RESEARCH

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ON SOME REPRESENTATIONS OF A.E. CONTINUOUS FUNCTIONS

Abstract

It is proved that the following conditions are equivalent:

(a) f is an almost everywhere continuous function.

(b) $f = g + h$, where g, h are strongly quasi-continuous.

(c) $f = c + gh$, where $c \in \mathbb{R}$ and g, h are s.q.c..

Let $\mathbb{R}$ be the set of all reals and let μ_e (μ) denote outer Lebesgue measure (Lebesgue measure) in $\mathbb{R}$. Denote by

$$d_u(A,x) = \limsup_{h\to 0} \mu_e(A \cap (x-h, x+h))/2h$$
$$(d_l(A,x) = \liminf_{h\to 0} \mu_e(A \cap (x-h, x+h))/2h)$$

the upper (lower) density of a set $A \subset \mathbb{R}$ at a point x. A point $x \in \mathbb{R}$ is called a density point of a set $A \subset \mathbb{R}$ if there exists a measurable (in the sense of Lebesgue) set $B \subset A$ such that $d_l(B,x) = 1$. The family $\mathcal{T}_d = \{A \subset \mathbb{R};\ A$ is measurable and every point $x \in A$ is a density point of $A\}$ is a topology called the density topology [1].

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be strongly quasi-continuous (in short s.q.c.) at a point x if for every set $A \in \mathcal{T}_d$ containing x and for every positive real η there is an open interval I such that $I \cap A \neq \emptyset$ and $|f(t) - f(x)| < \eta$ for all $t \in A \cap I$ [2].

If there is an open set U such that $d_u(U,x) > 0$ and the restricted function $f|(U \cup \{x\})$ is continuous at x, then f is s.q.c. at x. [3].

By an elementary proof, we obtain the following observation.

Key Words: continuity, strong quasicontinuity, density topology, sums of functions, products of functions

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Remark 1 *If all functions $f_n : \mathbb{R} \to \mathbb{R}$, $n = 1, 2, \ldots$, of some uniformly convergent sequence $(f_n)_n$ are s.q.c. at a point x , then its limit f is also s.q.c. at x.*

It is known [2, 3] that every s.q.c. function f is almost everywhere (with respect to μ) continuous. So, the sum and the product of two s.q.c. functions are almost everywhere continuous.

We will prove the following assertion.

Theorem 1 *If a function $f : \mathbb{R} \to \mathbb{R}$ is almost everywhere continuous, then there are two s.q.c. functions $g, h : \mathbb{R} \to \mathbb{R}$ such that $f = g + h$.*

Proof. Let cl denote the closure operation and let

$$B = \{y \in \mathbb{R}; \mu(\mathrm{cl}\,(f^{-1}(y)) > 0\}.$$

Since the function f is almost everywhere continuous, the set B is countable. Let $E(B)$ be the linear space over the field $\mathbb{Q}$ of all rationals generated by the set B. Since the set $E(B)$ is countable, there is a positive number $c \in \mathbb{R} \setminus E(B)$. Denote by $\mathbb{Z}$ the set of all integers and by $\mathbb{N}$ the set of all positive integers. Fix $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. If $(2k-1)c/4^n \leq f(x) < (2k+1)c/4^n$, then we define $f_n(x) = (2k-1)c/4^n$. Observe that every function f_n, $n \in \mathbb{N}$, is almost everywhere continuous and if $D(f_n)$ denotes the set of all discontinuity points of f_n, then $D(f_n)$ is a closed set of measure zero. Moreover, $D(f_n) \subset D(f_{n+1})$ for $n \in \mathbb{N}$. Let $C(f_n)$, $n \in \mathbb{N}$, be the set of all continuity points of the function f_n, i.e. $C(f_n) = \mathbb{R} \setminus D(f_n)$.

