $U \subseteq W$ *and* $V \in C^1(W, \mathbb{R})$ *such that*

$$\nabla V \cdot f < 0 \text{ on } U \tag{2}$$

then

(1) $V\phi(\cdot, \mathbf{x}) \in C^1(I_{\mathbf{x}}, \mathbb{R})$*, for all* $\mathbf{x} \in W$*;*

(2) if U *is open, then* $V\phi(\cdot, \mathbf{x})$ *is locally strictly decreasing on* $\phi(\cdot, \mathbf{x})^{-1}(U)$ *for all* $\mathbf{x} \in W$*;*

(3) if $U = V^{-1}(D)$*, with* $D \subseteq \mathbb{R}$ *open, then* $V\phi(\cdot, \mathbf{x})$ *is strictly decreasing on* $(V\phi(\cdot, \mathbf{x}))^{-1}(D)$*, for all* $\mathbf{x} \in W$*;*

(4) if $F \subset \mathbb{R}$ *is closed, then* $V\phi(\cdot, \mathbf{x})$ *is strictly decreasing on* $(V\phi(\cdot, \mathbf{x}))^{-1}(\mathbb{R} \setminus F) = \phi(\cdot, \mathbf{x})^{-1}(W \setminus V^{-1}(F))$*, for all* $\mathbf{x} \in W$*, and* V *is a strict Lyapunov function for* $V^{-1}(F)$*.*

Theorem 4.3. *Let* $W = \mathbb{R}^N$*. If there exist* $c_0 \in \mathbb{R}$ *and* $V \in C^1(\mathbb{R}^N, \mathbb{R})$ *such that*

(1) $\nabla V \cdot f < 0$ *on* $V^{-1}(c_0, +\infty)$*;*

(2) $V^{-1}(-\infty, a]$ *is compact for all* $a \in \mathbb{R}$*,*

then for every $\mathbf{x} \in \mathbb{R}^N$ *and* $c > c_0$ *there is a* $T = T(c, \mathbf{x}) > 0$ *such that* $t > T$ *implies* $V\phi(t, \mathbf{x}) < c$*.*

Corollary 4.4. *If* $c > c_0$*, then the solutions of system* (1) *are uniformly eventually bounded by the compact set* $V^{-1}(-\infty, c]$*. Consequently, the flow* ϕ *is global to the right (due to Theorem 4.3 and Theorem on extending solutions, see § 2.4), and* $\omega(\phi(\cdot, \mathbf{x})) \subseteq V^{-1}(-\infty, c]$ *for all* $\mathbf{x} \in \mathbb{R}^N$*.*

In hypothesis (2) of Theorems 4.1 and 4.3 one may require merely boundedness instead of compactness.

5 Examples.

We give three examples of differential systems (1) to which Theorem 4.3 can be applied. Therefore, by Theorem 4.3, these systems are dissipative in the sense of § 2.5.

5.1 The class of forced dissipative systems

The first example is the class of *forced dissipative systems*, which is important in Geophysical Fluid Dynamics. These systems result from the spectral expansion and truncation of the governing equations of most fluid dynamical problems [5], and are described by the equations of the form:

$$\dot{x}_i = \sum_{j,k=1}^{N} a_{ijk} x_j x_k - \sum_{j=1}^{N} b_{ij} x_j + c_i \quad i = 1, 2, \ldots, N. \tag{3}$$

where the real numbers a_{ijk}, b_{ij}, and c_i satisfy the following conditions:

(c1) $\sum_{i,j,k=1}^{N} a_{ijk} x_i x_j x_k = 0$, for all $x_1, x_2, \ldots, x_N$ in $\mathbb{R}$, and not all of a_{ijk} are zero;

(c2) $\sum_{i,j=1}^{N} b_{ij} x_i x_j > 0$, for all $x_1, x_2, \ldots, x_N$ in $\mathbb{R}$ such that at least one x_i is non–zero;

(c3) at least one of the coefficients c_i is non–zero.

