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ON THE GENERALIZED CONVERGENCE THEOREMS FOR THOMSON'S $\mathcal{B}$-INTEGRAL ON $\mathbb{R}^m$

1 Introduction

Thomson [T] defined the $\mathcal{B}$-integral using the derivation basis $\mathcal{B}$ and gave a pointwise convergence theorem (or the so-called equiintegrability theorem). The $\mathcal{B}$-integral is a generalization of the Henstock integral. Chew and Lee [CL] gave a controlled convergence theorem for the $\mathcal{B}$-integral involving $UACG^{**}$ which is an extension of the term $UACG^{**}_{\mathcal{B}}$ for the Henstock integral (cf. [L_2]). Kurzweil and Jarník [KJ] introduced an axiomatic concept of the Z-integral, which is a generalization of the Henstock integral on multi-dimensional Euclidean space, and proved the equivalence of the equiintegrability theorem and the controlled convergence theorem using $UZ - ACG^{\nabla}$. Lu and Lee [LL] characterized the Henstock integral by the $GSRS$ property and established a convergence theorem for the Henstock integral using $UGSRS$. In this paper, we extend the $\mathcal{B}$-integral on $\mathbb{R}$ to one on $\mathbb{R}^m$. After proving a weaker version of the equiintegrability theorem, the equivalence of five generalized convergence theorems will be established, which include the equiintegrability theorem, two versions of the generalized controlled convergence theorem which are based on $UACG^{\nabla}$ and $UACG^{**}_{\mathcal{B}}$ respectively, the generalized variational convergence theorem and the uniformly $MGSRS_{\mathcal{B}}$ (modified $GSRS$ with respect to $\mathcal{B}$) convergence theorem.

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2 Preliminaries

We will assume the reader is familiar with the Henstock integral and the AP-integral. The terminology used in this paper follows mainly Thomson's papers [T].

We denote by $\mathbb{R}$ the set of all real numbers. Let m be a fixed positive integer and let $\mathbb{R}^m$ denote m-dimensional Euclidean space. Assume that a norm of $\mathbb{R}^m$ has been defined, for example, $\|x\| = \max\{|\xi_i| : i = 1, 2, \ldots, m\}$ where $x = (\xi_1, \xi_2, \ldots, \xi_m) \in \mathbb{R}^m$. The open sphere with center x and radius r is $S(x, r) = \{y : \|x - y\| < r\}$. An interval I is a nondegenerate compact rectangle in $\mathbb{R}^m$; that is, the set $I = \prod_{i=1}^m [a_i, b_i]$ where $a_i, b_i \in \mathbb{R}$ and $a_i < b_i$, $i = 1, 2, \ldots, m$.

Let $I_0 \subset \mathbb{R}^m$ be a fixed closed interval. Let Ψ be the class of all subintervals of I_0. An element $(I, x) \in \Psi \times I$ is called an interval-point pair. The point x is called the associated point of the interval I. A derivation basis $\mathcal{B}$ of I_0 is a nonempty collection of subset β of $\Psi \times I_0$. Let $\beta \in \mathcal{B}$ and $E \subset I_0$. We write

$$\begin{array}{ll} \beta[E] = \{(I, x) \in \beta; x \in E\}, & \beta(E) = \{(I, x) \in \beta; I \subset E\}, \\ \mathcal{B}[E] = \{\beta[E]; \beta \in \mathcal{B}\}, & \mathcal{B}(E) = \{\beta(E); \beta \in \mathcal{B}\}. \end{array}$$

Obviously, $\beta[I_0] = \beta(I_0) = \beta$ and $\mathcal{B}[I_0] = \mathcal{B}(I_0) = \mathcal{B}$.

Definition 2.1 *Let $\mathcal{B}$ be a derivation basis, $\beta \in \mathcal{B}$ and $E \subset I_0$. Let $P = \{(I, x)\} \subset \beta$.*

(i) *P is said to be a partial β-partition of I_0, denoted by $P \in \mathcal{P}'(\beta)$, if $\{I : (I, x) \in P\}$ is a finite set of nonoverlapping subintervals of I_0.*

(ii) *P is said to be a E-tagged β-partition of I_0, denoted by $P \in \mathcal{P}'(\beta[E])$, if $P \in \mathcal{P}'(\beta)$ and $x \in E$ provided $(I, x) \in P$.*

(iii) *P is said to be a β-partition of I_0, denoted by $P \in \mathcal{P}(\beta)$, if $P \in \mathcal{P}'(\beta)$ and $\bigcup_{(I,x)\in P} I = I_0$.*

(iv) *Let $P', P'' \in \mathcal{P}'(\beta[E])$. Then P'' is said to be finer than P', denoted by $P'' \leq P'$, if for each $(I, x) \in P''$, there is $(J, y) \in P'$ such that $I \subset J$.*

If $P \in \mathcal{P}'(\beta[E]), E' \subset E$, then we write $P[E'] = \{(I, x) \in P; x \in E'\}$.

Throughout this paper, we always assume that the derivation basis $\mathcal{B}$ satisfies the following axioms:

Axiom 2.1

(1) *$\mathcal{B}$ ignores no point, i.e. for any $x \in I_0$ and any $\beta \in \mathcal{B}$, $\beta[\{x\}] \neq \emptyset$.*

(2) *$\mathcal{B}$ has partitioning property, i.e. for every $\beta \in \mathcal{B}$ and every $I \in \Psi$ there is $P \in \mathcal{P}(\beta(I))$.*

(3) *$\mathcal{B}$ is filtering down, i.e. for every β', $\beta'' \in \mathcal{B}$, there is $\beta \in \mathcal{B}$ such that $\beta \subset \beta' \cap \beta''$.*

(4) *$\mathcal{B}$ has δ-fine property, i.e. for every $\delta : I_0 \to (0,1)$, there exists $\beta \in \mathcal{B}$ which is δ-fine, that is, $I \subset S(x, \delta(x))$ provided $(I,x) \in \beta$.*

(5) *$\mathcal{B}$ has σ-local property, i.e. for any sequence of pairwise disjoint sets $\{X_n\}$ and any sequence $\{\beta_n\} \subset \mathcal{B}$ there is $\beta \in \mathcal{B}$ for which $\beta[X_n] \subset \beta_n$ for all n.*

Example 2.1

(1) *All of the basis Z_P, Z_Q, Z_R, Z_S in [K, Example 2.2], or Δ_i, Δ_i^* $(i = 1,2)$ in [O, Section 2.2] satisfy Axiom 2.2.*

(2) *On the real line, most of the derivation basis in [T] satisfy Axiom 2.1.*

The functions $f : I_0 \to \mathbb{R}$, $F : \Psi \to \mathbb{R}$ and $h : \Psi \times I_0 \to \mathbb{R}$ are called respectively a point function, a interval function, and a interval-point function. An interval function F is said to be additive if $F(I \cup J) = F(I) + F(J)$ for any pair of nonoverlapping intervals I and J with $I \cup J$ being an interval. In this paper, all point functions involved are always assumed to be measurable, all interval functions are additive, and all point sets involved are always assumed to be measurable and we denote the measure of a set $E \subset \mathbb{R}^m$ by $|E|$. In what follows we consider the product $f(x)|I|$ and interval function F as special cases of interval-point functions by agreeing that $f(I,x) = f(x)|I|$ and $F(I,x) = F(I)$.

