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ON THE DE GIORGI—LETTA INTEGRAL WITH RESPECT TO MEANS WITH VALUES IN RIESZ SPACES

Abstract

A monotone integral is given for scalar function, with respect to Riesz space values means, and also a necessary and sufficient condition to obtain a Radon-Nikodym density for two means.

1 Introduction

Integrals like Kurzweil-Stieltjes, Riemann sums and Bochner have been studied in vector lattices by Duchoň, Riečan and Vrábelová, ([11], [21], [22]), Wright ([26], [27]), McGill ([19]), Šipoš ([24]), Maličký ([18]), Cristescu ([8]), Haluška ([15]), Boccuto ([3], [4]), and others.

In this paper we extend to such spaces the monotone integral, given by Choquet in 1953 ([6]), and developed by De Giorgi-Letta ([9]), Greco ([13]), Brooks-Martellotti ([5]), and others ([10], [12], [16], etc.).

Given a mean $\mu : \mathcal{A} \to R$ and a measurable function $f : X \to \widetilde{\mathbb{R}}_0^+$, we say that f is integrable (in the monotone sense) if the following limit exists in R.

$$(o) - \lim_{a \to +\infty} \int_0^a \mu(\{x \in X : f(x) > t\})\, dt.$$

For this integral we obtain some elementary properties and we give some Vitali-type theorems. We note that in general this integral is different from the one

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introduced in [5] for Banach spaces. Finally we prove a version of Radon-Nikodym-type theorems for the introduced integral (see also [14]).

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2 Preliminaries

We begin with some definitions.

Definition 2.1. *A Riesz space R is called* Archimedean *if the following property holds.*

(2.1.1) *For every choice of $a, b \in R$, if $na \leq b$ for all $n \in \mathbb{N}$, then $a \leq 0$.*

Definition 2.2. *A Riesz space R is said to be* Dedekind complete *(resp. σ-*Dedekind complete*) if every nonempty (countable) subset of R, bounded from above, has supremum in R.*

The following results are well-known (see [1], [2]).

Proposition 2.3. *Every σ-Dedekind complete Riesz space is Archimedean.*

Theorem 2.4. *Given an Archimedean (Dedekind complete) Riesz space R, there exists a compact Stonian topological space Ω, unique up to homeomorphisms such that R can be embedded as a (solid) subspace of*

$$\mathcal{C}_\infty(\Omega) = \{f \in \widetilde{\mathbb{R}}^\Omega : f \textit{ is continuous, and } \{\omega : |f(\omega)| = +\infty\}$$

is nowhere dense in Ω}. Moreover if $(a_\lambda)_{\lambda \in \Lambda}$ is any family such that $a_\lambda \in R \;\forall\, \lambda$ and $a = \sup_\lambda a_\lambda \in R$ (where the supremum is taken with respect to R), then $a = \sup_\lambda a_\lambda$ with respect to $\mathcal{C}_\infty(\Omega)$ and the set $\{\omega \in \Omega : (\sup_\lambda a_\lambda)(\omega) \neq \sup_\lambda a_\lambda(\omega)\}$ is meager in Ω.

Definition 2.5. *A sequence $(r_n)_n$ is said to be order-convergent (or (o)-convergent) to r, if there exists a sequence $(p_n)_n \in R$ such that $p_n \downarrow 0$ and $|r_n - r| \leq p_n$, $\forall\, n \in \mathbb{N}$, and we will write $(o) - \lim_n r_n = r$.*

As $|r_n| \leq |r| + p_1 \;\forall\, n$, every (o)-convergent sequence is bounded. We note that, if R is a σ-Dedekind complete Riesz space, (o)-convergence can be formulated in the following equivalent ways (see also [25]).

Proposition 2.6. *A sequence $(r_n)_n$, bounded in R, (o)-converges to r if and only if $r = (o) - \limsup_n r_n = (o) - \liminf_n r_n$, where $(o) - \limsup_n r_n = \inf_n[\sup_{m \geq n} r_m]$, $(o) - \liminf_n \; r_n = \sup_n[\inf_{m \geq n} r_m]$.*

Proposition 2.7. *Let R be as above, Ω as in Theorem 2.4. A bounded sequence $(r_n)_n$, $r_n \in R$, (o)-converges to r if and only if the set $\{\omega \in \Omega : r_n(\omega) \not\to r(\omega)\}$ is meager in Ω.*

We recall some fundamental properties of the order convergence (see [25]).

Proposition 2.8. *If $(r_n)_n(o)$-converges to both r and s, then $r \equiv s$. If $(r_n)_n(o)$-converges to r, $(s_n)_n(o)$-converges to s and $\alpha \in \mathbb{R}$, then $(r_n + s_n)_n$, $(r_n \vee s_n)_n$, $(r_n \wedge s_n)_n$, $(\alpha r_n)_n$, $(|r_n|)_n(o)$-converge respectively to $r + s$, $r \vee s$, $r \wedge s$, αr, $|r|$.*

Definition 2.9. *A sequence $(r_n)_n$ is said to be (o)-Cauchy if there exists a sequence $(p_n)_n \in R$ such that $p_n \downarrow 0$ and $|r_n - r_m| \leq p_n$, $\forall\, n \in \mathbb{N}$, and $\forall\, m \geq n$.*

Definition 2.10. *A Riesz space R is called (o)-complete if every (o)-Cauchy sequence is (o)-convergent.*

The following result holds (see [17], [28]).

Proposition 2.11. *Every σ-Dedekind complete Riesz space is (o)-complete.*

We note that there are some cases, in which (o)-convergence is not "generated" by a topology. For example, $L^0(X, \mathcal{B}, \mu)$, where μ is a σ-additive non-atomic positive $\widetilde{\mathbb{R}}$-valued measure. We recall that, in such spaces, (o)-convergence coincides with almost everywhere convergence. (Also see [25].)