Following Lorenz's ideas [9], we take $V = \frac{1}{2} \sum_{i=1}^{N} x_i^2$; then

$$\nabla V \cdot f = - \sum_{i,j=1}^{N} b_{ij} (x_i + e_i)(x_j + e_j) + \sum_{i,j=1}^{N} b_{ij} e_i e_j \tag{4}$$

where $(e_1, e_2, \ldots, e_N)$ is the unique solution of the linear system $\sum_{j=1}^{N} (b_{ij} + b_{ji})\, e_j = c_i$, $i = 1, 2, \cdots, N$ (the solution is unique due to the dissipativity condition (c2) imposed on the coefficients b_{ij}). Denote $X = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N$. Relation (4) and the dissipativity condition (c2) show that the set

$$\mathcal{E} = \{X \colon \sum_{i,j=1}^{N} b_{ij} (x_i + e_i)(x_j + e_j) \leq \sum_{i,j=1}^{N} b_{ij} e_i e_j \} \tag{5}$$

is an N–dimensional solid ellipsoid. Take $c_0 = \max_{\mathcal{E}} V$. Then $X \in \mathcal{E}$ implies $V(X) \leq c_0$; by passing to the contrapositive proposition, $V(X) > c_0$ implies $X \in \mathbb{R}^N \setminus \mathcal{E}$. In other words, $\nabla V \cdot f < 0$ on $V^{-1}(c_0, +\infty)$, i.e. hypothesis (1) of Theorem 4.3 is fulfilled. Hypothesis (2) of Theorem 4.3 is obviously fulfilled due to the form of V.

5.2 Lorenz–63 system

The Lorenz–63 system [9]

$$\begin{aligned}\dot{x} &= s(y-x),\\ \dot{y} &= rx - y - xz,\\ \dot{z} &= -bz + xy,\end{aligned} \tag{6}$$

where $s, r, b > 0$, is a simplified model of atmospheric convection. By taking $V = \frac{1}{2}(rx^2 + sy^2 + s(z-2r)^2)$ (see [11]), and $c_0 = \max_{\mathcal{E}} V$, where $\mathcal{E}$ is the full ellipsoid $\mathcal{E} = \{(x,y,z) \in \mathbb{R}^3 \colon rx^2 + y^2 + b(z-r)^2 \leq br^2\}$, we can apply Theorem 4.3. In this case, $\nabla V \cdot f = s(-rx^2 - y^2 - b(z-r)^2 + br^2)$.

Lorenz–63 system is not of the form (3), but can be converted to it by the transformation $x' = x, y' = y, z' = z - r - s$ (see [9]).

5.3 Another example

The following system

$$\begin{aligned}\dot{x} &= x - y - xz^2 - \frac{x(x^2+y^2)}{3},\\ \dot{y} &= x + y - yz^2 - \frac{y(x^2+y^2)}{3},\\ \dot{z} &= -z - \frac{z^3}{3} - z(x^2+y^2),\end{aligned} \tag{7}$$

is an example of a P–competitive system with an orbitally stable solution (see [10] for more details). It is not, and cannot be converted into a forced dissipative system of the form (3), but it is dissipative in the sense of § 2.5.

In this case $V = \frac{1}{2}(x^2 + y^2 + z^2)$, and $c_0 = \max_{\mathcal{E}} V$, where $\mathcal{E} = \{(x,y,z) \in \mathbb{R}^3 \colon \frac{x^2}{3} + \frac{y^2}{3} + 2z^2 \leq 1\}$. Note that $\nabla V \cdot f = (x^2+y^2)\left(1 - \frac{x^2}{3} - \frac{y^2}{3} - 2z^2\right) - z^2 - \frac{z^4}{3}$.

