INROADS

Real Analysis Exchange
Vol. 21(1), 1995–96, pp. 317–319

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A CHARACTERIZATION OF ALMOST EVERYWHERE CONTINUOUS FUNCTIONS

Abstract

Let (X, d) be a separable metric space and $\mathcal{M}(X)$ the set of probability measures on the σ-algebra of Borel sets in X. In this paper we will show that a function f is almost everywhere continuous with respect to $\mu \in \mathcal{M}(X)$ if and only if $\lim_{n\to\infty} \int_X f\, d\mu_n = \int_X f\, d\mu$, for all sequences $\{\mu_n\}$ in $\mathcal{M}(X)$ such that μ_n converges weakly to μ.

Introduction and Main Result

Let (X, d) be a metric space. By $\mathcal{M}(X)$ we denote the set of probability measures on $\mathcal{B}_X$, where $\mathcal{B}_X$ is the σ-algebra generated by the closed subsets of X. Let $\mu \in \mathcal{M}(X)$. We say that a measurable function $f : X \to \mathbb{R}$ is *continuous almost everywhere* (μ) (continuous a.e. (μ)) if for the set D_f of discontinuity points of f we have $\mu(D_f) = 0$, where $D_f \in \mathcal{B}_X$. If $\{\mu_n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{M}(X)$ and $\mu \in \mathcal{M}(X)$, we say that $\{\mu_n\}_{n\in\mathbb{N}}$ *converges weakly to* μ ($\mu_n \rightharpoonup \mu$) if for any continuous bounded function f on X we have

$$\lim_{n\to\infty} \int_X f\, d\mu_n = \int_X f\, d\mu \tag{1}$$

In this paper we will show that a bounded measurable function f is continuous a.e. (μ) if and only if (1) is fulfilled for any sequence $\{\mu_n\}$ such that $\mu_n \rightharpoonup \mu$.

If $\mu_n \rightharpoonup \mu$, then for each bounded lower semicontinuous (upper semicontinuous) function h we have

$$\liminf_{n\to\infty} \int_X h\, d\mu_n \geq \int_X h\, d\mu \quad \left(\limsup_{n\to\infty} \int_X h\, d\mu_n \leq \int_X h\, d\mu\right)$$

Key Words: weak convergence, almost everywhere continuous functions, metric space
Mathematical Reviews subject classification: Primary: 28A60
Received by the editors October 17, 1994
*The author was supported by CONICOR and Universidad Nacional de Río Cuarto.

see [1, p. 17].

A set $B \in \mathcal{B}_X$ is called a continuity set for $\mu \in \mathcal{M}(X)$ if $\mu(\partial B) = 0$, where ∂B is the boundary of B. Let C_μ be the class of all continuity sets. Then C_μ is an algebra, see [3, p. 50]. For any function g on X we define the following functions:

$$\overline{g}(x) := \limsup_{r\to 0}\{g(y) : d(y,x) < r\}$$
$$\underline{g}(x) := \liminf_{r\to 0}\{g(y) : d(y,x) < r\}.$$

It is well known that $\underline{g}$ and $\overline{g}$ satisfy the following properties:

(i) $\underline{g} \le g \le \overline{g}$. Furthermore if g is bounded, then $\underline{g}$ and $\overline{g}$ are bounded.

(ii) $\underline{g}$ ($\overline{g}$) is a lower semicontinuous (upper semicontinuous) function. Therefore $\underline{g}$ and $\overline{g}$ are measurable functions.

(iii) $\underline{g}(x) = \overline{g}(x)$ if and only if x is a continuity point of g.

From (i) and (iii) it follows that $D_g = \{x : \overline{g}(x) - \underline{g}(x) > 0\}$. Then by (ii) $D_g \in \mathcal{B}_X$ for any measurable function g.

