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EXTENDABILITY AND ALMOST CONTINUITY

Sometimes it's as easy to prove a stronger result ...
Kenneth R. Kellum

Abstract

Every function $f\colon [0,1] \to [-1,1]$ can be expressed as the sum of three extendable functions, as the maximum of two minimums of extendable functions and as the limit of a transfinite sequence of extendable functions.

Let us establish some terminology to be used. Let $I = [0,1]$ and let X, Y be topological spaces. A function $f\colon X \to Y$ is:

- *Darboux* if it maps connected sets onto connected sets,
- *almost continuous* if every open neighborhood of f in $X \times Y$ contains a continuous function from X into Y,
- *connectivity* if the restriction $f|C\colon C \to Y$ is a connected subset of $X \times Y$ whenever C is a connected subset of X,
- *extendable* if there is a connectivity function $F\colon X \times I \to Y$ such that $F(x,0) = f(x)$ when $x \in X$.

For $X = Y = I$ we have the following chain of proper inclusions ([15], [12]):

continuous $\subset$ extendable $\subset$ almost continuous $\subset$ connectivity $\subset$ Darboux

It is known that a function $F\colon I \times I \to \mathbb{R}$ is connectivity iff it is

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- *peripherally continuous*, i.e., if for every $x \in I^2$ and all open neighborhoods U of x and V of $F(x)$, there exists an open neighborhood W of x in U such that $F(\operatorname{bd} W) \subset V$ [6].

Let $\mathcal{K}$ denote a class of functions from I into I and let $g \in \mathcal{K}$. A set $M \subset I$ is called *g-negligible with respect to* $\mathcal{K}$ if $f \in \mathcal{K}$ whenever $f\colon I \to I$ and $f = g$ on $I \setminus M$ ([1], see also [13]). This is the same as saying that every function $f\colon I \to I$ obtained by arbitrarily redefining g on M is still a member of $\mathcal{K}$.

Many theorems on representations of real functions as sums, products, maximums and minimums, or limits of sequences of almost continuous (and therefore also of connectivity and Darboux) functions can be proved using the Kellum & Garret's method of intersecting of blocking sets ([9], see also [11]). The analogous method does not work in the case of extendable functions. However, the latest results of Rosen on negligible sets with respect to the class of extendable functions can be applied to obtain some related results. Rosen showed how to express an arbitrary real function $f\colon I \to I$ as the pointwise limit of a sequence of extendable functions and an arbitrary $g\colon I \to [-1,1]$ as the sum of an infinite series of extendable functions [13]. We shall prove that every function $f\colon [0,1] \to [-1,1]$ can be expressed as the sum of three extendable functions, as the maximum of two minimums of extendable functions and as the limit of a transfinite sequence of extendable functions.

We need several lemmas. The first of them is obvious. (See e.g. [7].)

Lemma 1 *Assume that X, Y and Z are topological spaces, $h\colon X \to Y$ is a homeomorphism and $f\colon Y \to Z$ is connectivity. Then $f \circ h$ is a connectivity function.*

Corollary 1 *If a function $f\colon I \to I$ is extendable and $h\colon I \to I$ is a homeomorphism, then $f \circ h$ is extendable.*

Lemma 2 *Assume that J is a compact interval, $g\colon I \to J$ is an extendable function, $h\colon I \to I$ is a homeomorphism, and A is a g-negligible set with respect to the class of extendable functions. Then $h^{-1}(A)$ is $(g \circ h)$-negligible.*

Proof. Assume that $g_1 = g \circ h$ and $[f_1 \neq g_1] \subset h^{-1}(A)$ for some $f_1\colon I \to J$. Put $f = f_1 \circ h^{-1}$. Then $[f \neq g] \subset A$. Indeed, if $x \notin A$, then $h^{-1}(x) \notin h^{-1}(A)$. Therefore $f_1(h^{-1}(x)) = g_1(h^{-1}(x))$, so $f(x) = g(x)$. Thus f is extendable to a connectivity function and $f_1 = f \circ h$ is extendable, too. □

Lemma 3 *Assume that $A, B \subset (0,1)$ are of the first category. Then there exists a homeomorphism $h\colon I \to I$ such that $B \cap h(A) = \emptyset$.*

PROOF. We may assume that A and B are F_σ sets and $A = \bigcup_{n=1}^{\infty} F_n$, where F_n is closed and $F_n \subset F_{n+1}$ for $n \in \mathbb{N}$. Moreover, we may assume that each $x \in F_n$ is a point of bilateral accumulation of F_{n+1}, and $|J_0| \leq |J|/2$ whenever J is a component of the set $I \setminus F_n$ and J_0 is a component of $J \setminus F_{n+1}$ ($|J|$ denotes the length of an interval J). For every $k \in \mathbb{N}$ let $(I_{k,n})_n$ be a sequence of all components of the set $I \setminus F_k$.