Let $\beta \in \mathcal{B}, h : \Psi \times I_0 \to \mathbb{R}$ and $P = \{(I,x)\}$ a β-partial partition of I_0 be given. We write

$$\sigma(h,P) = (P)\sum h(I,x),$$
$$\sigma(|h|,P) = (P)\sum |h(I,x)|,$$

where $(P)\sum$ denotes the sum over P. If we set $\cup_{(I,x)\in P} I = U(P)$, then we may write

$$\sigma(P) = |U(P)|,$$
$$\sigma(P' \setminus P'') = |U(P') - \setminus U(P'')|,$$
$$\sigma(P' \nabla P'') = |U(P') \nabla U(P'')|,$$

where the symbol "∇" denotes the symmetric difference of two sets. Furthermore, let $\beta \in \mathcal{B}$ and $E \subset I_0$. The variation of h over $\beta[E]$ is

$$V(h, \beta[E]) = \sup\{\sigma(|h|, P) : P \in \mathcal{P}'(\beta[E])\}$$

and the variation of h over $\mathcal{B}[E]$ is

$$V(h, \mathcal{B}[E]) = \inf\{V(h, \beta[E]); \beta \in \mathcal{B}\}.$$

We can easily prove the following lemma.

Lemma 2.1 *Let h and h' be interval-point functions and let $\beta, \beta' \in \mathcal{B}$. Then*

(1) $0 \le V(h, \beta) \le +\infty$,

(2) *if* $\beta \subset \beta'$, *then* $V(h, \beta) \le V(h, \beta')$,

(3) *for any real number* $c \neq 0$, $V(ch, \beta) = |c| V(h, \beta)$,

(4) *if* $|h| \le |h'|$, *then* $V(h, \beta) \le V(h', \beta)$,

(5) $V(h + h', \beta) \le V(h, \beta) + V(h', \beta)$,

(6) *for any sequence of set* $E, E_1, E_2, \ldots$ *with* $E \subset \bigcup_{i=1}^{\infty} E_i$,

$$V(h, \beta[E]) \le \sum_{i=1}^{\infty} V(h, \beta[E_i]).$$

Definition 2.2 *A function $f : I_0 \to \mathbb{R}$ is said to be $\mathcal{B}$-integrable on I_0, if there exists a real number A such that for every $\varepsilon > 0$ there exists $\beta \in \mathcal{B}$, such that $|\sigma(f, P) - A| < \varepsilon$ whenever $P \in \mathcal{P}(\beta)$. In this case, we write $A = (\mathcal{B}) \int_{I_0} f$.*

Remark 2.1 *Since $\mathcal{B}$ is filtering down, we can easily check that such a number A is unique.*

Lemma 2.2 (The fundamental lemma of the $\mathcal{B}$-integral) *Let $f : I_0 \to \mathbb{R}$. Then the following are equivalent.*

(a) *f is $\mathcal{B}$-integrable on I_0.*

(b) *There exists an additive interval function $F \colon \Psi \to \mathbb{R}$ such that for every $\varepsilon > 0$ there exists $\beta \in \mathcal{B}$ such that $V(f - F, \beta) < \varepsilon$ in which case, $F(I) = (\mathcal{B}) \int_I f$ for every $I \in \Psi$ and we called F a primitive of f.*

PROOF. The proof is similar to that for the $\mathcal{B}$-integral in [T, Chapter III, Lemma 4.4, p. 152]. □

Lemma 2.3 (Cauchy criterion) *A function $f : I_0 \to \mathbb{R}$ is $\mathcal{B}$-integrable on I_0 iff for every $\varepsilon > 0$ there exists $\beta \in \mathcal{B}$ such that $|\sigma(f, P') - \sigma(f, P'')| < \varepsilon$ whenever $P', P'' \in \mathcal{P}(\beta)$.*

The proof is elementary.

3 Convergence Theorem.

Definition 3.1 *A sequence of measurable functions $f_n : I_0 \to \mathbb{R}$, $n = 1, 2, \ldots,$ is said to be $\mathcal{B}$-equiintegrable on I_0, if there exists a sequence of additive interval functions $\{F_n\}$, such that for every $\varepsilon > 0$ there exists $\beta \in \mathcal{B}$ such that $V(f_n - F_n, \beta) < \varepsilon$ for all n, or more precise, $|\sigma(f_n - F_n, P)| < \varepsilon$ for all n whenever $P \in \mathcal{P}(\beta)$.*

Definition 3.2 *A sequence of additive interval functions $\{F_n\}$ is said to satisfy the uniformly $\mathcal{B}$-strong Lusin condition in I_0, $\{F_n\} \in USL_{\mathcal{B}}$, if for any $Z \subset I_0$ of measure zero and for every $\varepsilon > 0$ there exists $\beta \in \mathcal{B}$ such that $V(F_n, \beta[Z]) < \varepsilon$ for all n*

If we only consider a function F in the above definition, then we say that F satisfies the $\mathcal{B}$-strong Lusin condition, denoted by $F \in SL_{\mathcal{B}}$.

We remark that the $SL_{\mathcal{B}}$ condition is a modification of the Strong Lusin condition (SL), which appears in [L_4], [G_1] and [LV]. Another variant of (SL) is in [KJ], where the authors use the term "well behaved on sets of measure zero".