Step 1. Since the set $D(f_1)$ is closed and of measure zero, for $k \in \mathbb{Z}$ and $j \in \mathbb{N}$ there are disjoint closed intervals $I_{1,k,j} = [a_{1,k,j}, b_{1,k,j}] \subset C(f_1)$, such that for every $k \in \mathbb{Z}$ and for every $x \in D(f_1)$ we have $d_u\big(\cup_{j\in\mathbb{N}} I_{1,k,j}, x\big) = 1$ and if there exists the limit $\lim_{l\to\infty} a_{1,k_l,j_l}$, then $\lim_{l\to\infty} a_{1,k_l,j_l} = \lim_{l\to\infty} b_{1,k_l,j_l} \in D(f_1)$. Let

$$g_1(x) = \begin{cases} (2k+1)c/4 & \text{if } x \in I_{1,2k,j},\ j \in \mathbb{N} \\ f_1(x) & \text{otherwise} \end{cases}$$

and for $x \in \mathbb{R}$ let $h_1(x) = f_1(x) - g_1(x)$. Observe that the functions g_1, h_1 are s.q.c. and $f_1 = g_1 + h_1$.

Step 2. First, we find disjoint sets $F_{2,2k,j,l} \subset \mathrm{int}\,(I_{1,2k,j}) \setminus D(f_2)$, $k \in \mathbb{Z}$, $j \in \mathbb{N}$, $l = 1, \ldots, 7$, being the unions of finite families of disjoint closed intervals and such that

- $\mu(F_{2,2k,j,l}) = \mu(I_{1,2k,j})/10$ for $k \in Z$, $j \in \mathbb{N}$, $l = 1, \ldots, 7$;

Moreover, we find a family of disjoint closed intervals $I_{2,k,j} = [a_{2,k,j}, b_{2,k,j}] \subset C(f_2) \setminus \cup_{k\in\mathbb{Z}; j\in\mathbb{N}; l\leq 7} F_{2,2k,j,l}$, $k \in \mathbb{Z}$, $j \in \mathbb{N}$ such that

- for every $k \in \mathbb{Z}$ and for every $x \in D(f_2)$ we have $d_u(\bigcup_{j\in\mathbb{N}} I_{2,k,j}, x) = 1$;

- if there exists the limits $\lim_{l\to\infty} a_{2,k_l,j_l}$, then $\lim_{l\to\infty} a_{2,k_l,j_l} = \lim_{2,k_l,j_l} b_{2,k_l,j_l} \in D(f_2)$;
- for all $k_1, k_2 \in \mathbb{Z}$ and $j_1, j_2 \in \mathbb{N}$ we have $I_{1,k_1,j_1} \cap I_{2,k_2,j_2} = \emptyset$ or $I_{2,k_2,j_2} \subset \operatorname{int}(I_{1,k_1,j_1})$.

Let

$$g_2(x) = \begin{cases} f_2(x) & \text{if } x \in D(f_2) \\ g_1(x) + lc/16 & \text{if } x \in I_{2,l,j},\ j \in \mathbb{N},\ l = 1, \dots, 7 \\ g_1(x) + lc/16 & \text{if } x \in F_{2,2k,j,l},\ k \in \mathbb{Z},\ j \in \mathbb{N},\ l \leq 7 \\ g_1(x) & \text{otherwise} \end{cases}$$

and for $x \in \mathbb{R}$ let $h_2(x) = f_2(x) - g_2(x)$. Then the functions g_2, h_2 are s.q.c. and $f_2 = g_2 + h_2$. Moreover, $|g_1 - g_2| \leq c/2$ and $|h_1 - h_2| \leq |f_1 - f_2| + |g_1 - g_2| \leq c/2 + c/2 = c$.

Step n $(n > 2)$. There are s.q.c. functions g_{n-1}, h_{n-1} such that
- $g_{n-1} + h_{n-1} = f_{n-1}$ and
- $g_{n-1}(\mathbb{R}) \cup h_{n-1}(\mathbb{R}) \subset \{kc/4^{n-1};\ k \in \mathbb{Z}\}$.

If $(g_{n-1})^{-1}(kc/4^{n-1}) \neq \emptyset$ for some $k \in \mathbb{Z}$, then for there are disjoint closed intervals $I_{n,k,l,j} \subset \operatorname{int}((g_{n-1})^{-1}(kc/4^{n-1})) \cap C(f_n)$, $l, j \in \mathbb{N}$, such that
- for every $l \in \mathbb{N}$ and for every $x \in D(f_n) \cap (g_{n-1})^{-1}(kc/4^{n-1})$ we have $d_u(\bigcup_{j\in\mathbb{N}} I_{n,k,l,j}, x) > 0$ and
- if a sequence of points x_i, $i \in \mathbb{N}$, belonging to different intervals I_{n,k,l_i,j_i} converges to a point x, then $x \in D(f_n)$.