Since the set $I \setminus (B \cup \{0,1\})$ is a dense G_δ subset of I, there exists a Cantor set $E_1 \subset I \setminus (B \cup \{0,1\})$ such that $|J| \leq 1/2$ for each component J of the complement of $I \setminus E_1$. Let $h_1 \colon I \to I$ be an increasing homeomorphism such that

- $h_1(F_1) = E_1$ (hence $h_1(F_1) \cap B = \emptyset$),
- h_1 is linear on each $\overline{I}_{1,n}$,
- $|h_1(I_{1,n})| \leq 1/2$ for every $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$ let $J_{1,n} = h_1(I_{1,n})$ and let $h_{2,n} \colon \overline{I}_{1,n} \to \overline{J}_{1,n}$ be an increasing homeomorphism such that

- $h_{2,n}(I_{1,n} \cap F_2) \cap B = \emptyset$,
- $h_{2,n}(\operatorname{bd} I_{1,n}) = h_{1,n}(\operatorname{bd} I_{1,n})$,
- $h_{2,n}$ is linear on the closure of every component of $I_{1,n} \setminus F_2$,
- $|h_{2,n}(J)| \leq |J_{1,n}|/2$ for every component J of $I_{1,n} \setminus F_2$.

Note that $h_2 = \bigcup_n h_{2,n}$ is an increasing homeomorphism of I onto I and $E_2 = h_2(F_2)$ is disjoint from B. Assume that for $k \in \mathbb{N}$ we have defined an increasing homeomorphism $h_k \colon I \to I$ such that $E_k = h_k(F_k)$ is disjoint from B. For each $n \in \mathbb{N}$ let $J_{k,n} = h_k(I_{k,n})$ and let $h_{k+1,n} \colon \overline{I}_{k,n} \to \overline{J}_{k,n}$ be an increasing homeomorphism such that

- $h_{k+1,n}(I_{k,n} \cap F_{k+1}) \cap B = \emptyset$,
- $h_{k+1,n}(\operatorname{bd} I_{k,n}) = h_{k,n}(\operatorname{bd} I_{k,n})$,
- $h_{k+1,n}$ is linear on the closure of every component of $I_{k,n} \setminus F_{k+1}$,
- $|h_{k+1,n}(J)| \leq |J_{k,n}|/2$ for every component J of $I_{k,n} \setminus F_{k+1}$. n

Let $h_{k+1} = \bigcup_n h_{k+1,n}$ and $E_{k+1} = h_{k+1}(F_{k+1})$. Observe that the function $\widetilde{h} = \bigcup_k (h_k | F_k)$ is increasing, the set A is dense in I, and for each $x \in I$ we can define $h(x) = \lim_{t \to x} \widetilde{h}(t)$. Then h is an increasing homeomorphism from I onto I. Moreover, $h(A) \cap B = \emptyset$. □

Theorem 1 *For every function $f\colon I \to [-1,1]$ there exist three extendable functions $f_i\colon I \to [-1,1]$, $i = 0,1,2$, such that $f = f_0 + f_1 + f_2$.*[1]

PROOF. Let $F\colon I^2 \to I$ be a connectivity function such that for some $x_0 \in I$ the restriction $d = F|(I \times \{x_0\})$ is a connectivity function whose graph is dense in $I \times \{x_0\} \times I$; see [12, Example 2]. (Such functions have been constructed in fact in [2] and [4].) Obviously d is an extendable function, so there exists a dense extendable function $g\colon I \to [-1,1]$ (cf, [13]). There exists a first category set $A \subset I$ such that $I \setminus A$ is g-negligible [13, Theorem 1(iii)]. Since every nowhere dense subset of I is g-negligible [13, Theorem 1(i)], we can assume that $A \subset (0,1)$. By Lemma 3, there exist: a homeomorphism $h_1\colon I \to I$ such that $h_1(A) \cap A = \emptyset$, and a homeomorphism $h_2\colon I \to I$ such that $h_2(A) \cap (A \cup h_1(A)) = \emptyset$. By Corollary 1, the functions $g_i = g \circ h_i^{-1}$, $i = 1,2$, are extendable. Define $f_i\colon I \to [-1,1]$ for $i = 0,1,2$ by

$$f_0(x) = \begin{cases} g(x) & \text{for } x \in A, \\ -g_1(x) & \text{for } x \in h_1(A), \\ f(x) & \text{for } x \notin A \cup h_1(A), \end{cases}$$

$$f_1(x) = \begin{cases} -g(x) & \text{for } x \in A, \\ g_1(x) & \text{for } x \in h_1(A), \\ -g_2(x) & \text{for } x \in h_2(A), \\ 0 & \text{for } x \notin A \cup h_1(A) \cup h_2(A), \end{cases}$$

$$f_2(x) = \begin{cases} f(x) & \text{for } x \in A \cup h_1(A), \\ g_2(x) & \text{for } x \in h_2(A), \\ 0 & \text{for } x \notin A \cup h_1(A) \cup h_2(A). \end{cases}$$

By Lemma 2, the functions f_i, $i = 0,1,2$, are extendable. Moreover, $f = f_0 + f_1 + f_2$. □

Theorem 2 *For every function $f\colon I \to I$ there exist four extendable functions f_i, $i = 0,1,2,3$, such that $f = \max(\min(f_0, f_1), \min(f_2, f_3))$.*[2]

PROOF. Let $g\colon I \to I$ be a dense extendable function and let A be a first category set such that $I \setminus A$ is g-negligible. Let $h_1\colon I \to I$ be a homeomorphism such that $h_1(A) \cap A = \emptyset$. Let $h_2\colon I \to I$ be a homeomorphism such that $h_2(A) \cap (A \cup h_1(A)) = \emptyset$. Let $h_3\colon I \to I$ be a homeomorphism such that

[1] The analogous result for almost continuous functions is proved in [3].