Lemma 3.1 *Let $\{f_n\}$ be a sequence of point functions which is pointwise bounded on I_0. Then for any $Z \subset I_0$ of measure zero and for any $\varepsilon > 0$ there is $\beta \in \mathcal{B}$ such that $V(f_n, \beta[Z]) < \varepsilon$ for all n.*

PROOF. Suppose that $Z \subset I_0$ is of measure zero. For each positive integer i set

$$Z_i = \{x \in Z;\ i - 1 \leq \sup_n\{|f_n(x)|) < i\}.$$

For every $\varepsilon > 0$, choose an open set O_i so that $Z_i \subset O_i$ and $|O_i| < \varepsilon 2^{-i} i^{-1}$. Take $\delta : I_0 \to (0, 1)$ such that $S(x, \delta(x)) \subset O_i$, when $x \in Z_i$ for each i. Since $\mathcal{B}$ is δ-fine, we can choose $\beta \in \mathcal{B}$ which is δ-fine, and then for all n

$$V(f_n, \beta[Z]) \leq \sum_i V(f_n, \beta[Z_i]) \leq \sum_1 i|O_i| < \varepsilon.$$

□

We remark that the proofs of Lemmas 3.1 and 3.4 below use some techniques employed by Gordon in [G_2].

Lemma 3.2 *Let $\{f_n\}$ be pointwise bounded on I_0. If $\{f_n\}$ is $\mathcal{B}$-equiintegrable on I_0 with primitives $\{F_n\}$, then $\{F_n\} \in USL_{\mathcal{B}}$.*

PROOF. Suppose that $Z \subset I_0$ is of measure zero. For any $\varepsilon > 0$, by Lemma 3.1, there exists $\beta' \in \mathcal{B}$ such that $V(f_n, \beta'[Z]) < \varepsilon/2$ for all n. By the assumption, there is $\beta'' \in \mathcal{B}$ such that $V(f_n - F_n, \beta'') < \varepsilon/2$ for all n. Since $\mathcal{B}$ is filtering down, there is $\beta \in \mathcal{B}$ such that $\beta \subset \beta' \cap \beta'$. So we have

$$\begin{aligned} V(F_n, \beta[Z]) &\le V(f_n - F_n, \beta[Z]) + V(f_n, \beta[Z]) \\ &\le V(f_n - F_n, \beta'') + V(f_n, \beta'[Z]) < \varepsilon \end{aligned}$$

for all n, i.e., $\{F_n\} \in USL_{\mathcal{B}}$. □

Theorem 3.1 (Weak equiintegrability theorem) *Let*

(1) *$f_n(x) \to f(x)$ a.e. on I_0,*

(2) *$\{f_n\}$ is pointwise bounded on I_0,*

(3) *$\{f_n\}$ is $\mathcal{B}$-equiintegrable on I_0.*

Then f is $\mathcal{B}$-integrable on I_0 and $(\mathcal{B})\int_{I_0} f = \lim_{n\to\infty}(\mathcal{B})\int_{I_0} f_n$.

PROOF. Let Z have measure zero such that $\{f_n\}$ converges to f everywhere on $I_0 \setminus Z$. Set

$$g_n(x) = \begin{cases} f_n(x) & \text{for } x \in I_0 \setminus Z \\ 0 & \text{otherwise} \end{cases} \qquad \text{and } g(x) = \begin{cases} f(x) & \text{for } x \in I_0 \setminus Z \\ 0 & \text{otherwise.} \end{cases}$$

For any $\varepsilon > 0$, since (2) holds, by Lemma 3.1, there exists $\beta' \in \mathcal{B}$ such that $V(f_n, \beta'[Z]) < \varepsilon$ for all n and that $V(f, \beta'[Z]) < \varepsilon$. Since (3) holds, there exists $\beta'' \in \mathcal{B}$ such that $V(f_n - F_n, \beta'') < \varepsilon$ for all n, where F_n is the primitive of f_n. Since $\mathcal{B}$ is filtering down, there is $\beta \in \mathcal{B}$ such that $\beta \subset \beta' \cap \beta''$. Now taking $P', P'' \in \mathcal{P}(\beta)$, we have

$$\begin{aligned} |\sigma(f_n, P') - \sigma(f_n, P'')| &\le |\sigma(f_n - F_n, P')| + |\sigma(f_n - F_n, P'')| \\ &\le 2V(f_n - F_n, \beta) \le 2V(f_n - F_n, \beta'') < 2\varepsilon \end{aligned}$$

for all n. Hence we have

$$\begin{aligned} |\sigma(g_n, P') - \sigma(g_n, P'')| \le& |\sigma(f_n, P') - \sigma(f_n, P'') \\ &+ |\sigma(f_n, P'[Z])| + |\sigma(f_n, P''[Z])| \\ \le& 2\varepsilon + 2V(f_n, \beta'[Z]) < 4\varepsilon \end{aligned}$$

for all n. Letting $n \to \infty$, we obtain $|\sigma(g, P') - \sigma(g, P'')| \leq 4\varepsilon$. By Lemma 2.3, we have that g is $\mathcal{B}$-integrable on I_0, and so is f. Since $\mathcal{B}$ is filtering down, we may assume that for the same β, $V(f - F, \beta) < \varepsilon$ where F denotes the primitive of f. Since the number of x in $P[I_0 \setminus Z]$ is finite, we can find a positive integer N such that $|f_n(x) - f(x)| < \varepsilon/|I_0|$ for all $(I, x) \in P[I_0 \setminus Z]$ whenever $n \geq N$. Hence

$$\begin{aligned} |F_n(I_0) - F(I_0)| \leq & |\sigma(f_n - F_n, P)| + |\sigma(f - F, P)| \\ & + \sigma(|f_n - f|, P[I_0 \setminus Z]) + \sigma(|f_n|, P[Z]) + \sigma(|f|, P[Z]) \\ \leq & V(f_n - F_n, \beta) + V(f - F, \beta) + \sigma(P[I_0 \setminus Z]) \cdot \varepsilon/|I_0| \\ & + V(f_n, \beta[Z]) + V(f, \beta[Z]) \leq 5\varepsilon \end{aligned}$$

for $n \geq N$. That is, $F_n(I_0) \to F(I_0)$ as $n \to \infty$. □

We remark that in Theorem 3.1, if we replace the condition (1) by

$$f_n(x) \to f(x) \text{ everywhere on } I_0, \tag{1'}$$

then condition (2) can be omitted and the conclusion still holds, because we can prove that (1′) implies (2).

