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ON THE DENSITY TOPOLOGY WITH RESPECT TO AN EXTENSION OF LEBESGUE MEASURE

Abstract

We prove that, for every complete extension μ of Lebesgue measure, the μ-density topology is the Hashimoto topology generated by the density topology and the σ-ideal of μ-null sets (cf. [1]).

Let μ be any complete extension of Lebesgue measure l on the real line $\mathbb{R}$. Let $\mathcal{S}_\mu$ denote the σ-field of μ-measurable sets, $\mathcal{I}_\mu$ — the σ-ideal of μ-null sets. We denote by $\mathcal{L}$ the σ-field of Lebesgue measurable sets. Let μ^* and μ_* be, respectively, the outer measure and the inner measure induced by μ. We recall that a point $x \in \mathbb{R}$ is a μ-density point of a μ-measurable set X if

$$\lim_{h \to 0+} \frac{\mu(X \cap [x-h, x+h])}{2h} = 1.$$

For each set $X \in \mathcal{S}_\mu$, let

$$\Phi_\mu(X) = \{x \in \mathbb{R} : x \text{ is a } \mu\text{-density point of } X\}.$$

Let

$$\mathcal{T}_\mu^* = \{X \in \mathcal{S}_\mu : X \subset \Phi_\mu(X)\}.$$

Theorem 1 (cf. [2]) *The family $\mathcal{T}_\mu^*$ is a topology in $\mathbb{R}$.*

PROOF. It is clear that the sets $\emptyset$ and $\mathbb{R}$ are members of the family $\mathcal{T}_\mu^*$. Also, the family $\mathcal{T}_\mu^*$ is closed under finite intersections. Our task is to prove that, for each family $\{X_t\}_{t \in T} \subset \mathcal{T}_\mu^*$, we have $\bigcup_{t \in T} X_t \in \mathcal{T}_\mu^*$. It suffices to prove

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that $\bigcup_{t\in T} X_t$ is μ-measurable because the inclusion $\bigcup_{t\in T} X_t \subset \Phi_\mu\big(\bigcup_{t\in T} X_t\big)$, when $\bigcup_{t\in T} X_t \in \mathcal{S}_\mu$, is obvious. Let $X = \bigcup_{t\in T} X_t$ and suppose that X is bounded. Let K be a segment such that $X \subset K$. We show that, for each $\varepsilon > 0$, there exists a μ-measurable set C such that $X \subset C$ and $\mu^*(C \setminus X) < \varepsilon$. It suffices to snow that $X \in \mathcal{S}_\mu$. Fix $0 < \varepsilon < l(K)$. Putting

$$\mathcal{K} = \Big\{\Delta \subset K : \mu_*(\Delta \cap X) > \Big(1 - \frac{\varepsilon}{l(K)}\Big)\mu(\Delta)\Big\},$$

where Δ denotes an interval, we can easily check that the family $\mathcal{K}$ forms a Vitali covering of the set X. Hence we have a sequence $\{\Delta_n\}_{n\in\mathbb{N}} \subset K$ of pairwise disjoint intervals such that $l\big(X \setminus \bigcup_{n=1}^\infty \Delta_n\big) = 0$. We show that $\mu^*\big(\bigcup_{n=1}^\infty \Delta_n \setminus X\big) < \varepsilon$. For every positive integer n, there exists a μ-measurable set B_n such that $B_n \subset \Delta_n \cap X$ and $\mu(B_n) > (1 - \frac{\varepsilon}{l(K)})\mu(\Delta_n)$. Hence

$$\sum_{n=1}^\infty \mu^*(\Delta_n \setminus X) \le \sum_{n=1}^\infty \mu(\Delta_n \setminus B_n) = \sum_{n=1}^\infty (\mu(\Delta_n) - \mu(B_n)) < \frac{\varepsilon}{l(K)} \sum_{n=1}^\infty \mu(\Delta_n) \le \varepsilon.$$