3 The Monotone Integral

Definition 3.1. *Let X be any set, R a Dedekind complete Riesz space, $\mathcal{A} \subset \mathcal{P}(X)$ an algebra. A map $\mu : \mathcal{A} \to R$ is said to be mean if $\mu(A) \geq 0$, $\forall\, A \in \mathcal{A}$ and $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A \cap B = \emptyset$. A mean μ is countably additive (or σ-additive) if $\mu(\cap_n A_n) = \inf_n \mu(A_n)$, whenever $(A_n)_n$ is a decreasing sequence in $\mathcal{A}$, such that $\cap_n A_n \in \mathcal{A}$.*

Given a mapping $f : X \to \tilde{\mathbb{R}}_0^+$ and a mean μ as above for all $A \in \mathcal{A}$ and $t \in \mathbb{R}_0^+$, set $E_{t,A}^f$ (or simply $E_{t,A}$, when no confusion can arise) $\equiv \{x \in A : f(x) > t\}$; $E_t^f(E_t) \equiv \{x \in X : f(x) > t\}$; and, for every $t > 0$, let $u_{A,f}(t) \equiv \mu(E_{t,A}^f)$; $u_f(t) = u(t) \equiv \mu(E_t)$.

Definition 3.2. *With the same notation as above, we say that a function $f : X \to \tilde{\mathbb{R}}_0^+$ is measurable if $E_t^f \in \mathcal{A}$, $\forall\, t \in \mathbb{R}^+$.*

Now we define a Riemann (Lebesgue)-type integral, for maps, defined in an interval of the real line and taking values in a Dedekind complete Riesz space. (For similar integrals existing in the literature, also see [21] and [20].)

Definition 3.3. *Let $a, b \in \mathbb{R}$, $a < b$, and R be as above. We say that a map $g : [a, b] \to R$ is a step function if there exist $n+1$ points $x_0 \equiv a < x_1 < \ldots < x_n \equiv b$ such that g is constant in each interval of the type $]x_{i-1}, x_i[$ $(i = 1, \ldots, n)$. We say that g is simple if there exist n elements of R, $a_1, \ldots, a_n$, and n pairwise disjoint measurable sets E_i such that $g = \sum_{i=1}^n a_i \chi_{E_i}$. If g is a step (simple) function, we put $\int_a^b g(t)\,dt \equiv \sum_{i=1}^n (x_i - x_{i-1}) \cdot g(\xi_i) \left[\sum_{i=1}^n |E_i| \cdot g(\xi_i)\right]$, where ξ_i is an arbitrary point of $]x_{i-1}, x_i[$ $[E_i]$.*

Definition 3.4. *Let $u : [a, b] \to R$ be a bounded function. We call the upper integral (resp. lower integral) of u the element of R given by*

$$\inf_{v \in V_u} \int_a^b v(t)\,dt \left[\sup_{s \in S_u} \int_a^b s(t)\,dt\right],$$

where

$$V_u \equiv \{v : v \text{ is a step (simple) function, } v(t) \geq u(t),\ \forall\, t \in [a, b]\}$$
$$S_u \equiv \{s : s \text{ is a step (simple) function, } s(t) \leq u(t),\ \forall\, t \in [a, b]\}.$$

We say that u is Riemann (Lebesgue) integrable (or (R), resp. (L)-integrable) if its lower integral coincides with its upper integral and, in this case, we call integral of u (and write $\int_a^b u(t)\,dt$) their common value.

It is easy to check that this integral is well-defined, and is a linear monotone functional, with values in R.

The following result holds.

Proposition 3.5. *Every bounded monotone map $u : [a, b] \to R$ is Riemann integrable.*

PROOF. The proof is almost identical to the classical one. □

Now we define an integral for extended real-valued functions with respect to R-valued means.

Definition 3.6. *Let X, R, μ, $f : X \to \widetilde{\mathbb{R}}_0^+$, $u = u_f$ be as above. We say that f is integrable if the quantity*

$$\textbf{(3.6.1)} \quad \int_0^{+\infty} u(t)\,dt \equiv \sup\nolimits_{a>0} \int_0^a u(t)\,dt = (o) - \lim\nolimits_{a \to +\infty} \int_0^a u(t)\,dt,$$

exists in R where the integral in (3.6.1) is intended as in Definition 3.4. If f is integrable, we denote the element in (3.6.1) by $\int_X f\,d\mu$. A measurable function $f : X \to \mathbb{R}$ is integrable if both f^+, f^- are integrable and, in this case we set $\int_X f\,d\mu = \int_X f^+\,d\mu - \int_X f^-\,d\mu$.

Remark 3.7. *We can extend Definition* 3.6 *when* $\mu : \mathcal{A} \to R$ *is any finitely additive bounded map. A measurable function* f *is integrable if and only if* f *is integrable with respect to* μ^+, μ^-, *where for every* $A \in \mathcal{A}$

$$\mu^+(A) \equiv \vee_{B \subset A, B \in \mathcal{A}} \, \mu(B),$$
$$\mu^-(A) \equiv - \wedge_{B \subset A, B \in \mathcal{A}} \mu(B),$$

and $\mu = \mu^+ - \mu^-$. *In this case, we set* $\int_X f\, d\mu \equiv \int_X f\, d\mu^+ - \int_X f\, d\mu^-$. *(Also see* [7].*)*

An immediate consequence of Definition 3.6 and monotonicity of μ is the following assertion.

Proposition 3.8. *If* f *is integrable, then for each* $A \in \mathcal{A}$, $\sup_{a>0} \int_0^a u_{A,f}(t)\, dt$ *exists in* R *and is denoted by* $\int_A f\, d\mu$.