Definition 3.3

(1) *Let $X \subset I_0$. A sequence of additive interval functions $\{F_n\}$ is said to be uniformly $AC^{\nabla}_{\mathcal{B}}(X)$, $\{F_n\} \in UAC^{\nabla}_{\mathcal{B}}(X)$, if for every $\varepsilon > 0$ there exists $\beta \in \mathcal{B}$ and $\eta > 0$ such that $|\sigma(F_n, P' - P'')| < \varepsilon$ whenever $P', P'' \in \mathcal{P}'(\beta[X])$ with $|U(P')\nabla U(P'')| < \eta$, where $\sigma(F_n, P' - P'') = \sigma(F_n, P') - \sigma(F_n, P'')$.*

(2) *$\{F_n\}$ is said to be $UACG^{\nabla}_{\mathcal{B}}$ on I_0, $\{F_n\} \in UACG^{\nabla}_{\mathcal{B}}$, if there exists a sequence of measurable sets $X_k \subset I_0$ such that $\cup_{k=1}^{\infty} X_k = I_0$ and $\{F_n\} \in UAC^{\nabla}_{\mathcal{B}}(X_k)$ for each k.*

If we consider $P' \leq P''$ (or $P'' = \emptyset$) in Definition 3.3 (1), then we say that $\{F_n\}$ is $UACC^{**}_{\mathcal{B}}(X)$ (or $UAC^{*}_{\mathcal{B}}(X)$). Analogously, we can also define $UAG^{**}_{\mathcal{B}}$ and $UACG^{*}_{\mathcal{B}}$ respectively.

Definition 3.4

(1) *A sequence of additive interval functions $\{F_n\}$ is said to be $\mathcal{B}$-variational convergent on $X \subset I_0$, $\{F_n\} \in VC_{\mathcal{B}}(X)$, if for every $\varepsilon > 0$ there exists $\beta \in \mathcal{B}$ and a positive integer N such that $V(F_\ell - F_n, \beta) < \varepsilon$ for all $\ell, n \geq N$.*

(2) $\{F_n\}$ *is said to be generalized* $\mathcal{B}$*-variational convergent on* I_0, $\{F_n\} \in GVC_{\mathcal{B}}$, *if there exists a sequence of measurable sets* $X_k \subset I_0$ *such that* $\cup_{k=1}^{\infty} X_k = I_0$ *and* $\{F_n\} \in VC(X_k)$ *for each* k.

Definition 3.5 *A sequence of measurable functions* $\{f_n\}$ *is said to have uniformly modified* $GSRS_{\mathcal{B}}$ *property on* I_0, $\{f_n\} \in UMGSRS_{\mathcal{B}}$, *if for every* $\varepsilon > 0$ *there exists a measurable set* $E \subset I_0$ *and* $\beta \in \mathcal{B}$ *such that* $\{f_n\}$ *is uniformly bounded on* E *and* $|\sigma(f_n, P[I_0 \setminus E])| < \varepsilon$ *for all* n *whenever* $P \in \mathcal{P}(\beta)$.

If we only consider one function f *instead of* $\{f_n\}$, *then* f *is said to have* $MGSRS_{\mathcal{B}}$ *property on* I_0.

We remark that the concepts of $GSRS$ (globally small Riemann sum, see [L$_1$]) and $FSRS$ (functional small Riemann sums, see [LL]) were first defined by S. P. Lu in an attempt to characterize the Henstock integral and establish a convergence theorem. Here $MGSRS_{\mathcal{B}}$ is an extension of $GSRS$ and $FSRS$.

Lemma 3.3 *Let* $\{f_n\}$ *be a sequence of measurable functions. If*

(1) $\{f_n\}$ *converges a.e. on* I_0 *and*

(2) $\{f_n\}$ *is bounded uniformly on* I_0,

then $\{f_n\}$ *is McShane equiintegrable on* I_0 *([LY], Theorem 1). In other words, for any* $\varepsilon > 0$ *there exists* $\delta : I_0 \to (0,1)$ *such that for every* δ*-fine McShane partition* $P = \{(I,x)\}$ *of* I_0, *we have* $|\sigma(f_n - F_n, P)| < \varepsilon$ *for all* n, *where* F_n *is the primitive of* f_n. *Furthermore,* $\{f_n\}$ *is also* $\mathcal{B}$*-equiintegrable on* I_0.

Recall that a partition $P = \{(I,x)\}$ is said to be a δ-fine McShane partition if I_0 is the union of intervals I, $I \subset S(x, \delta(x))$ and x may not belong to I.
PROOF. Let

$$f(x) = \begin{cases} \lim_{n\to\infty} f_n(x) & \text{when it exists at } x \in I_0 \\ 0 & \text{otherwise} \end{cases}$$

and let $|f_n(x)| \leq K$ for all $x \in I_0$ and all n, where K is a certain positive constant. Then f is also measurable and bounded by K on I_0. Hence f_n and f are all McShane integrable. By the Lebesgue Dominated Convergence Theorem we have $F_n(I_0 \to F(I_0)$ as $n \to \infty$, where F_n and F are the primitives of f_n and f respectively. It follows that for any $\varepsilon > 0$ there exists $\delta : I_0 \to (0,1)$ and a positive integer N such that $|\sigma(f - F, P)| < \varepsilon$ whenever P is a δ-fine McShane partition of I_0 and $|F_n(I_0) - F(I_0)| < \varepsilon$ for all $n \geq N$. By Egoroff's theorem we can choose an open set $O \subset I_0$ with $|O| < \varepsilon/K$ and a positive integer N' such that $|f_n(x) - f(x)| < \varepsilon/|I_0|$ for all $n \geq N'$ and all $x \in I_0 \setminus O$.

Diminish δ if necessary such that $S(x,\delta(x)) \subset O$ for $x \in O$. For any δ-fine McShane partition $P = \{(I,x)\}$ of I_0, we have

$$\begin{aligned}|\sigma(f_n - F_n, P)| \leq &\sigma(|f_n|, P[O]) + \sigma(|f|, P[O]) + \sigma(|f_n - f|, P[I_0 \setminus O]) \\ &+ |\sigma(f - F, P)| + |F(I_0) - F_n(I_0)| \\ < &K\sigma(P[O]) + K\sigma(P[O]) + \sigma(P[I_0 \setminus O])\varepsilon/|I_0| + \varepsilon + \varepsilon < 5\varepsilon\end{aligned}$$

for all $n \geq \max(N, N')$. Since the numbers $n < \max(N, N')$ are finite, we can diminish δ again so that $|\sigma(f_n - F_n, P)| < 5\varepsilon$ for all $n < \max(N, N')$ whenever P is a δ-find McShane partition. Hence the first conclusion holds.