6 Proofs of the main results.

PROOF OF THEOREM 3.1 BY CONTRADICTION. Assume there exist $t_1 < t_2$ in $f^{-1}(D)$ such that

$$f(t_1) \leq f(t_2). \tag{8}$$

By hypothesis (3), for t_i (i=1,2) there is an ϵ_i (i=1,2) such that f is strictly decreasing at t_i on intervals $I(t_i, \epsilon_i) \subset f^{-1}(D)$, (i=1,2). Note that these intervals are disjoint (otherwise, if $t_1 < \xi < t_2$ is a common point, then $f(t_1) > f(\xi) > f(t_2)$ contradicting (8)).

We claim that for t_1 and t_2 there is a point τ_1 in $f^{-1}(D)$ such that $t_1 < \tau_1 < t_2$ and $f(\tau_1) = f(t_2)$. To prove this, we first pick up $t_1' > t_1$ in $I(t_1, \epsilon_1)$ and $t_2' < t_2$ in $I(t_2, \epsilon_2)$. By hypothesis (3) and by inequality (8) we get

$$f(t_1') < f(t_1) \leq f(t_2) < f(t_2').$$

Since f is Darboux by hypothesis (1), there exists τ_1 such that $t_1' < \tau_1 < t_2'$ and $f(\tau_1) = f(t_2)$. Moreover, τ_1 is in $f^{-1}(D)$ because t_2 is in $f^{-1}(D)$.

By repeating the reasoning, we construct the sequence $\{\tau_n\}_{n\geq 1}$ in $f^{-1}(D)$ such that for all $n \geq 1$:

$$\tau_{n+1} < \tau_n, \quad t_1 < \tau_n < t_2, \quad f(\tau_n) = f(t_2). \tag{9}$$

The sequence $\{\tau_n\} \subset \mathbb{R}$ is strictly decreasing and bounded in $[t_1, t_2]$, hence it is convergent. Let us denote its limit by $\tau \in [t_1, t_2]$:

$$\tau_n \to \tau. \tag{10}$$

Since f is right continuous at τ by hypothesis (2), we have $f(\tau_n) \to f(\tau)$, and because $f(\tau_n) = f(t_2)$ for all n, we get $f(\tau) = f(t_2)$; thus, τ is in $f^{-1}(D)$. From hypothesis (3), it follows that there is an $\epsilon > 0$ such that f is strictly decreasing at τ on $I(\tau, \epsilon)$. This fact together with (9) and (10) leads to a contradiction: on one hand, $I(\tau, \epsilon)$ must contain a term τ_n, for which $f(\tau_n) = f(t_2)$, while, on the other hand, $f(t_2) = f(\tau) > f(\tau_n)$. □

PROOF OF THEOREM 4.2. **(1)** For every $\mathbf{x} \in W$, the function $V\phi(\cdot, \mathbf{x})$ is C^1 because it is the composition of the two C^1 functions V and $\phi(\cdot, \mathbf{x})$. **(2)** For every $\mathbf{x} \in W$, denote $v = V\phi(\cdot, \mathbf{x})$. Then, for every $t \in \phi(\cdot, \mathbf{x})^{-1}(U)$, we have $\phi(t, \mathbf{x}) \in U$, and $v'(t) = (\nabla V \cdot f)(\phi(t, \mathbf{x})) < 0$, by hypothesis (2). Since U is open and $v' < 0$ on the open set $\phi(\cdot, \mathbf{x})^{-1}(U)$, v is locally strictly decreasing on $\phi(\cdot, \mathbf{x})^{-1}(U)$. **(3)** If $U = V^{-1}(D)$, with D open in $\mathbb{R}$, then $\phi(\cdot, \mathbf{x})^{-1}(U) = (V\phi(\cdot, \mathbf{x}))^{-1}(D)$, and the statement follows from Theorem 3.2 for $f = V\phi(\cdot, \mathbf{x})$, and $\mathbf{x} \in W$. **(4)** If $F \subset \mathbb{R}$ is closed, then $\mathbb{R} \setminus F$ is open, and the statement follows from the previously proved statement (3). □