Putting $C = \big(X \setminus \bigcup_{n=1}^\infty \Delta_n\big) \cup \big(\bigcup_{n=1}^\infty \Delta_n\big)$, we have that C has the desired property. Clearly the set C is Lebesgue measurable. □

Lemma 1 *Every $\mathcal{T}_\mu^*$-open set X has the form $Y \setminus Z$ where $Y \in \mathcal{L}$ and $Z \in \mathcal{I}_\mu$.*

PROOF. Let $X \in \mathcal{T}_\mu^*$. Then $X \in \mathcal{S}_\mu$ and $X \subset \Phi_\mu(X)$. According to the proof of Theorem 1, for any $\varepsilon > 0$ there exists a Lebesgue measurable set $C \supset X$ such that $\mu(C \setminus X) < \varepsilon$. As a simple consequence we obtain that $X = Y \setminus Z$ where $Y \in \mathcal{L}$ and $Z \in \mathcal{I}_\mu$. □

Lemma 2 *If μ is a complete extension of Lebesgue measure, such that, for each set $X \in \mathcal{S}_\mu$, $\mu(X \triangle \Phi_\mu(X)) = 0$, then*

$$\{Y \in \mathcal{S}_\mu : Y \subset \Phi_\mu(Y)\} = \{Y \subset \mathbb{R} : Y = \Phi_\mu(Z) \setminus U; Z \in \mathcal{S}_\mu, U \in \mathcal{I}_\mu\}.$$

PROOF. If $Y \in \mathcal{S}_\mu$ and $Y \subset \Phi_\mu(Y)$, then $Y = \Phi_\mu(Y) \setminus (\Phi_\mu(Y) \setminus Y)$ has the desired form. If $Y = \Phi_\mu(Z) \setminus U$ where $Z \in \mathcal{S}_\mu$ and $\mu(U) = 0$, then we see that $\Phi_\mu(Z) \in \mathcal{S}_\mu$ and $\Phi_\mu(\Phi_\mu(Z) \setminus U) = \Phi_\mu(\Phi_\mu(Z)) = \Phi_\mu(Z) \supset \Phi_\mu(Z) \setminus U$. □

We call the topology $\mathcal{T}_\mu^*$ a μ-density topology. In case $\mu = l$ the topology $\mathcal{T}_\mu^*$ is the $\mathcal{T}_d$-topology called the density topology (cf. [4]).

Lemma 3 *If μ is a complete extension of Lebesgue measure, such that $\mathcal{S}_\mu = \mathcal{L} \triangle \mathcal{I}_\mu = \{X \triangle Y : X \in \mathcal{L}, Y \in \mathcal{I}_\mu\}$, then*

$$\mathcal{T}^*_\mu = \{Y \subset \mathbb{R} : Y = \Phi_l(Z) \setminus U,\ Z \in \mathcal{L},\ U \in \mathcal{I}_\mu\}.$$

PROOF. We see that, for each set $X \in \mathcal{S}_\mu$, $\mu(X \triangle \Phi_\mu(X)) = 0$. Thus, applying Lemma 2 and taking account of the fact that, for every μ-measurable set A there exists a Lebesgue measurable set B such that $\Phi_\mu(A) = \Phi_l(B)$, we obtain the desired equality. □

Lemma 4 *If μ is any complete extension of Lebesgue measure, then there exists a complete extension of Lebesgue measure μ', such that*

(1) $\mathcal{S}_{\mu'} = \mathcal{L} \triangle \mathcal{I}_\mu$,

(2) $\mathcal{I}_{\mu'} = \mathcal{I}_\mu$,

(3) $\mathcal{T}^*_{\mu'} = \mathcal{T}^*_\mu$.