Since $\mathcal{B}$ has δ-fine property, the second conclusion follows. □

Lemma 3.4 *Let $\{f_n\}$ be a pointwise bounded sequence of $\mathcal{B}$-integrable functions on I_0, and suppose that F_n is the primitive of f_n, $n = 1, 2, \ldots$. If $\{F_n\} \in GVC_{\mathcal{B}}$, then $\{F_n\} \in USL_{\mathcal{B}}$.*

PROOF. Suppose that $Z \subset I_0$ is of measure zero and let $\varepsilon > 0$. Put $Z = \cup_i Z_i$ where $\{Z_i\}$ are pairwise disjoint and $\{F_n\} \in VC_{\mathcal{B}}(Z_i)$ for each i. Fix i and let $\varepsilon_i = \varepsilon 2^{-i}3^{-1}$. By the definition of $\{F_n\} \in VC_{\mathcal{B}}(Z_i)$ and Lemma 3.1, there exists $\beta_i' \in \mathcal{B}$ and a positive integer $N(i)$ such that $V(F_\ell - F_n, \beta_i'[Z_i]) < \varepsilon$ for all $\ell, n > N(i)$ and $V(f_n, \beta_i'[Z_i]) < \varepsilon$ for all n. Since f_n $(n = 1, 2, \ldots, N(i))$ are $\mathcal{B}$-integrable to F_n on I_0, there exists $\beta_i'' \in \mathcal{B}$ so that $V(f_n - F_n, \beta_i'') < \varepsilon_i$ for $n = 1, 2, \ldots, N(i)$. By the fact that $\mathcal{B}$ is filtering down, there exists $\beta \in \mathcal{B}$ such that $\beta[Z_i] \subset \beta_i' \cap \beta_i''$ for all i. For $n = 1, 2, \ldots, N(i)$, we have

$$\begin{aligned}V(F_n, \beta[Z_i] &\leq V(F_n - f_n, \beta[Z_i]) + V(f_n, \beta[Z_i]) \\ &\leq V(F_n - f_n, \beta_i'') + V(f_n, \beta_i'[Z_i]) < 2\varepsilon_i\end{aligned}$$

and for $n > N(i)$, we have $V(F_n, \beta[Z_i]) \leq V(F_n - F_{N(i)}, \beta[Z_i]) + V(F_{N(i)}, \beta[Z_i]) < 3\varepsilon_i$. Hence, $V(F_n, \beta[Z]) \leq \sum_i V(F_n, \beta[Z_i]) < \sum_i 3\varepsilon_i < \varepsilon$ for all n, and we obtain $\{F_n\} \in USL_{\mathcal{B}}$. □

Lemma 3.5 (Lu's lemma of [LL]) *If f is $\mathcal{B}$-integrable on I_0 then there is a sequence of measurable sets $\{X_k\}$ with $X_k \subset X_{k+1}$ for k, $I_0 = \bigcup_k X_k$, such that f is bounded on each X_k and $(\mathcal{L})\int_{X_k} f = (\mathcal{B})\int_{I_0} f$ for all k, where $(\mathcal{L})\int$ denotes the Lebesgue integral.*

PROOF. This follows from the proof of Lemma 2 of [LL] if the Henstock integral is replaced by the $\mathcal{B}$-integral. □

The following theorem is the main result in this section.

Theorem 3.2 *Let f_n be $\mathcal{B}$-integrable on I_0 with the primitive $F_n, n=1, 2, \ldots,$ let $\{f_n\}$ be pointwise bounded on I_0 and suppose that $\{f_n\}$ converges a.e. on I_0. Then the following conditions are equivalent.*

I: $\{f_n\}$ *is* $\mathcal{B}$*-equiintegrable on* I_0.

II: $\{F_n\} \in UACG_{\mathcal{B}}^{\nabla}$.

III: $\{F_n\} \in UACG_{\mathcal{B}}^{**}$.

IV: $\{F_n\} \in GVC_{\mathcal{B}}$.

V: $\{f_n\} \in UMGSRS_{\mathcal{B}}$.

And consequently, any one of I, II, III, IV, V implies that the limit function f *of* $\{f_n\}$ *is* $\mathcal{B}$*-integrable on* I_0 *and that* $(\mathcal{B})\int_{I_0} f = \lim_{n\to\infty}(\mathcal{B})\int_{I_0} f_n$.

PROOF. I implies II: For all positive integer i let

$$X_i = \{x \in I_0; \{f_n(x)\} \text{ converges and } |f_n(x)| \le i \text{ for all } n\}$$

and let $Z = I_0 \setminus \cup_i X_i$. Then $|Z| = 0$. Since $\{F_n\} \in USL_{\mathcal{B}}$ by Lemma 3.2, we get $\{F_n\} \in UAC_{\mathcal{B}}^{\nabla}(Z)$. It remains to show that $\{F_n\} \in UAC_{\mathcal{B}}^{\nabla}(X_i)$ for each i.

Fix i and write, for convenience, $X = X_i$. Put

$$f_{X,n}(x) = \begin{cases} f_n(x) & \text{for } x \in X, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{f_{X,n}\}$ is bounded uniformly on I_0. Let $\varepsilon > 0$. Since $\{f_n\}$ is $\mathcal{B}$-equiintegrable on I_0 and so is $\{f_{X,n}\}$, by Lemma 3.3, there exists $\beta \in \mathcal{B}$ such that $V(F_n - f_n, \beta) < \varepsilon$ and $V(F_{X,n} - f_{X,n}, \beta) < \varepsilon$, where $F_{X,n}$ denotes the primitive of $f_{X,n}$. It follows from $f_n(x) = f_{X,n}(x)$ when $x \in X$ that $\sigma(F_n, P' \setminus P'') | \le 4\varepsilon + |\sigma(F_{X,n}, P' \setminus P'')|$, whenever $P', P'' \in \mathcal{P}'(\beta[X])$. On the other hand, since $\{f_{X,n}\}$ is also McShane equiintegrable on I_0, so $\{F_{X,n}\}$ is uniform absolutely continuous, UAC on I_0 ([LY]), Theorem 3), there is $\eta > 0$ such that whenever $\sigma(P' \nabla P'') < \eta$, we have

$$|\sigma(F_{X,n}, P' \setminus P'')| \le \sum_{I \in U(P') \setminus U(P'')} |F_{X,n}(I)| + \sum_{I \in U(P'') \setminus U(P')} |F_{X,n}(I)| < \varepsilon.$$

The last two estimates give $\{F_n\} \in UAC_{\mathcal{B}}^{\nabla}(X)$. It follows that $\{F_n\} \in UACG_{\mathcal{B}}^{\nabla}$.