PROOF OF THEOREM 4.1 BY CONTRADICTION. We prove the theorem only for $M = \mathbb{R}^N$ (in the other case the proof is similar). Assume that there is an $\mathbf{x_0}$ in $\mathbb{R}^N$ and an open set U, $U \supset F$, such that for every $T \in (0, t^+(\mathbf{x_0}))$ there is a $t > T$ with $\phi(t, \mathbf{x_0}) \in \mathbb{R}^N \setminus U$. Then there is a strictly increasing sequence $\{t_k\}$, $t_k \to \theta$, with $\theta \leq t^+(\mathbf{x_0})$, such that

$$t_k \in \phi(\cdot, \mathbf{x_0})^{-1}(\mathbb{R}^N \setminus U), \quad \text{for all } k \geq 1. \tag{11}$$

Since $F \subset U$, it follows that

$$t_k \in \phi(\cdot, \mathbf{x_0})^{-1}(\mathbb{R}^N \setminus F), \quad \text{for all } k \geq 1. \tag{12}$$

From (12) and hypothesis (1), we deduce $V\phi(t_k, \mathbf{x_0}) \leq V\phi(t_1, \mathbf{x_0})$, for all $k \geq 1$. In other words, for all $k \geq 1$, $\phi(t_k, \mathbf{x_0}) \in V^{-1}(-\infty, V\phi(t_1, \mathbf{x_0})]$, which is a compact set by hypothesis (2). Thus, by replacing $\{t_k\}$ with one of its subsequences if needed, we may assume that $\{\phi(t_k, \mathbf{x_0})\}$ converges, and we denote by $\mathbf{x}^*$ its limit:

$$\phi(t_k, \mathbf{x_0}) \to \mathbf{x}^*. \tag{13}$$

Since U is open, and includes F, statements (11) and (13) imply

$$\mathbf{x}^* \in \mathrm{cl}(\mathbb{R}^N \setminus U) = \mathbb{R}^N \setminus U \subset \mathbb{R}^N \setminus F,$$

hence

$$\mathbf{x}^* \in \mathbb{R}^N \setminus F. \tag{14}$$

Since V is continuous, from (13) it follows that

$$V\phi(t_k, \mathbf{x_0}) \to V(\mathbf{x}^*). \tag{15}$$

Now, (15) and the monotonicity of sequence $\{V\phi(t_k, \mathbf{x_0})\}$ lead to

$$V\phi(t_k, \mathbf{x_0}) > V(\mathbf{x}^*) \quad \text{for all } k \geq 1. \tag{16}$$

In what follows, we will get a contradiction of (16). To this end, we follow the steps below:
STEP 1: the set $\phi(\cdot, \mathbf{x}^*)^{-1}(\mathbb{R}^N \setminus F)$ contains 0 and also an $s > 0$ which can be taken arbitrarily small; therefore,

$$V\phi(s, \mathbf{x}^*) < V(\mathbf{x}^*); \tag{17}$$

STEP 2: there is a k_1 such that

$$V\phi(s + t_k, \mathbf{x_0}) < V(\mathbf{x}^*) \quad \text{for all } k \geq k_1; \tag{18}$$

STEP 3: there is a k_2 such that

$$s + t_k \in \phi(\cdot, \mathbf{x_0})^{-1}(\mathbb{R}^N \setminus F) \quad \text{for all } k \geq k_2; \tag{19}$$

STEP 4: take $k > \max\{k_1, k_2\}$ and $m \geq 1$ such that $t_m > s + t_k$; then,

$$V\phi(t_m, \mathbf{x_0}) < V\phi(s + t_k, \mathbf{x_0}). \tag{20}$$

Now relations (18) and (20) contradict (16).