PROOF. If μ is any extension of Lebesgue measure, then, for each $A \in \mathcal{I}_\mu$, we have that $l_*(A) = 0$. Thus, applying the Marczewski method of the extension of Lebesgue measure (cf. [3]), we can consider the σ-field

$$\mathcal{S}_{\mu'} = \{X \triangle Y : X \in \mathcal{L},\ Y \in \mathcal{I}_\mu\}.$$

Putting $\mu'(X \triangle Y) = l(X)$ where $X \in \mathcal{L}$ and $Y \in \mathcal{I}_\mu$, we correctly define a measure which is an extension of Lebesgue measure. We have thus proved condition (1). In fact, the measure μ' is the restriction of the measure μ to $\mathcal{S}_{\mu'}$. Now, we prove that $\mathcal{I}_\mu = \mathcal{I}_{\mu'}$. Let X be such that $\mu(X) = 0$. Then $X = \emptyset \triangle X \in \mathcal{S}_{\mu'}$ and $\mu'(X) = 0$. Let X be such that $\mu'(X) = 0$. Then $X \in \mathcal{S}_{\mu'}$. Hence $X = Y \triangle Z$ where $Y \in \mathcal{L}$ and $\mu(Z) = 0$. We have that $0 = \mu'(X) = l(Y)$. This implies that $\mu(Y) = 0$ and, finally, $\mu(X) = 0$. The demonstration of the fact that $\mathcal{T}^*_\mu = \mathcal{T}^*_{\mu'}$ will complete the proof.

We show that $\mathcal{T}^*_\mu \subset \mathcal{T}^*_{\mu'}$. Let $X \in \mathcal{T}^*_\mu$. Then, by Lemma 1, $X = Y \setminus Z$ where $Y \in \mathcal{L}$ and $Z \in \mathcal{I}_{\mu'}$. By condition 2, $Z \in \mathcal{I}_\mu$. This implies that $X \in \mathcal{S}_{\mu'}$ and $\Phi_{\mu'}(X) = \Phi_{\mu'}(Y \setminus Z) = \Phi_{\mu'}(Y) = \Phi_l(Y) = \Phi_\mu(Y) = \Phi_\mu(Y \setminus Z) = \Phi_\mu(X) \supset X$. Hence $X \in \mathcal{T}^*_{\mu'}$. Let $X \in \mathcal{T}^*_{\mu'}$. Then $X \in \mathcal{S}_{\mu'}$ and $X \subset \Phi_{\mu'}(X)$. At the same time $X \in \mathcal{S}_\mu$ and $\Phi_{\mu'}(X) = \Phi_\mu(X)$. Thus $X \in \mathcal{T}^*_\mu$. □

Combining Lemmas 3 and 4, we have the following.

Theorem 2 *Fo any complete extension μ of Lebesgue measure, the μ-density topology $\mathcal{T}^*_\mu$ is the Hashimoto topology of the form*

$$\mathcal{T}^*_\mu = \{X \subset \mathbb{R} : X = Y \setminus Z,\ Y \in \mathcal{T}_d,\ \mu(Z) = 0\}.$$

Corollary 1 *If μ_1 and μ_2 are complete extensions of Lebesgue measure and the families of μ_1-null sets and μ_2-null sets are identical, then $\mathcal{T}^*_{\mu_1} = \mathcal{T}^*_{\mu_2}$.*

Lemma 5 *For any complete extension μ of Lebesgue measure, the family $K(\mathcal{T}^*_\mu)$ of all $\mathcal{T}^*_\mu$-meager sets is identical with the σ-ideal $\mathcal{I}_\mu$.*