Let

$$g_n(x) = \begin{cases} f_n(x) & \text{if } x \in D(f_n) \\ g_{n-1}(x) + lc/4^n & \text{if } x \in I_{n,k,l,j},\ j \in \mathbb{N},\ k \in Z,\ l = 1, \dots, 7 \\ g_{n-1}(x) & \text{otherwise} \end{cases}$$

and let $h_n(x) = f_n(x) - g_n(x), \quad x \in \mathbb{R}$. Then the functions g_n, h_n are s.q.c. and $f_n = g_n + h_n$. Moreover, $|g_n - g_{n-1}| \leq 2c/4^{n-1}$ and $|h_n - h_{n-1}| \leq c/4^{n-2}$. The sequences $(g_n)_n$ and $(h_n)_n$ uniformly converge to some functions g and h respectively, which are, by Remark 1, s.q.c.. Moreover,

$$g + h \lim_{n\to\infty} g_n + \lim_{n\to\infty} h_n = \lim_{n\to\infty} (g_n + h_n) = \lim_{n\to\infty} f_n = f.$$

This finishes the proof. □

Remark 2 *If the function f from Theorem 1 is of Baire α class ($\alpha > 0$), then the functions g and h can be the same.*

Remark 3 *From the proof of Theorem 1 it follows immediately that if I is an open interval and if $f : I \to \mathbb{R}$ is an almost everywhere continuous function, then there are two s.q.c. functions $g, h : I \to \mathbb{R}$ such that $f = g + h$.*

Now we will examine the products of s.q.c. functions.

Theorem 2 *Let $f:\mathbb{R}\to\mathbb{R}$ be an almost everywhere continuous function such that $\mu(\mathrm{cl}\,(f^{-1}(0))\setminus\mathrm{int}\,(f^{-1}(0)))=0$. Then there are two s.q.c. functions g, h such that $f=gh$.*

PROOF. Denote by A the set $\{x; f(x)>0\}$, by B the set $\{x; f(x)<0\}$ and observe that $\mu(\mathbb{R}\setminus\mathrm{int}\,(A)\setminus\mathrm{int}\,(B)\setminus\mathrm{int}\,(f^{-1}(0)))=0$. If I is a component of the set $\mathrm{int}\,(A)$, then the function $x\to\ln(f(x))$ for $x\in I$, is an almost everywhere continuous function, and by Remark 3, there are two s.q.c. functions $g_I, h_I: I\to\mathbb{R}$ such that $\ln(f(x))=g_I(x)+h_I(x)$ for $x\in I$. Consequently, the reduced function $f|I=(e^{\ln(f)})/I=e^{g_I}e^{h_I}$ is the product of two s.q.c. functions. Analogously, if J is a component of the set $\mathrm{int}\,(B)$, then the function $-f|J$ is the product of two s.q.c. functions and consequently, the function $f|J$ is also the product of two s.q.c. functions. So, there are two s.q.c. functions $g_1, h_1:(\mathrm{int}\,(A)\cup\mathrm{int}\,(B))\to\mathbb{R}$ such that $f|(\mathrm{int}\,(A)\cup\mathrm{int}\,(B))=g_1h_1$. Let F be the set of all points $x\in\mathrm{cl}\,(\mathrm{int}\,(f^{-1}(0)))$ at which $d_l(\mathrm{int}\,(f^{-1}(0)),x)=1$ and $f(x)\neq 0$. There are families of closed intervals $I_{k,n}=[a_{k,n},b_{k,n}]\subset\mathrm{int}\,(f^{-1}(0))$, $k,n\in\mathbb{N}$, such that

- $I_{k_1,n_1}\cap I_{k_2,n_2}=\emptyset$ if $(k_1,n_1)\neq(k_2,n_2)$, $k_1,k_2,n_1,n_2\in\mathbb{N}$,
- if $\exists$ the limit $\lim_{l\to\infty}a_{k_l,n_l}$, then $\lim_{l\to\infty}a_{k_l.n_l}=\lim_{l\to\infty}b_{k_l,n_l}\in\mathrm{cl}\,(F)$,
- for every point $x\in F$ and for every $k\in\mathbb{N}$ we have $d_u(\bigcup_n I_{k,n},x)>0$.