Proposition 3.9. *With the same notation as above, if* f *is integrable, then*

$$\int_A f\, d\mu = \int_X f \cdot \chi_A \, d\mu, \quad \forall\, A \in \mathcal{A}.$$

PROOF. For each fixed $t > 0$ and $x \in X$, we have $[f \cdot \chi_A(x) > t]$ if and only if $[x \in A]$ and $[f(x) > t]$. So, $u_{X, f\cdot\chi_A} \equiv u_{A,f}$. Thus, the assertion follows. $\square$

It is easy to check that this integral is a linear R-valued functional and that, for every positive integrable map f, $\int_{\cdot} f d\mu$ is a mean.

We now list a number of technical results.

Proposition 3.10. *If* f *is integrable, then* $(o) - \lim_{t\to+\infty} \mu(E_t) = 0$ *and hence* $\mu(E_\infty) = 0$, *where* $E_\infty \equiv \{x \in X : f(x) = +\infty\}$.

PROOF. For every $t > 0$, we have

$$0 \le \mu(E_\infty) \le \mu(E_t) = \frac{\int_{E_t} t\, d\mu}{t} \le \frac{\int_{E_t} f\, d\mu}{t} \le \frac{\int_X f\, d\mu}{t}.$$

Taking the infimum, we obtain $0 \le \mu(E_t) \le \inf_{t>0} \frac{\int_X f\, d\mu}{t} = 0$. $\square$

Proposition 3.11. *Let* $f : X \to \tilde{\mathbb{R}}_0^+$ *be measurable. Then,* f *is integrable if and only if* $\sup_n \int_X (f \wedge n)\, d\mu \in R$, *and in this case* $\sup_n \int_X (f \wedge n)\, d\mu = \int_X f\, d\mu$.

PROOF. Fix $n \in \mathbb{N}$ and pick $t < n$. Then $f(x) \wedge n > t$ if and only if $f(x) > t$ and so $\int_0^n u_f(t)\, dt = \int_0^n u_{f\wedge n}(t)\, dt = \int_0^{+\infty} u_{f\wedge n}(t)\, dt = \int_X (f \wedge n)\, d\mu$. So the first part of the assertion follows immediately. Moreover taking the suprema, we get $\sup_n \int_X (f \wedge n)\, d\mu = (o) - \lim_{n\to+\infty} \int_0^n u_f(t)\, dt = \int_X f\, d\mu$. $\square$

Proposition 3.12. *Let $f : X \to \mathbb{R}_0^+$ be measurable and bounded and set S_f (resp. V_f) $\equiv \{g : X \to \mathbb{R} : g \le f,\ g$ is simple$\}$, (resp. $\{h : X \to \mathbb{R} : h \ge f,\ h$ is simple$\}$). Then $\int_X f\,d\mu = \sup_{g \in S_f} \int_X g\,d\mu = \inf_{h \in V_f} \int_X h\,d\mu$, and f is integrable.*

Proof. It suffices to prove the part involving S_f. Let $L = \sup_{x \in X} f(x)$ and, for every fixed $n \in \mathbb{N}$, let $s_n(0) \equiv u(0)$, and $s_n(t) \equiv u\left(\frac{L\,i}{2^n}\right)$ whenever $t \in \left]\frac{L(i-1)}{2^n}, \frac{L\,i}{2^n}\right]$ $(i = 1, \ldots, 2^n)$. We have $\int_0^L s_n(t)\,dt = \sum_{i=1}^{2^n} \frac{L}{2^n} u\left(\frac{L}{2^n} i\right)$. Put

$$U_i^{(n)} \equiv \left\{x \in X : f(x) > \frac{Li}{2^n}\right\};$$

$$g_n \equiv \sum_{i=1}^{2^n} \frac{L}{2^n} \chi_{U_i^{(n)}}, \forall\, n \in \mathbb{N},\ i = 1, 2, \ldots, 2^n.$$

Then (Also see [9].) $\int_X g_n\,d\mu = \sum_{i=1}^{2^n} \frac{L}{2^n} \mu(U_i^{(n)}) = \sum_{i=1}^{2^n} \frac{L}{2^n} u\left(\frac{L}{2^n} i\right)$. Taking the supremum, we get

$$\int_X f\,d\mu = \int_0^L u(t)\,dt = \sup_n \int_X g_n\,d\mu = (o) - \lim_n \int_X g_n\,d\mu.$$

If $g \in S_f$, then $\int_X g\,d\mu \le \int_X f\,d\mu$, and so $\int_X f\,d\mu = \sup_{n \in \mathbb{N}} \int_X g_n\,d\mu \le \sup_{g \in S_f} \int_X g\,d\mu \le \int_X f\,d\mu$, which completes the proof. □

Proposition 3.13. *If $f : X \to \widetilde{\mathbb{R}}_0^+$ is integrable, then $\displaystyle\int_X f\,d\mu = \sup_{g \in S_f} \int_X g\,d\mu$. Conversely, if $f \ge 0$ is such that the quantity $\sup_{g \in S_f} \int_X g\,d\mu$ exists in R, then f is integrable and $\int_X f\,d\mu = \sup_{g \in S_f} \int_X g\,d\mu$.*

Proof. The assertion follows by Propositions 3.11 and 3.12. □

The following result is easy also.

Proposition 3.14. *Let $f : X \to \widetilde{\mathbb{R}}_0^+$ be an integrable map, $g : X \to \widetilde{\mathbb{R}}_0^+$ measurable such that $0 \le g(x) \le f(x),\ \ \forall\, x \in X$. Then g is integrable, and $\displaystyle\int_X g\,d\mu \le \int_X f\,d\mu$.*

Now we note that if $\mu : X \to R$ is a mean and $\mathcal{C}_\infty(\Omega)$ is as in Theorem 2.4, then there exists a nowhere dense set $\Omega' \subset \Omega$ such that $\mu(A)(\omega)$ is real, $\forall\, \omega \notin \Omega'$, $\forall\, A \in \mathcal{A}$.