Theorem 1 *Let (X, d) be a separable metric space and $\mu \in \mathcal{M}(X)$. We suppose that f is a bounded measurable function on X. The following are equivalent:*

(a) *f is continuous a.e. (μ).*

(b) $\lim_{n\to\infty} \int_X f\,d\mu_n = \int_X f\,d\mu$, *for any sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}(X)$ such that $\mu_n \rightharpoonup \mu$.*

PROOF. We suppose that f is continuous a.e. (μ) and $\mu_n \rightharpoonup \mu$. Then by (i)–(iii) we obtain

$$\limsup_{n\to\infty} \int_X f\,d\mu_n \le \limsup_{n\to\infty} \int_X \overline{f}\,d\mu_n \le \int_X \overline{f}\,d\mu = \int_X f\,d\mu$$

and

$$\liminf_{n\to\infty} \int_X f\,d\mu_n \ge \liminf_{n\to\infty} \int_X \underline{f}\,d\mu_n \ge \int_X \underline{f}\,d\mu = \int_X f\,d\mu.$$

Hence we get (b).

Now we suppose that (b) holds. We will show that there is a sequence $\{\mu_n\}$ in $\mathcal{M}(X)$ such that $\mu_n \rightharpoonup \mu$ and $\lim_{n\to\infty} \int_X f\,d\mu_n = \int_X \overline{f}\,d\mu$. Hence, by (i), $\overline{f} = f$ a.e. (μ). Since a similar fact holds with $\underline{f}$, we will get that $\overline{f} = \underline{f}$ a.e. (μ); i.e., f is continuous a.e. (μ).

For each $x \in X$ and $r \in \mathbb{R}$, $B(X,r)$ denotes the open ball of radius r and center x. Let $n \in \mathbb{N}$ and $x \in X$. Since the set $\{r : \mu(\partial B(x,r) > 0)\}$ is at most countable, there exists a positive real number r_x^n such that $r_x^n \leq \frac{1}{n}$ and $B(x,r_x^n) \in C_\mu$. As X is a separable metric space and $X = \cup_{x\in X} B(x,r_x^n)$, there exists a sequence $\{x_j^n\}_{j\in\mathbb{N}}$ in X such that $X = \cup_{j=1}^{\infty} B(x_j^n, r_{x_j^n}^n)$. We define the sets A_j^n by $A_1^n := B(x_1^n, r_{x_1^n}^n)$ and $A_j^n := B(x_j^n, r_{x_j^n}^n) \setminus \{B(x_1^n, r_{x_1^n}^n) \cup \cdots \cup B(x_{j-1}^n, r_{x_{j-1}^n}^n)\}$ for $j > 1$. The sequence $\{A_j^n\}_{j\in\mathbb{N}}$ satisfies

(i) $X = \cup_{j=1}^{\infty} A_j^n$ and $A_j^n \in C_\mu$. Furthermore $A_j^n \cap A_i^n = \emptyset$ if $i \neq j$.

(ii) $\mathrm{diam}(A_j^n) \leq \frac{2}{n}$.

For each A_j^n let $z_j^n \in A_j^n$ such that $\sup\{f(x) : x \in A_j^n\} - \frac{1}{n} < f(z_j^n)$. We define the measures μ_n on $\mathcal{B}(X)$ by $\mu_n := \sum_{j=1}^{\infty} \mu(A_j^n)\delta_{z_j^n}$ where as usual δ_x is the Dirac delta measure. Clearly we have that $\lim_{n\to\infty} \int_X g\,d\mu_n = \int_X g\,d\mu$ for each bounded uniformly continuous function g. Hence by [3, p. 40] we get that $\mu_n \rightharpoonup \mu$.

Let A^0 denote the interior of the set A. It is easy to prove that if $x \in (A_j^n)^0$, then $\overline{f}(x) < \overline{f}(z_j^n) + \frac{1}{n}$. As $A_j^n \in C_\mu$, we have that

$$\int_X \overline{f}\,d\mu - \int_X f\,d\mu_n = \sum_{j=1}^{\infty} \int_{(A_j^n)^0} \{\overline{f} - \overline{f}(z_j^n)\}\,d\mu \leq \frac{1}{n} \sum_{j=1}^{\infty} \mu(A_j^n) = \frac{1}{n}\,.$$

Therefore

$$\int_X \overline{f}\,d\mu \geq \int_X f\,d\mu = \lim_{n\to\infty} \int_X f\,d\mu_n \geq \int_X \overline{f}\,d\mu_n.$$

Hence

$$\int_X \overline{f}\,d\mu = \int_X f\,d\mu.$$

So the proof is complete. □

Remark. *Obviously the last Theorem is true for μ_n, μ finite and positive measures.*

References

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