Now we prove the steps above. STEP 1. $0 \in \phi(\cdot, \mathbf{x}^*)^{-1}(\mathbb{R}^N \setminus F)$ follows from $\phi(0, \mathbf{x}^*) = \mathbf{x}^*$ (§ 2.3, property (ϕ1) of a flow) and (14). The existence of an $s > 0$ in $\phi(\cdot, \mathbf{x}^*)^{-1}(\mathbb{R}^N \setminus F)$ follows from the continuity of the function $\sigma \mapsto \phi(\sigma, \mathbf{x}^*)$ at $\sigma = 0$ and from the fact that $\phi(0, \mathbf{x}^*) = \mathbf{x}^* \in \mathbb{R}^N \setminus F$, which is an open set. STEP 2. From STEP1 and the continuity of $\xi \mapsto V\phi(s, \xi)$ at $\xi = \mathbf{x}^*$ we deduce that $V\phi(s, \mathbf{y}) < V(\mathbf{x}^*)$ for all $\mathbf{y}$ in a sufficiently small neighborhood of $\mathbf{x}^*$; the statement follows now from (13) and $\phi(s, \phi(t_k, \mathbf{x_0})) = \phi(s+t_k, \mathbf{x_0})$ (§ 2.3, property (ϕ2) of a flow). STEP 3. The function $\xi \mapsto \phi(s, \xi)$ is continuous at $\xi = \mathbf{x}^*$, and $\phi(s, \mathbf{x}^*) \in \mathbb{R}^N \setminus F$ which is open; hence, $\phi(s, \mathbf{y}) \in \mathbb{R}^N \setminus F$ for all $\mathbf{y}$ in a sufficiently small neighborhood of $\mathbf{x}^*$. The statement follows from (13) and $\phi(s, \phi(t_k, \mathbf{x_0})) = \phi(s + t_k, \mathbf{x_0})$ (§ 2.3, property (ϕ2) of a flow). STEP 4. Pick $k_0 > \max\{k_1, k_2\}$; since $s > 0$ can be taken such that $s < \theta - t_{k_0}$, and $\{t_k\}$ is strictly increasing and tends to θ, there exists a positive integer m such that $s + t_{k_0} < t_m < \theta$. Due to (12) and (19), both t_m and $s + t_{k_0}$ are in $\phi(\cdot, \mathbf{x_0})^{-1}(\mathbb{R}^N \setminus F)$, set on which $V\phi(\cdot, \mathbf{x_0})$ is strictly decreasing (hypothesis (1)). □

Acknowledgment. The author wishes to thank the referee for the constructive critique of the first draft.

References

[1] S. K. Berberian, *Fundamentals of Real Analysis*, Universitext, Springer–Verlag, 1999.

[2] A. M. Bruckner, *Differentiation of Real Functions*, Lecture Notes in Mathematics 659, Springer–Verlag, 1978.

[3] J. L. Corne, and N. Rouche, *Attractivity of Closed Sets Proved by Using a Family of Liapunov Functions*, J. Differential Equations, **13** (1973), 231–246.

[4] V. Ene, *Monotonicity and local systems*, Real Anal. Exchange, **17** (1991/92), 291–313.

[5] M. Ghil, and S. Childress, *Topics in Geophysical Fluid Dynamics: Atmospheric Dynamics, Dynamo Theory, and Climate Dynamics*, Springer, New York, 1987.

[6] Ph. Hartman, *Ordinary Differential Equations*, John Wiley & Sons, New York, 1964.

[7] M. W. Hirsch, and S. Smale, *Differential equations, Dynamical Systems, and Linear Algebra*, Academic Press, 1974.

[8] J. P. LaSalle, *Stability Theory for Ordinary Differential Equations*, J. Differential Equations, **4** (1968), 57–65.

[9] E. N. Lorenz, *Deterministic Nonperiodic Flow*, J. Atmospheric Sci. **20** (1963), 130–141.

[10] R. Ortega and L. A. Sanchez, *Abstract Competitive Systems and Orbital Stability in* $\mathbb{R}^3$, Proc. Amer. Math. Soc., **128** (2000), 2911–2919.

[11] C. Sparrow, *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*, Springer, New York, 1982.

[12] B. S. Thomson, *Real Functions*, Lecture Notes in Mathematics 1170, Springer, Berlin, 1985.

[13] B. S. Thomson, J. B. Bruckner, and A. M. Bruckner, *Elementary Real Analysis*, Second Edition, www.classicalrealanalysis.com, 2008.