PROOF. Let $X \in \mathcal{I}_\mu$. We prove that X is $\mathcal{T}^*_\mu$-nowhere dense. Let U be a nonempty $\mathcal{T}^*_\mu$-open set. Then, by Theorem 2, $U = Y \setminus Z$ where $Y \in T_d$ and $Z \in \mathcal{I}_\mu$. Putting $U' = Y \setminus (Z \cup X)$, we have that U' is a nonempty $\mathcal{T}^*_\mu$-open subset of U disjoint from X. This means that X is a member of the family $K(\mathcal{T}^*_\mu)$. Now, let $X \in K(\mathcal{T}^*_\mu)$. It suffices to consider the fact that X is a $\mathcal{T}^*_\mu$-nowhere dense set. Thus the $\mathcal{T}^*_\mu$-closure $\overline{X}$ is also a $\mathcal{T}^*_\mu$-nowhere dense set. We see that the set $\mathbb{R} \setminus \overline{X} = Y \setminus Z$ where $Y \in \mathcal{T}_d$ and $Z \in \mathcal{I}_\mu$. Hence $\overline{X} = (\mathbb{R} \setminus Y) \cup Z$ and $\mathbb{R} \setminus Y$ is a $\mathcal{T}_d$-closed set such that $\mathrm{Int}(\mathbb{R} \setminus Y) = \emptyset$ with respect to the $\mathcal{T}_d$-topology. This means that $\mathbb{R} \setminus Y$ is $\mathcal{T}_d$-nowhere dense and, thus, it is a Lebesgue null set (cf. ([5]). Finally, we conclude that $X \in \mathcal{I}_\mu$. □

Lemma 6 *For any complete extension μ of Lebesgue measure, each set having the Baire property with respect to $\mathcal{T}^*_\mu$ is the sum of a $\mathcal{T}^*_\mu$-open set and a $\mathcal{T}^*_\mu$-nowhere dense and closed set. Moreover, the family $\mathcal{B}(\mathcal{T}^*_\mu)$ of all sets having the Baire property with respect to $\mathcal{T}^*_\mu$ is identical with the family $\mathcal{L} \triangle \mathcal{I}_\mu$.*

PROOF. Let $X \in \mathcal{B}(\mathcal{T}^*_\mu)$. Thus $X = (U \setminus Y) \cup Z$ where $U \in \mathcal{T}^*_\mu$ and $Y, Z \in \mathcal{K}(\mathcal{T}^*_\mu)$. By Lemma 5, the sets $Y, Z \in \mathcal{I}_\mu$ and, at the same time, they are closed and nowhere dense in the topology $\mathcal{T}^*_\mu$. Hence $U \setminus Y$ is $\mathcal{T}^*_\mu$-open and the set X has the desired representation.

We prove that $\mathcal{B}(\mathcal{T}^*_\mu) = \mathcal{L} \triangle \mathcal{I}_\mu$. Let $X \in \mathcal{B}(\mathcal{T}^*_\mu)$. Thus $X = U \setminus Y$ where $U \in \mathcal{T}^*_\mu$ and $Y \in \mathcal{I}_\mu$. By Theorem 2, $U = W \setminus Z$ where $W \in \mathcal{T}_d$ and $Z \in \mathcal{I}_\mu$. Hence $X \in \mathcal{L} \triangle \mathcal{I}_\mu$.

Now, we show that $\mathcal{L} \triangle \mathcal{I}_\mu \subset \mathcal{B}(\mathcal{T}^*_\mu)$. By Lemma 5, $\mathcal{I}_\mu \subset \mathcal{B}(\mathcal{T}^*_\mu)$. We prove that $\mathcal{L} \subset \mathcal{B}(\mathcal{T}^*_\mu)$. Let $X \in \mathcal{L}$. Thus $X = (\Phi_l(X) \setminus (\Phi_l(X) \triangle X)) \cup ((\Phi_l(X) \triangle X) \setminus \Phi(X))$. We conclude that the set X is the sum of a $\mathcal{T}^*_\mu$-open set and a μ-null set. This implies that $X \in \mathcal{B}(\mathcal{T}^*_\mu)$. Finally, we have $\mathcal{L} \triangle \mathcal{I}_\mu \subset \mathcal{B}(\mathcal{T}^*_\mu)$. □

Corollary 2 $\mathcal{B}(\mathcal{T}^*_\mu) = \mathcal{B}(\mathcal{T}_d) \triangle \mathcal{I}_\mu$.