Proposition 3.15. *Let $R \subset \mathcal{C}_\infty(\Omega)$ be a Dedekind complete Riesz space where Ω' is as above and set $\mu_\omega(A) \equiv \mu(A)(\omega)$, $\forall\, \omega \notin \Omega'$. Assume that $f : X \to \mathbb{R}$ is an integrable map. Then there exists a meager set $N \subset \Omega$ such that f is integrable with respect to μ_ω and $\int_A f\, d\mu_\omega = \left(\int_A f\, d\mu\right)(\omega)$, $\forall\, \omega \in N^c$, $\forall\, A \in \mathcal{A}$.*

PROOF. Without loss of generality, we can assume that f is nonnegative. First suppose that f is bounded. There exists a sequence of simple functions $(s_n)_n$ such that $s_n \uparrow f$ and $\int s_n\, d\mu \uparrow \int f\, d\mu$. So we have, for every $n \in \mathbb{N}$, up to the complement of a meager set, depending only on X

$$\begin{aligned}
0 &\leq \left| \int_A f\, d\mu_\omega - \left(\int_A f\, d\mu\right)(\omega) \right| \\
&\leq \left| \int_A f\, d\mu_\omega - \int_A s_n\, d\mu_\omega \right| + \left| \int_A s_n\, d\mu_\omega - \left(\int_A f\, d\mu\right)(\omega) \right| \\
&= \left| \int_A f\, d\mu_\omega - \int_A s_n\, d\mu_\omega \right| + \left| \left(\int_A s_n\, d\mu\right)(\omega) - \left(\int_A f\, d\mu\right)(\omega) \right| \\
&\leq \int_X f - s_n\, d\mu_\omega + \left(\int_X f - s_n\, d\mu\right)(\omega).
\end{aligned}$$

Then

$$\begin{aligned}
0 &\leq \left| \int_A f\, d\mu_\omega - \left(\int_A f\, d\mu\right)(\omega) \right| \\
&\leq \limsup_n \int_X f - s_n\, d\mu_\omega + \limsup_n \left(\int_X f - s_n\, d\mu\right)(\omega) \\
&= \inf_n \int_X f - s_n\, d\mu_\omega + \inf_n \left(\int_X f - s_n\, d\mu\right)(\omega) = 0.
\end{aligned}$$

Assume now that f is integrable. By the previous step, there exists a meager set N^* such that, $\forall\, n \in \mathbb{N}$, $\forall\, \omega \notin N^*$, $\forall\, A \in \mathcal{A}$

$$\int_A (f \wedge n)\, d\mu_\omega = \left(\int_A f \wedge n\, d\mu\right)(\omega).$$

The proof is now analogous to the first part. It will be enough to replace s_n with $f \wedge n$. $\square$

Now we prove the following theorem.

Theorem 3.16. *Let $f : X \to \tilde{\mathbb{R}}_0^+$ be an integrable map. Then there exists a meager set N such that for every $A \in \mathcal{A}$ and for every $\omega \notin N$,*

$$\left(\int_A f\,d\mu\right)(\omega) \in (\mu(A)\,\overline{\mathrm{co}}\,\{f(x) : x \in A\})(\omega).$$

PROOF. By Proposition 3.15 and classical results we have, up to the complement of a meager set

$$\begin{aligned}\left(\int_A f\,d\mu\right)(\omega) &= \int_A f\,d\mu_\omega \in \mu_\omega(A)\,\overline{\mathrm{co}}\,\{f(x), x \in A\} \\ &= \overline{\mathrm{co}}\,\{f(x)\mu_\omega(A), x \in A\} = (\mu(A)\,\overline{\mathrm{co}}\,\{f(x), x \in A\})(\omega).\end{aligned}$$ □

For the definition of absolute continuity and related remarks, see ([4]).

Proposition 3.17. *If $f : X \to \tilde{\mathbb{R}}_0^+$ is integrable, then the integral $\int_\cdot f\,d\mu$ is absolutely continuous; that is, $(o) - \lim_n \int_{A_n} f\,d\mu = 0$ whenever $(A_n)_n$ is a sequence in $\mathcal{A}$ such that $(o) - \lim_n \mu(A_n) = 0$.*

PROOF. The assertion is trivial when f is bounded. So we prove absolute continuity in the general case. Fix $n, k \in \mathbb{N}$, and pick $(A_n)_n$, with $(o) - \lim_n \mu(A_n) = 0$. We have

$$\begin{aligned}0 \le \int_{A_n} f\,d\mu &= \int_{A_n} (f \wedge k)\,d\mu + \int_{A_n} f - (f \wedge k)\,d\mu \\ &\le \int_{A_n} (f \wedge k)\,d\mu + \int_X f - (f \wedge k)\,d\mu.\end{aligned}$$

As $(o) - \lim_k \int_X f - (f \wedge k)\,d\mu = 0$ and $(o) - \lim_n \int_{A_n} (f \wedge k)\,d\mu = 0$ for each $k \in \mathbb{N}$, there exist a sequence $(r_k)_k$ in R, $r_k \downarrow 0$, and a double sequence $(r'_{n,k})_{n,k}$ in R, $r'_{n,k} \downarrow 0$ ($n \to +\infty$, $k = 1, 2, \dots$) such that

$$0 \le \int_{A_n} f\,d\mu \le r'_{n,k} + r_k, \quad \forall\, n, k \in \mathbb{N}.$$

It follows that

$$0 \le (o) - \limsup_{n\to+\infty} \int_{A_n} f\,d\mu \le ((o) - \limsup_{n\to+\infty} r'_{n,k}) + r_k = r_k, \quad \forall\, k \in \mathbb{N}.$$

By the arbitrariness of k, we get $(o) - \limsup_{n\to+\infty} \int_{A_n} f\,d\mu = 0$ and hence $(o) - \lim_{n\to+\infty} \int_{A_n} f\,d\mu = 0$. □

Now we will prove a Vitali-type theorem for our integral.