Next, enumerate all non zero rationals in a sequence $w_1,\dots,w_k,\dots$ such that $w_i\neq w_j$ for $i\neq j$, $i,j\in\mathbb{N}$, and let $H=\mathbb{R}\setminus\mathrm{int}\,(A)\setminus\mathrm{int}\,(B)\setminus\mathrm{int}\,(f^{-1}(0))\setminus F$. There are disjoint closed intervals $J_{k,n}=[c_{k,n},d_{k,n}]\subset\mathrm{int}\,(A)\cup\mathrm{int}\,(B)$, $k,n\in\mathbb{N}$, such that

- the functions g_1,h_1 are continuous at all points $c_{k,n}$ and $d_{k,n}$, $k,n\in\mathbb{N}$,
- if there exist the limit $\lim_{l\to\infty}c_{k_l,n_l}$, then $\lim_{l\to\infty}c_{k_l,n_l}=\lim_{l\to\infty}d_{k_l,n_l}\in\mathrm{cl}\,(H)$,
- for every point $x\in H$ and for every $k\in\mathbb{N}$ we have $d_u(\bigcup_n J_{k,n},x)>0$.

Since the function g_1 is almost everywhere continuous on its domain, for every interval $\mathrm{int}\,(J_{k,n})$, $k,n\in\mathbb{N}$, there are a positive real $r(k,n)$ and a finite family of disjoint closed intervals $K_{k,n,i}\subset\mathrm{int}\,(J_{k,n})$, $i=1,\dots,i(k,n)$, such that

- $|g_1(x)|>r(k,n)$ for $x\in K_{k,n,i}$, $k,n\in\mathbb{N}$, $i=1,\dots.i(k,n)$,
- $\mathrm{osc}_{K_{k,n,i}}g_1<r(k,n)/nw_k$ for $k,n\in\mathbb{N}$, $i=1,\dots,i(k,n)$,
- for every point $x\in H$ and for every $k\in\mathbb{N}$ we have $d_u(\bigcup_{n\in\mathbb{N};i\leq i(k,n)}K_{k,n,i},x)>0$.

In every interval $\mathrm{int}\,(K_{k,n,i})$, $k,n\in\mathbb{N}$, $i=1,\dots,i(k,n)$, fix a point $x_{k,n,i}$.

Put

$$g(x) = \begin{cases} w_k & \text{if } x \in I_{2k,n},\ k,n \in \mathbb{N} \\ 0 & \text{if } x \in I_{2k-1,n},\ k,n \in \mathbb{N} \\ 0 & \text{otherwise on } f^{-1}(0) \\ g_1(x)w_k/g_1(x_{2k,n,i}) & \text{if } x \in K_{2k,n,i},\ k,n \in \mathbb{N},\ i \le i(k,n) \\ g_1(x) & \text{otherwise on } \operatorname{int}(A) \cup \operatorname{int}(B) \\ g_1(x)h_1(x) & \text{if } x \in F \cup H \end{cases}$$

and

$$h(x) = \begin{cases} 0 & \text{if } x \in I_{2k,n},\ k,n \in \mathbb{N} \\ 1 & \text{if } x \in I_{2k-1,n},\ k,n \in \mathbb{N} \\ 0 & \text{otherwise on } f^{-1}(0) \\ h_1(x)g_1(x_{2k,n,i})/w_k & \text{if } x \in K_{2k,n,i},\ k,n \in \mathbb{N},\ i \le i(k,n) \\ h_1(x) & \text{otherwise on } \operatorname{int}(A) \cup \operatorname{int}(B) \\ 1 & \text{if } x \in F \cup H. \end{cases}$$