II implies III is direct.

III implies IV: Clearly, we have

$$|I_0 \setminus \bigcup_{k=1}^{\infty} \{x \in I_0 : |f_n(x)| \le k \text{ for all } n\}| = 0.$$

Hence, for each positive integer i there exists $E_i \subset I_0$ and some positive integer k_i such that $|f_n(x)| \leq k_i$ on E_i for all n, and that $|I_0 \setminus E_i| < 1/2i$. Since each E_i is measurable, by Egoroff's Theorem, there is a closed set $H_i \subset E_i$ with $|E_i \setminus H_i| < 1/2i$, such that $\{f_n\}$ converges uniformly on H_i. It follows that $|I_0 \setminus \bigcup_i H_i| = 0$. On the other hand, by III we can choose a sequence of closed sets $\{K_j\}$ such that $\{F_n\} \in UAC^{**}_{\mathcal{B}}(K_j)$ for each j with $|I_0 \setminus \bigcup_j K_j| = 0$. For positive integers i and j let $X_{ij} = H_i \cap K_j$. Then each X_{ij} is a closed set and $|Z| = 0$ where $Z = I_0 \setminus \cup_{i,j} X_{ij}$.

By $\{F_n\} \in UACG^{**}_{\mathcal{B}}$, let $Z = \cup_k Z_k$ where $\{Z_k\}$ are pairwise disjoint and $\{F_n\} \in UAC^{**}(Z_k)$ for each k. Let $\varepsilon > 0$ and let $\varepsilon_k = \varepsilon 2^{-k-2}$. For each k there exists $\eta_k > 0$ and $\beta_k \in \mathcal{B}$ such that $|\sigma(F_n, P)| < \varepsilon_k$ for all n whenever $P \in \mathcal{P}'(\beta_k[Z_k])$ with $\sigma(P) < \eta_k$. Choose an open set O_k such that $Z_k \subset O_k$ and $|O_k| < \eta_k$. Take $\delta : I_0 \to (0, 1)$ such that $S(x, \delta(x)) \subset O_k$ when $x \in Z_k$ for each k. Since $\mathcal{B}$ has the δ-fine property and σ-local character, there exists $\beta \in \mathcal{B}$ which is δ-fine such that $\beta[Z_k] \subset \beta_k$ for each k. Suppose that $P \in \mathcal{P}'(\beta[Z])$. Since $\sigma(P[Z_k]) \leq |O_k| < \eta_k$ for each k, we have, for any positive integers ℓ, n,

$$|\sigma(F_\ell - F_n, P)| \leq |\sigma(F_\ell, P)| + |\sigma(F_n, P)| < \sum_k |\sigma(F_\ell, P[Z_k])| + \sum_k |\sigma(F_n, P([Z_k])| < 2\sum_k \varepsilon_k \leq \varepsilon/2,$$

and hence $\sigma(|F_\ell - F_n|, P) < \varepsilon$. It follows that, for all positive integers ℓ, n, $V(F_m - F_n, \beta[Z]) \leq \varepsilon$, and we obtain $\{F_n\} \in VC_{\mathcal{B}}(Z)$. It remains to show that $\{F_n\} \in VC_{\mathcal{B}}(X_{ij})$ for each i, j.

Now fix i, j and write, for convenience, $X = X_{ij}$. For given $\varepsilon > 0$, since $\{F_n\} \in UAC^{**}_{\mathcal{B}}(X)$, there exist $\beta' \in \mathcal{B}$ and $\eta > 0$, both independent of n, such that the remaining conditions for $UAC^{**}_{\mathcal{B}}(X)$ hold. Choose an open set O such that $X \subset O$ and with $|O \setminus X| < \eta$. Next, take $\delta : I_0 \to (0, 1)$ such that $S(x, \delta(x)) \subset O$ when $x \in X$ and $S(x, \delta(x)) \subset I_0 \setminus X$ otherwise. Define

$$f_{X,n}(x) = \begin{cases} f_n(x) & \text{for } x \in X \\ 0 & \text{otherwise} \end{cases}$$

for all n. It follows from Lemma 3.3 that $\{f_{X,n}\}$ is $\mathcal{B}$-equiintegrable on I_0. In other word, there exists $\beta'' \in \mathcal{B}$ such that $V(F_{X,n} - f_{X,n}, \beta'') < \varepsilon$ for all n, where $F_{X,n}$ is the primitive of $f_{X,n}$. Since $\mathcal{B}$ is filtering down, there is $\beta \in \mathcal{B}$ such that $\beta \subset \beta' \cap \beta''$. For each n, there exists $\beta_n \in \mathcal{B}$ with $\beta_n \subset \beta$ and β_n is δ-fine, such that $V(F_n - f_n, \beta_n) < \varepsilon$. Suppose that $P \in \mathcal{P}'(\beta[X])$. Take a β_n-partition of each I in P and denote the total partition by P'. Then $P' \in \mathcal{P}'(\beta_n)$ with $\sigma(P') = \sigma(P)$. Note that $P'[X]$ and $P \in \mathcal{P}'(\beta[X])$ with

$P'[X] \le P$ and that

$$\sigma(P \setminus P'[X]) = \sigma(P'[O \setminus X]) \le |O \setminus X| < \eta$$

by $P'[O \setminus X] \in \mathcal{P}(\beta(O \cap (I_0 \setminus X)))$, we have, for all n,

$$\begin{aligned}|\sigma(F_n - F_{X,n}, P)| =& |\sigma(F_n - F_{X,n}, P')| \le |\sigma(F_n - f_n, P'[X])| \\ &+ |\sigma(F_n, P'[O \setminus X])| + |\sigma(F_{X,n} - f_{X,n}, P')| \\ \le& V(F_n - f_n, \beta_n) + |\sigma(F_n, P \setminus P'[X])| \\ &+ V(F_{X,n} - f_{X,n}, \beta) \\ <& \varepsilon + \varepsilon + \varepsilon = 3\varepsilon\end{aligned}$$

By the processes described above, we have $V(F_n - F_{X,n}, \beta[X]) \le 6\varepsilon$ for all n. Since $\{f_n\}$ converges uniformly on X, there is a positive integer N such that $|f_n(x) - f_\ell(x)| < \varepsilon/|I_0|$ for all $x \in X$ whenever $\ell, n \ge N$. Hence

$$\begin{aligned}V(F_\ell - F_n, \beta[X]) <& V(F_\ell - F_{X,\ell}, \beta[X]) \\ &+ V(F_{X,\ell} - f_{X,\ell}, \beta[X]) + V(f_\ell - f_n, \beta[X]) \\ &+ V(F_{X,n} - f_{X,n}, \beta[X]) + V(F_n - F_{X,n}, \beta[X]) \\ <& 6\varepsilon + \varepsilon + \varepsilon + \varepsilon + 6\varepsilon\end{aligned}$$

for all $m, n \ge N$. Therefore $\{F_n\} \in VC_{\mathcal{B}}(X)$.