Definition 3.18. *Let* $(f_n : X \to \widetilde{\mathbb{R}})_n$ *be a sequence of integrable functions. We say that* $(f_n)_n$ *is uniformly integrable if* $\sup_n \int_X |f_n|\, d\mu \in R$*and* $(o) - \lim_n \sup_{k \geq n} \left(\int_{A_n} |f_k|\, d\mu\right) = 0,$*whenever* $(o) - \lim_k \mu(A_k) = 0$.

Definition 3.19. *Under the same hypotheses and notation as above, we say that* $(f_n)_n$ *converges in* L^1 *to* f *if* $(o) - \lim_n \int_X |f_n - f|\, d\mu = 0$.

Remark 3.20. *It is easy to check that* $(f_n)_n$ *converges in* L^1 *to* f *if and only if* $\int_A f\, d\mu = (o) - \lim_{n\to+\infty} \int_A f_n\, d\mu$ *uniformly with respect to* $A \in \mathcal{A}$.

Theorem 3.21. *(Vitali's theorem) Under the same notation as above, let* $(f_n)_n$ *be a uniformly integrable sequence of functions, convergent in measure to* f*. Then* f *is integrable and* $(f_n)_n$ *converges in* L^1 *to* f.

Conversely, every sequence (f_n) *of integrable functions, convergent in* L^1 *to an integrable map* f*, is convergent in measure to* f *and uniformly integrable.*

PROOF. To obtain the integrability of $|f|$, it is enough to prove that

$$\sup S_{|f|} \equiv \sup \left\{ \int_X \varphi\, d\mu : 0 \leq \varphi \leq |f| \quad \text{and} \quad \varphi \text{ is simple} \right\} \in R, \tag{1}$$

by virtue of Proposition 3.13. Let $\varphi \in S_{|f|}$, $\varphi = \sum_{j=1}^k c_j\ \chi_{B_j}$. Fix $j = 1, 2, \ldots, k$ and for every $n \in \mathbb{N}$, set $A_n \equiv E_1^{|f-f_n|}$. If $x \in {A_n}^c \cap B_j$, we have $\varphi(x) = c_j \leq |f_n(x)| + 1$ and hence $\int_{B_j \cap A_n^c} \varphi(x)\, d\mu \leq \int_{B_j} |f_n(x)|\, d\mu + \mu(B_j)$. As to $A_n \cap B_j$, we have $\int_{B_j \cap A_n} \varphi(x)\, d\mu \leq c_j \mu(A_n)$. Thus

$$\int_{B_j} \varphi(x)\, d\mu \leq \int_{B_j} |f_n(x)|\, d\mu + \mu(B_j) + c_j \mu(A_n),$$
$$\int_X \varphi(x)\, d\mu \leq \int_X |f_n(x)|\, d\mu + \mu(X) + \mu(A_n) \sum_{j=1}^k c_j.$$

By convergence in measure, $(o) - \lim_{n\to+\infty} \mu(A_n) \sum_{j=1}^k c_j = 0$ and since n is arbitrary, $\int_X \varphi\, d\mu \leq \sup_n \int_X |f_n|\, d\mu + \mu(X) \in R$.Since the right hand side does not depend on φ, (1) follows. So $|f|$ is integrable. By Proposition 3.14, f^+ and f^- are integrable and so is f.

Now fix $\varepsilon > 0$ and $n \in \mathbb{N}$. As f_n is integrable by hypothesis, so is $f - f_n$.

We have

$$\begin{aligned}\int_X |f_n - f|\,d\mu &\le \int_{\{x\in X:|f_n-f|\le\varepsilon\}} |f_n - f|\,d\mu + \int_{\{x\in X:|f_n-f|>\varepsilon\}} |f_n - f|\,d\mu \\ &\le \int_X \varepsilon\,d\mu + \int_{\{x\in X:|f_n-f|>\varepsilon\}} |f_n|\,d\mu + \int_{\{x\in X:|f_n-f|>\varepsilon\}} |f|\,d\mu \\ &\le \varepsilon\cdot\mu(X) + \sup_{k\ge n}\int_{\{x\in X:|f_n-f|>\varepsilon\}} |f_k|\,d\mu + \int_{\{x\in X:|f_n-f|>\varepsilon\}} |f|\,d\mu.\end{aligned}$$

As $(o)-\lim_n \mu(\{x\in X : |f-f_n|>\varepsilon\}) = 0$, by virtue of uniform integrability of $(f_k)_k$, integrability of f and absolute continuity of the integral we get

$$(o)-\lim_{n\to+\infty}\left[\sup_{k\ge n}\int_{\{x\in X:|f_n-f|>\varepsilon\}} |f_k|\,d\mu + \int_{\{x\in X:|f_n-f|>\varepsilon\}} |f|\,d\mu\right] = 0.$$

So there exists a sequence $(r_n)_n$ in R, $r_n \downarrow 0$ such that

$$0 \le \int_X |f_n - f|\,d\mu \le \varepsilon\cdot\mu(X) + r_n, \quad \forall\, n\in\mathbb{N}.$$

Thus we obtain

$$\begin{aligned}0 \le&(o)-\limsup_{n\to+\infty}\int_X |f_n - f|\,d\mu \le \varepsilon\cdot\mu(X) + (o)-\limsup_{n\to+\infty} r_n \\ =&\varepsilon\cdot\mu(X) + \inf_{n\in\mathbb{N}} r_n = \varepsilon\cdot\mu(X).\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we get $(o)-\lim_{n\to+\infty}\int_X |f_n - f|\,d\mu = 0$.