Since $g(K_{k,n,i}) \subset (w_k - 1/n, w_k + 1/n)$ for all $k,n \in \mathbb{N}$, $i = 1,\dots,i(k,n)$, and $d_u(\bigcup_{n\in\mathbb{N},\ i\le i(k,n)} K_{k,n,i}, x) > 0$ for each $x \in H$ and for each $k \in \mathbb{N}$, the function g is s.q.c. at every point $x \in H$. Evidently, it is also s.q.c. otherwise on $\mathbb{R}$. Analogously, h is a s.q.c. function. Obviously, $f = gh$ and the proof is completed. □

Theorem 3 *If $f : \mathbb{R} \to \mathbb{R}$ is an almost everywhere continuous function, then there are a constant $c \in \mathbb{R}$ and two s.q.c. functions g, h such that $f = c + gh$.*

Proof. Let $c \in \mathbb{R}$ be a number such that $\mu(\operatorname{cl}(f^{-1}(c))) = 0$. Then the function $f_1 = f - c$ satisfies the suppositions of Theorem 2 and consequently, there are two s.q.c. functions g, h such that $f_1 = gh$. So, $f = c + gh$ and the proof is completed. □

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is quasi-continuous (cliquish) at a point x if for every positive η and for every open set U containing x there is a nonempty open set $V \subset U$ such that $|f(t) - f(x)| < \eta$ for all $t \in V$ ($\operatorname{osc}_V f < \eta$) [6].

Remark 4 *Since for every cliquish function $f : \mathbb{R} \to \mathbb{R}$ there is a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ such that $f \circ h$ is almost everywhere continuous, by Theorems 1 and 3 we obtain immediately that for every cliquish function f there are a constant $c \in \mathbb{R}$ and quasi-continuous functions f_1, f_2, f_3, f_4 such that $f = f_1 + f_2$ and $f = c + f_3 f_4$.*

Remark 5 *If a function $f : \mathbb{R} \to \mathbb{R}$ is the product of a finite family of s.q.c. functions g_k, $k = 1,\dots,n$, then it satisfies the following condition:*

(H) if $A \subset \operatorname{cl}(f^{-1}(0)) - f^{-1}(0)$ *is such that* $d_l(f^{-1}(0), x) = 1$ *for every* $x \in A$*, then* A *is nowhere dense in* $f^{-1}(0)$*.*

PROOF. Denote by B the set of all density points of the set $f^{-1}(0)$ belonging to $f^{-1}(0)$. If $B \neq \emptyset$ and A is not nowhere dense in $f^{-1}(0)$, then there is a point $x \in A$ and a positive integer $i \leq n$ such that x is a density point of the set $(f_i)^{-1}(0)$. Since $f_i(x) \neq 0$ and f_i is a s.q.c. function, we obtain a contradiction. If $B = \emptyset$, then A is the same. This completes the proof. □

Remark 6 *There is almost everywhere continuous functions which are not the products of finite families of s.q.c. functions.*

PROOF. Such as, for example, the function

$$f(x) = \begin{cases} 1/n & \text{if } x = w_n,\ n \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

(see also [4, 5]), since it does not satisfy condition (H) in Remark 5. □

Problems

1. Let $f : \mathbb{R} \to \mathbb{R}$ be an almost everywhere continuous function. Is the function f the sum of two Darboux s.q.c. functions?
2. Let $f : \mathbb{R} \to \mathbb{R}$ be an almost everywhere continuous function satisfying the condition (H) from Remark 5. Is f the product of two s.q.c. functions?
3. Characterize the products of two Darboux s.q.c. functions.

References

[1] A. M. Bruckner, *Differentiation of real functions*, Lectures Notes in Math., **659** (1978), Springer-Verlag.

[2] Z. Grande, *Measurability, quasicontinuity and cliquishness of functions of two variables*, Real Analysis Exchange **20** No. 2 (1994–95), 744–752.

[3] Z. Grande, *On strong quasi-continuity of functions of two variables*, Real Analysis Exchange **21** No. 1 (1995–96).

[4] Z. Grande, *Sur les fonctions cliquish*, Cas. Pest. Mat., **110** (1985), 225–236.

[5] T. Natkaniec, *Products of quasi-continuous functions*, Math. Slovaca, **40** (1990), 401–405.

[6] T. Neubrunn, *Quasi-continuity*, Real Analysis Exchange, **14** No. 2 (1988–89), 259–306.