IV implies I: Let $\{F_n\} \in GVC_{\mathcal{B}}$. Then there is a sequence of measurable sets $\{E_i\}$ such that $I_0 = \bigcup_i E_i$ and that $\{F_n\} \in VC_{\mathcal{B}}(E_i)$ for each i. Use Egoroff's theorem to write $I_0 = \bigcup_j C_j \cup Z$ where each C_j is measurable, $\{f_n\}$ converges uniformly on each C_j, and $|Z| = 0$. By reducing a doubly indexed sequence to a sequence, $I_0 = \bigcup_k X_k \cup Z$ where $\{f_n\}$ converges uniformly on each X_k and $\{F_n\} \in VC_{\mathcal{B}}(X_k)$ for each k and we may assume that $\{X_k\} \cup \{Z\}$ are pairwise disjoint.

For a given $\varepsilon > 0$ and for each k, let $\varepsilon_k = \varepsilon 2^{-k}$. By the definition of X_k there is $\beta'_k \in \mathcal{B}$ and a positive integer $N(k)$ such that for all $\ell, n \ge N(k)$, we have $V(F_\ell - F_n, \beta'_k[X_k]) < \varepsilon_k$ and $|f_n(x) - f_\ell(x)| < \varepsilon/|I_0|$ for all $x \in X_k$. For $n = 1, 2, \ldots, N(k)$, there exists $\beta''_k \in \mathcal{B}$ such that $V(f_n - F_n, \beta''_k) < \varepsilon_k$. By Lemmas 3.4 and 3.1, there exists $\beta_0 \in \mathcal{B}$ such that $V(F_n, \beta_0[Z]) < \varepsilon$ and $V(f_n, \beta_0[Z]) < \varepsilon$. By the fact that $\mathcal{B}$ has σ-local character and is filtering down, we can choose $\beta \in \mathcal{B}$ so that $\beta[X_k] \subset \beta'_k \cap \beta''_k$ and $\beta[Z] \subset \beta_0$. Fix k. If $n \le N(k)$, we have $V(f_n - F_n, \beta[X_n]) \le V(f_n - F_n, \beta''_k) < \varepsilon_k$ and if $n > N(k)$, we have

$$\begin{aligned}V(f_n - F_n, \beta[X_n]) \le& V(f_n - f_{N(k)}, \beta[X_k]) \\ &+ V(f_{N(k)} - F_{N(k)}, \beta[X_k]) + V(F_{N(k)} - F_n, \beta[X_k]) \\ <& \varepsilon_k + \varepsilon_k + \varepsilon_k.\end{aligned}$$

Hence, for all n,

$$\begin{aligned} V(f_n - F_n, \beta) &\leq \sum_k V(f_n - F_n, \beta[X_k]) + V(F_n, \beta[Z]) + V(f_n, \beta[Z]) \\ &\leq \sum_k 3\varepsilon_k + \varepsilon + \varepsilon \leq 5\varepsilon. \end{aligned}$$

Therefore $\{f_n\}$ is $\mathcal{B}$-equiintegrable on I_0.

I implies V: Let f be the limit function of $\{f_n\}$. Since I holds, by Theorem 3.1, we have that f is $\mathcal{B}$-integrable on I_0. By Lu's Lemma, there exists a measurable set X such that f is bounded on X and $(\mathcal{L})\int_X f = (\mathcal{B})\int_{I_0} f$. Let Y be the subset of X on which $\{f_n\}$ converges everywhere. Thus $|X \setminus Y| = 0$ and $(\mathcal{L})\int_x f = (\mathcal{L})\int_Y f$. Put

$$f_Y(x) = \begin{cases} f(x) & \text{for } x \in Y \\ 0 & \text{otherwise} \end{cases} \qquad \text{and } f_{Y,n}(x) = \begin{cases} f_n(x) & \text{for } x \in Y \\ 0 & \text{otherwise.} \end{cases}$$

Then $\{f_{Y,n}\}$ converges to f_Y everywhere on I_0, and hence bounded uniformly on I_0. By Lemma 3.4, $\{f_{Y,n}\}$ is $\mathcal{B}$-equiintegrable on I_0. So, there is $\beta \in \mathcal{B}$ such that $|\sigma(f_n, P) - F_n(I_0)| < \varepsilon$ and $|\sigma(f_{Y,n}, P) - F_{Y,n}(I_0)| < \varepsilon$ whenever $P \in \mathcal{P}(\beta)$. Furthermore, there is a positive integer N such that $|F_n(I_0) - F(I_0)| < \varepsilon$ and $|F_{Y,n}(I_0) - F_Y(I_0)| < \varepsilon$ for all $n \geq N$, where F, $F_{Y,n}$ and F_Y denote the primitives of f, $f_{Y,n}$ and f_Y, respectively.

Let $P \in \mathcal{P}(\beta)$ and $n \geq N$. Note that

$$F(I_0) = (\mathcal{B})\int_{I_0} f = (\mathcal{L})\int_Y f = (\mathcal{L})\int_{I_0} f_Y = F_Y(I_0).$$

We have

$$\begin{aligned} |\sigma(f_n, P[I_0 \setminus Y])| =& |\sigma(f_n, P) - \sigma(f_{Y,n}, P)| \\ \leq& |\sigma(f_n, P) - F_n(I_0)| + |F_n(I_0) - F(I_0)| \\ &+ |F_Y(I_0) - F_{Y,n}(I_0)| + |F_{Y,n}(I_0) - \sigma(f_{Y,n}, P)| < 4\varepsilon. \end{aligned}$$

That is, $\{f_n\} \in UMGSRS_{\mathcal{B}}$.