Conversely, suppose that $(f_n)_n$ converges in L^1 to f. Fix $\varepsilon > 0$ and set

$$E_\varepsilon^{|f-f_n|} \equiv \{x\in X : |f_n(x) - f(x)| > \varepsilon\}, \quad \forall\, n\in\mathbb{N}.$$

Then

$$\frac{\int_X |f_n - f|\,d\mu}{\varepsilon} \ge \frac{\int_{E_\varepsilon^{|f-f_n|}} |f_n - f|\,d\mu}{\varepsilon} \ge \mu(E_\varepsilon^{|f-f_n|}) \ge 0,$$

and hence $(o)-\lim_n\ \mu(E_\varepsilon^{|f-f_n|}) = 0$.

Now we prove uniform integrability. By convergence in L^1, it follows immediately that $\sup_k \int_X |f_k|\,d\mu \in R$. Let $(A_n)_n$ be a sequence in $\mathcal{A}$ such that $(o)-\lim_n \mu(A_n) = 0$. Fix $n\in\mathbb{N}$. For every $k\ge n$ we have

$$\int_{A_n} |f_k|\,d\mu \le \int_{A_n} |f_k - f|\,d\mu + \int_{A_n} |f|\,d\mu \le \int_X |f_k - f|\,d\mu + \int_{A_n} |f|\,d\mu.$$

By convergence in L^1, there exists a sequence $(r_k)_k$ in R, $r_k \downarrow 0$ such that $\int_X |f_k - f|\, d\mu \le r_k \le r_n$. Thus $\sup_{k \ge n} \int_{A_n} |f_k|\, d\mu \le r_n + \int_{A_n} |f|\, d\mu$. So

$$0 \le (o) - \limsup_{n \to +\infty} \sup_{k \ge n} \int_{A_n} |f_k|\, d\mu \le \inf_n r_n + (o) - \limsup_{n \to +\infty} \int_{A_n} |f|\, d\mu = 0$$

and hence $(o) - \lim_{n \to +\infty} \sup_{k \ge n} \int_{A_n} |f_k|\, d\mu = 0$. □

A consequence of Vitali's theorem is the following theorem.

Theorem 3.22. *(Lebesgue dominated convergence theorem) Let $(f_n)_n$, f_n be a sequence of measurable functions and suppose that there exists an integrable map h such that $|f_n(x)| \le |h(x)|$ for all $n \in \mathbb{N}$ and almost everywhere with respect to x. Furthermore assume that $(f_n)_n$ converges in measure to f. Then for every $n \in \mathbb{N}$, f_n is integrable and $(f_n)_n$ converges in L^1 to f.*

PROOF. Without loss of generality, we suppose that

$$|f_n(x)| \le |h(x)|, \quad \forall\, n \in \mathbb{N}, \quad \forall\, x \in X.$$

By integrability of $|h|$ and Proposition 3.14, f_n is integrable for every $n \in \mathbb{N}$. Moreover by virtue of absolute continuity of the integral of h, the hypotheses of Theorem 3.21 hold. So the assertion follows. □

As a consequence of Theorem 3.22, we prove the following theorem, that is a sufficient condition for the convergence in L^1, inspired by a well-known result of Scheffé's ([23]):

Theorem 3.23. *With the same notation as above, let $(f_n)_n : X \to \widetilde{\mathbb{R}}_0^+$ be a sequence of integrable functions, convergent in measure to a nonnegative integrable mapping f. Assume that $\int_X f_n\, d\mu (o)$-converges to $\int_X f\, d\mu$. Then $(f_n)_n$ converges in L^1 to f.*

PROOF. Let $h_n(x) = f_n(x) - f(x)$, $\forall\, x \in X$. Thus $0 \le [h_n(x)]^- \le f(x)$, $\forall\, x$. Let $H_n(x) = [h_n(x)]^-$, $\forall\, x$. Then f, H_n are integrable for every n and $(H_n)_n$ converges in measure to 0. By Theorem 3.22, we have $0 = (o) - \lim_n \int_X [h_n(x)]^-\, d\mu$ and so $(o) - \lim_n \int_X [h_n(x)]^+\, d\mu = (o) - \lim_n \int_X h_n\, d\mu = 0$, by hypothesis. Finally we get

$$\begin{aligned}(o) - \lim_n \int_X |h_n|\, d\mu =& (o) - \lim_n \int_X [h_n(x)]^+\, d\mu \qquad (2)\\ &+ (o) - \lim_n \int_X [h_n(x)]^-\, d\mu = 0.\end{aligned}$$

□

We now state a version of the monotone convergence theorem.

Theorem 3.24. *With the same notation as above, let $(f_n)_n$ be an increasing sequence of non negative integrable maps, convergent in measure to an integrable function f. Then $\int_X f\,d\mu = (o) - \lim_n \int_X f_n\,d\mu$ and therefore $f_n \to f$ in L^1.*

Proof. It is an immediate consequence of Vitali's Theorem.

4 Countably Additive Case

If μ is countably additive, convergence almost everywhere implies convergence in measure; this can be proved along classical lines. Hence we simply state the results. So both Levi's theorem and Fatou's lemma hold.

Proposition 4.1. *Let R be a Dedekind complete Riesz space, $\mathcal{A} \subset \mathcal{P}(X)$ a σ-algebra, and assume that $\mu : \mathcal{A} \to R$ is a σ-additive mean. Set*

$$A_n^{\varepsilon} \equiv \{x \in X : |f_n(x) - f(x)| > \varepsilon\}, \quad \forall\, \varepsilon > 0.$$

Then, f_n converges almost everywhere to f if and only if $\mu(\limsup_n \ A_n^{\varepsilon}) = 0$, $\forall\, \varepsilon > 0$.

It is easy to prove the following.