[2] The analogous result on almost continuous functions is proved in [10].

$h_3(A) \cap (A \cup h_1(A) \cup h_2(A)) = \emptyset$. Moreover, let $h_0 = \mathrm{id}_I$ and let $g_i = g \circ h_i^{-1}$ for $i = 0, 1, 2, 3$. By Corollary 1, the functions g_i are extendable. Define

$$f_i(x) = \begin{cases} g_i(x) & \text{for } x \in h_i(A), \\ f(x) & \text{for } x \notin h_i(A). \end{cases}$$

By Lemma 2, all functions f_i are extendable. It is easy to verify that $f = \max(\min(f_0, f_1), \min(f_2, f_3))$. □

Recall that a function $f\colon I \to I$ is a discrete limit of a net $(f_\sigma)_{\sigma\in\Sigma}$, where $(\Sigma, \preceq)$ is a directed set, if for each $x \in I$ there exists $\sigma_0 \in \Sigma$ such that $f_\sigma(x) = f(x)$ whenever $\sigma_0 \prec \sigma$. Moreover, if $\Sigma = \omega_1$, then f is called the limit of a transfinite sequence $(f_\alpha)_{\alpha<\omega_1}$ [14].

Let $\mathrm{add}\,(\mathcal{K})$ denote the additivity of the ideal of all first category sets in I, i.e., the least cardinal κ for which there exists a family $\mathcal{F}$ of first category sets such that $\mathrm{card}\,(\mathcal{F}) = \kappa$ and the set $\bigcup \mathcal{F}$ is of the second category.

Theorem 3 *Assume that $(\Sigma, \preceq)$ is a directed set with $\mathrm{card}\,(\Sigma) \le \mathrm{add}\,(\mathcal{K})$. Then each function $f\colon I \to I$ is the discrete limit of a net $(f_\sigma)_{\sigma\in\Sigma}$ of extendable functions.*[3]

PROOF. Let $g\colon I \to I$ be a dense extendable function and let A be a first category set such that $I \setminus A$ is g-negligible. Put $h_0 = \mathrm{id}_I$. By Lemma 3, for every ordinal $\alpha < \kappa = \mathrm{card}\,(\Sigma)$ there exists a homeomorphism $h_\alpha\colon I \to I$ such that $h_\alpha(A) \cap \bigcup_{\beta<\alpha} h_\beta(A) = \emptyset$. By Corollary 1, all functions $g_\alpha = g \circ h_\alpha^{-1}$ are extendable. Let $\varphi\colon \Sigma \to \kappa$ be a bijection. By Lemma 2, the following functions are extendable:

$$f_\sigma(x) = \begin{cases} g_{\varphi(\sigma)}(x) & \text{for } x \in h_{\varphi(\sigma)}(A), \\ f(x) & \text{for } x \notin h_{\varphi(\sigma)}(A). \end{cases}$$

It is easy to verify that f is the discrete limit of $(f_\sigma)_{\sigma\in\Sigma}$. □

In particular, if $\Sigma = \omega_0$, we obtain Theorem 3 of [13].

Corollary 2 *Each function $f\colon I \to I$ is the pointwise limit of a sequence of extendable functions $f_n\colon I \to I$.*

For $\Sigma = \omega_1$ we have the following corollary.

Corollary 3 *Each function $f\colon I \to I$ is the transfinite limit of a sequence of extendable functions $f_\alpha\colon I \to I$, $\alpha < \omega_1$.*[4]

[3] The analogous result on almost continuous functions is proved in [11].

[4] The analogous result for almost continuous functions is proved in [5].

Remark There is only one problem to obtain analogous results for any function $f\colon I \to \mathbb{R}$: we are unable to construct an extendable connectivity function g which is dense in $I \times \mathbb{R}$. **Added in the proof.**

(1) I was recently informed by Marek Balcerzak that Lemma 3 in my paper follows easily from Lemma 2 in *The homeomorphic transformation of c-sets into d-sets* by W. J. Gorman III (Proc. Amer. Math. Soc., **17** (1966), 825–830).

(2) Chris Ciesielski informed me during the latest Summer Symposium in Real Analysis, Erice, June 1995, that he with Irek Recław and, independently, Harvey Rosen had constructed an extendable function which is dense in $I \times \mathbb{R}$.

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