V implies I: By V, for every $\varepsilon > 0$ there exist a measurable set $X \subset I_k$, a positive integer N and $\beta \in \mathcal{B}$ such that $\{f_n\}$ is bounded uniformly on X and that $|\sigma(f_n, P[I_0 \setminus X])| < \varepsilon$ for all $n \geq N$ whenever $P \in \mathcal{P}(\beta)$. Let $Y \subset X$ on which $\{f_n\}$ converges everywhere. Put

$$f_{Y,n}(x) = \begin{cases} f_n(x) & \text{for } x \in Y \\ 0 & \text{otherwise} \end{cases}$$

for $n \geq N$. Then $\{f_{Y,n}\}_{n\geq N}$ is $\mathcal{B}$-equiintegrable on I_0 by Lemma 3.3. Hence, since $\mathcal{B}$ is filtering down, we may assume for the some $\beta \in \mathcal{B}$, we have $|F_{Y,n}(I_0) - \sigma(f_{Y,n}, P)| < \varepsilon$ for all $n \geq N$ whenever $P \in \mathcal{P}(\beta)$, where $F_{Y,n}$ stands for the primitive of $f_{Y,n}$. Choose an open set O such that $X \setminus Y \subset O$ and that $|O| < \varepsilon/K$, where K is the uniform bound of $\{f_n\}_{n\geq N}$ on X. Take $\delta : I_0 \to (0,1)$ such that $S(x, \delta(x)) \subset O$ when $x \in X \setminus Y$. Since $\mathcal{B}$ has δ-fine property, we may assume β is δ-fine.

Further, for each $n \geq N$, we have $|F_n(I_0) - F_{Y,n}(I_0)| < 4\varepsilon$. Indeed, since $f_{Y,n}, f_n$ are $\mathcal{B}$-integrable on I_0, there is $\beta_n \in \mathcal{B}$ with $\beta_n \subset \beta$ such that $|\sigma(f_n, P) - F_n(I_0)| < \varepsilon$ and $|\sigma(f_{Y,n}, P) - F_{Y,n}(I_0)| < \varepsilon$ whenever $P \in \mathcal{P}(\beta_n)$, it follows that

$$\begin{aligned}|F_n(I_0) - F_{Y,n}(I_0)| \leq &|F_n(I_0) - \sigma(f_n, P)| + |\sigma(f_n, P[I_0 \setminus X])| \\ &+ |\sigma(f_n, P[X \setminus Y])| + |\sigma(f_{Y,n}, P) - F_{Y,n}(I_0)| \\ < &\varepsilon + \varepsilon + K|O| + \varepsilon = 4\varepsilon.\end{aligned}$$

Now take any $P \in \mathcal{P}(\beta)$. For each $n \geq N$, we have

$$\begin{aligned}|\sigma(f_n, P) - F_n(I_0)| \leq &|\sigma(f_n, P[I_0 \setminus X] + |\sigma(f_n, P[X \setminus Y]| \\ &+ |\sigma(f_{Y,n}, P) - F_{Y,n}(I_0)| + |F_{Y,n}(I_0) - F_n(I_0)| \\ < &\varepsilon + K|O| + \varepsilon + 4\varepsilon < 7\varepsilon.\end{aligned}$$

Furthermore, since the number of $n < N$ is finite, by the fact that $\mathcal{B}$ is filtering down, we an assume for the same β we have $|\sigma(f_n, P) - F_n(I_0)| < 7\varepsilon$ for all $n \in N$ whenever $P \in \mathcal{P}(\beta)$. □

Corollary 3.1 *Let f_n be $\mathcal{B}$-integrable on I_0 with the primitive F_n, $n=1,2,\ldots$, and suppose that $\{f_n\}$ converges to a function f a.e. on I_0. Then any one of II, III, IV, V implies that f is $\mathcal{B}$-integrable on I_0 and that*

$$(\mathcal{B}) \int_{I_0} f = \lim_{n\to\infty} (\mathcal{B}) \int_{I_0} f_n.$$

PROOF. We can redefine $\{f_n\}$ on $Z = \{x; f_n(x)$ doesn't convergent to $f(x)\}$ so that $\{f_n\}$ is pointwise bounded on Z. And the primitives $\{F_n\}$ of them are still invariant. It follows from Theorem 3.2 that the conclusion holds. □

Remark 3.1 *It is interesting to point out that we can't prove Theorem 3.2 with $UACC^{**}_{\mathcal{B}}$ replaced by $UACC^{*}_{\mathcal{B}}$. However in [L_3] we proved such a result when $\mathcal{B}$ is an approximate derivation basis on the real line (cf. [T], p.103). Note that in the last case, the Lebesgue density theorem has been used.*

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References

[CL] T. S. Chew and P. Y. Lee, *Nonabsolute integration using Vitali covers*, to appear, 1992.

[KJ] J. Kurzweil and J. Jarník, *Equiintegrability and controlled convergence of Perron-type integrable functions*, Real Analysis Exchange **17** (1991–1992), 110–139.

[G_1] R. A. Gordon, *A descriptive characterization of the generalized Riemann integral*, Real Analysis Exchange **15** (1989–90), 397–400.

[G_2] R. A. Gordon, *On the equivalence of two convergence theorems for the Henstock-integral*, Real Analysis Exchange **18** (1992–93), 261–266.

[L_1] P. Y. Lee, *Lanzhou lectures on Henstock integration*, World Scientific, 1989.

[L_2] P. Y. Lee, *Integration of highly oscillatory functions in the plane*, Proceedings of the Asian mathematics conference, 1990.

[L_3] Y. J. Lin, *On the equivalence of four convergence theorems for the AP-integral*, Real Analysis Exchange **19** (1) (1993–94), 155–164.

[L_4] P. Y. Lee, *On ACG^* functions*, Real Analysis Exchange **15** (1989–90), 754–759.

[LV] P. Y. Lee and R .O. Vyborny, *Kurzweil-Henstock integration and the strong Lusin condition*, Real Analysis Exchange **17** (1991–92), 25–26.

[LL] S. P. Lu and P. Y. Lee, *Globally small Riemann sums and the Henstock integral*, Real Analysis Exchange **16** (2) (1990–91) 537–545.

[LY] S. P. Lu and K. R. Yang, *Some convergence theorems for the Lebesgue integral*, Ann. Math. Res. **25** (2) (1992), 90–94 (in Chinese).

[O] K. M. Ostaszewski, *Henstock integral in the plane*, Memoirs of AMS, Vol. 63, No. 353, 1986.

[T] B. S. Thomson, *Derivation bases on the real line, I and II*, Real Analysis Exchange **8** (1982–83), 67–208, 278–442.