Proposition 4.2. *Let R, $\mathcal{A}$ and μ be as above, and assume that μ is σ-additive. Then for each sequence (A_n) in $\mathcal{A}$ one has*

$$\mu(\liminf_n \ A_n) \le \liminf_n \mu(A_n) \le \limsup_n \mu(A_n) \le \mu(\limsup_n A_n).$$

A straightforward consequence of Proposition 4.2 is the following.

Theorem 4.3. *Let f_n, f and μ be as above. If (f_n) converges to f almost everywhere, (f_n) converges to f in measure.*

From Theorems 3.24 and 4.3, and by the proceeding as in the classical case, the next theorem follows.

Theorem 4.4. *With the same notation and hypotheses as above, let $(f_n)_n$ be an increasing sequence of nonnegative measurable maps. Then $f(x) \equiv \lim_n f_n(x)$ is integrable if and only if $\lim_n \int_X f_n\,d\mu \in R$ and in this case*

$$\int_X f\,d\mu = (o) - \lim_n \int_X f_n\,d\mu.$$

A consequence of Beppo Levi's Theorem is the following version of Fatou's Lemma.

Theorem 4.5. *Let X, R, μ be as above, $(f_n)_n$ a sequence of nonnegative integrable maps, $f(x) \equiv \liminf_n f_n(x)$, $\forall\, x \in X$. If $\liminf_n \int_X f_n\, d\mu \in R$, then f is integrable and $\liminf_n \int_X f_n\, d\mu \geq \int_X f\, d\mu$.*

5 Radon-Nikodym Theorem

In this section we give a Greco-type condition for the existence of a Radon-Nikodym derivative for the monotone integral introduced in the previous section (see [14]). We show that the Radon-Nikodym problem, in general, has no solutions. Indeed, there exist two $\mathbb{R}^2$-valued σ-additive means μ and ν, with $\nu \ll \mu$, such that there is no function $f : X \equiv \{0, 1\} \to \mathbb{R}$ such that $\nu = \int_X f\, d\mu$.

Let $X \equiv \{0, 1\}$, $\mathcal{A} \equiv \mathcal{P}(X)$, $R \equiv \mathbb{R}^2$ (endowed with componentwise ordering), $\mu, \nu : \mathcal{P}(X) \to \mathbb{R}^2$ defined by setting

$$\mu(\{0\}) = (1, 0), \quad \mu(\{1\}) = (0, 1), \quad \nu(\{0\}) = (0, 1), \quad \nu(\{1\}) = (1, 0).$$

It is easy to check that μ and ν are σ-additive, ν is absolutely continuous with respect to μ and μ is absolutely continuous with respect to ν. However there is no function $f : X \to \mathbb{R}$ such that $\nu(A) = \int_A f\, d\mu$, $\forall\, A \in \mathcal{P}(X)$ for otherwise, we would have $(1, 0) = \nu(\{1\}) = \int_{\{1\}} f\, d\mu = f(1)\mu(\{1\}) = (0, f(1))$, which is a contradiction.

Furthermore it is easy to see that for every $r > 0$ there exists no Hahn decomposition for the map $\nu - r\mu$. $\square$

Now we introduce two preliminary lemmas.

Proposition 5.1. *Let $\mu, \nu : \mathcal{A} \to R$ be two means with $\nu \ll \mu$. If there exists an $\mathcal{A}$-measurable function $f : X \to \widetilde{\mathbb{R}}_0^+$ such that, for every $E \in \mathcal{A}$*

$$\nu(E) = \int_E f\, d\mu,$$

then, for every $r > 0$, the set $A_r = \{x \in X : f(x) > r\}$ satisfies

(5.1.1) *$\nu(E) \geq r\mu(E)$ for every $E \in A_r \cap \mathcal{A}$,*

(5.1.2) *$\nu(E) \leq r\mu(E)$ for every $E \in A_r^c \cap \mathcal{A}$,*

(5.1.3) $(o) - \lim_{r \to +\infty} \nu(A_r) = 0$.

PROOF. $A_r \in \mathcal{A}$ for every $r > 0$ since f is measurable. Moreover for every $r > 0$ and for every $E \in A_r \cap \mathcal{A}$, $F \in A_r^c \cap \mathcal{A}$ we have

$$\nu(E) = \int_E f\, d\mu \geq \int_E r\, d\mu = r\mu(E)$$
$$\nu(F) = \int_F f\, d\mu \leq \int_F r\, d\mu = r\mu(F).$$

This proves (5.1.1) and (5.1.2).

(5.1.3) is a consequence of (5.1.1). In fact (5.1.1) yields

$$\mu(A_r) \leq \frac{\nu(A_r)}{r} \leq \frac{\nu(X)}{r}, \quad \forall\, r > 0.$$

So $(o) - \lim_{r\to+\infty} \mu(A_r) = 0$, and hence $(o) - \lim_{r\to+\infty} \nu(A_r) = 0$. □

Proposition 5.2. *Let μ, $\nu : \mathcal{A} \to R$ be two means with $\nu \ll \mu$. Let $D \equiv \Big\{ \frac{i}{2^n}, i, n \in \mathbb{N} \Big\}$. If there exists a decreasing family $(A_r)_{r\in D}$ such that $A_0 = X$ and satisfying* (5.1.1) *and* (5.1.2)*, then the function $f : X \to [0, +\infty]$, defined by $f(x) \equiv \sup\{r \in D : x \in A_r\}$, is integrable and $\nu(E) = \int_E f\, d\mu$, $\forall\, E \in \mathcal{A}$.*