Corollary 3 $\mathcal{B}(\mathcal{T}^*_\mu) = Borel(\mathcal{T}^*_\mu)$.

Corollary 4 $Borel(\mathcal{T}^*_\mu) = Borel(\mathcal{T}^*_\mu) \triangle \mathcal{I}_\mu$.

Based on Lemmas 5 and 6, we have the following assertion.

Theorem 3 *For any complete extension μ of Lebesgue measure, the topology $\mathcal{T}_\mu^*$ is the von Neumann topology associated with the measure μ if and only if $\mathcal{S}_\mu = \mathcal{L} \triangle \mathcal{I}_\mu$.*

Lemma 7 *If μ is any complete extension of Lebesgue measure such that for every $X \in \mathcal{S}_\mu$, $X \setminus \Phi_\mu(X) \in \mathcal{I}_\mu$, then the topology $\mathcal{T}_\mu^*$ is the von Neumann topology associated with μ.*

Proof. By Lemma 5, we see that the family $\mathcal{K}(\mathcal{T}_\mu^*)$ of meager sets with respect to the topology $\mathcal{T}_\mu^*$ is identical with μ. Thus we need only prove that the family $\mathcal{B}(\mathcal{T}_\mu^*)$ of Baire sets is identical with $\mathcal{S}_\mu$. By Lemma 6, $\mathcal{B}(\mathcal{T}_\mu^*) \subset \mathcal{S}_\mu$. Let $X \in \mathcal{S}_\mu$. By assumption, we have that $X \setminus \Phi_\mu(X) \in \mathcal{I}_\mu$ and moreover, it is easy to see that $\Phi_\mu(X) \setminus X \in \mathcal{I}_\mu$. Hence $\Phi_\mu(X)$ is the μ-measurable set. Putting $X = (X \cap \Phi_\mu(X)) \cup (X \setminus \Phi_\mu(X))$, we have that $X \setminus \Phi_\mu(X) \in \mathcal{K}(\mathcal{T}_\mu^*)$ and $X \cap \Phi_\mu(X) \subset \Phi_\mu(X) = \Phi_\mu(X \cap \Phi_\mu(X))$. The last assertion means that $X \cap \Phi_\mu(X) \in \mathcal{T}_\mu^*$ and we have obtained that the set X has the Baire property with respect to the topology $\mathcal{T}_\mu^*$. □

Theorem 4 *If μ is any complete extension of Lebesgue measure, then the condition that for every $X \in \mathcal{S}_\mu$, $X \triangle \Phi_\mu(X) \in \mathcal{I}_\mu$, holds if and only if $\mathcal{S}_\mu = \mathcal{L} \triangle \mathcal{I}_\mu$.*

Proof. The necessity is the consequence of Theorem 3 and Lemma 7.

The sufficiency is the consequence of the Lebesgue density theorem. □

Now, we turn our attention to the family of continuous functions with respect to μ-density topology.

Applying Theorem 4 in [4], we conclude the following property.

Proposition 1 *If $(X, \mathcal{T})$ is a topological space and $\mathcal{I}$ is an arbitrary σ-ideal of subsets of X free from nonempty $\mathcal{T}$-open sets and such that the family of sets $\mathcal{T} - \mathcal{I} = \{Y \subset X : Y = W \setminus Z,\ \ W \in \mathcal{T},\ \ Z \in \mathcal{I}\}$ forms a topology (called the Hashimoto topology), then the family of all real continuous functions in the topology $\mathcal{T}$ is identical with the family of all real continuous functions in the topology $\mathcal{T} - \mathcal{I}$.*

As an easy application of this proposition and Theorem 2 we have the following.

Theorem 5 *For any complete extension μ of Lebesgue measure, the family of all real functions which are continuous in the μ-density topology is identical with the family of all approximately continuous functions.*

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