PROOF. f is $\mathcal{A}$-measurable, since, $\forall\, t > 0$, $\{x \in X : f(x) > t\} = \cup_{r\in D, r>t} A_r$. Let $f_n \equiv \frac{1}{2^n} \sum_{k=1}^{n2^n} \chi_{A_{\frac{k}{2^n}}}$, for every $n \in \mathbb{N}$. Then $f \wedge n - f \wedge \frac{1}{2^n} \leq f_n \leq f$, $\forall\, n$. By construction for every $E \in \mathcal{A}$,

$$\begin{aligned}\int_E f_n\, d\mu =& \frac{1}{2^n} \sum_{k=1}^{n2^n} \mu(A_{\frac{k}{2^n}}) \\ =& \sum_{k=1}^{n2^n-1} \frac{k}{2^n} \Big[\mu(A_{\frac{k}{2^n}} \cap E) - \mu(A_{\frac{k+1}{2^n}} \cap E)\Big] + n\mu(A_n \cap E) \\ \leq& \sum_{k=1}^{n2^n-1} \Big[\nu(A_{\frac{k}{2^n}} \cap E) - \nu(A_{\frac{k+1}{2^n}} \cap E)\Big] + n\nu(A_n \cap E) \leq \nu(E).\end{aligned}$$

So $\sup_n \int_X f_n\, d\mu \leq \nu(X) \in R$ and thus

$$\sup_n \int_X (f \wedge n)\, d\mu \leq \sup_n \int_X (f_n + 1)\, d\mu \leq \nu(X) + \mu(X).$$

So by Proposition 3.11, f is integrable and hence, by Proposition 3.8, $f \cdot \chi_E$

is integrable, $\forall\, E \in \mathcal{A}$. Thus

$$\begin{aligned}(o)-\lim_n \left[\int_E (f\wedge n)\,d\mu - \int_E \left(f \wedge \frac{1}{2^n}\right)\,d\mu\right] =& (o)-\lim_n \int_E (f\wedge n)\,d\mu \\ =& \int_E f\,d\mu,\end{aligned}$$

and therefore $(o)-\lim_n \int_E f_n\,d\mu = \int_E f\,d\mu$ and $\int_E f\,d\mu \le \nu(E), \quad \forall\, E \in \mathcal{A}$. On the other hand,

$$\begin{aligned}\int_E f_n\,d\mu =& \sum_{k=1}^{n2^n-1} \frac{k+1}{2^n}\left[\mu(A_{\frac{k}{2^n}} \cap E) - \mu(A_{\frac{k+1}{2^n}} \cap E)\right] + n\mu(A_n \cap E) + \\ & - \frac{1}{2^n}\sum_{k=1}^{n2^n-1}\left[\mu(A_{\frac{k}{2^n}} \cap E) - \mu(A_{\frac{k+1}{2^n}} \cap E)\right] \\ \ge& \nu(A_{\frac{1}{2^n}} \cap E) - \nu(A_n \cap E) - \frac{1}{2^n}\left(\mu(A_{\frac{k}{2^n}}) - \mu(A_n \cap E)\right).\end{aligned}$$

Taking the (o)-limits as $n \to \infty$, we obtain $\int_E f\,d\mu = \nu(E)$. □

A consequence of Proposition 5.1 and 5.2 is the following Radon-Nikodym Theorem.

Theorem 5.3. *Let $\mu, \nu : \mathcal{A} \to R$ be two means with $\nu \ll \mu$. Then the following are equivalent:*

(5.3.a) *there exists an $\mathcal{A}$-measurable function $f : X \to \widetilde{\mathbb{R}}_0^+$ such that, for every $E \in \mathcal{A}$ we have $\nu(E) = \int_E f\,d\mu$,*

(5.3.b) *there exists a family $(A_r)_{r>0}$ of measurable sets such that for every $r > 0$*

(5.3.b.1) *$\nu(E) \ge r\mu(E)$ for every $E \in A_r \cap \mathcal{A}$,*

(5.3.b.2) *$\nu(E) \le r\mu(E)$ for every $E \in A_r^c \cap \mathcal{A}$.*

The following is a different formulation of Theorem 5.3.

Theorem 5.4. *Let $\mu, \nu : \mathcal{A} \to R$ be two means with $\nu \ll \mu$. Then the following are equivalent:*

(5.4.a) *there exists a $\mathcal{A}$-measurable function $f : X \to \widetilde{\mathbb{R}}_0^+$ such that, for every $E \in \mathcal{A}$ we have $\nu(E) = \int_E f\,d\mu$,*

(5.4.b) *for every $r > 0$ the measure $\nu - r\mu$ admits a Hahn decomposition, namely there exist two disjoint measurable sets (B_r, C_r) such that, $\forall E \in \mathcal{A}$*

$$\begin{aligned}(\nu - r\mu)^+(E) =&(\nu - r\mu)(E \cap B_r)\\ (\nu - r\mu)^-(E) =&(\nu - r\mu)(E \cap C_r).\end{aligned}$$

Proof. (5.4.a) $\Longrightarrow$ (5.4.b)

By Theorem 5.3, there exists a family $(A_r)_{r>0}$ of measurable sets such that, for every $r > 0$

(5.3.b.1) $\nu(E) \geq r\mu(E)$ for every $E \in A_r \cap \mathcal{A}$,

(5.3.b.2) $\nu(E) \leq r\mu(E)$ for every $E \in A_r^c \cap \mathcal{A}$.

Set $B_r \equiv A_r$, $C_r \equiv A_r^c$. For every $E \in A_r \cap \mathcal{A}$ we have

$$\begin{aligned}(\nu - r\mu)^+(E) =&(\nu - r\mu)^+(E \cap A_r) + (\nu - r\mu)^+(E \cap A_r^c)\\ =&(\nu - r\mu)^+(E \cap A_r) = (\nu - r\mu)(E \cap A_r)\end{aligned}$$

from (5.3.b.1), since $(\nu - r\mu)(F) \leq 0$, $\forall F \in E \cap A_r^c \cap \mathcal{A}$. So we obtain, for every $E \in \mathcal{A}$, $(\nu - r\mu)^+(E) = (\nu - r\mu)(E \cap B_r)$. Analogously for each $E \in \mathcal{A}$, $(\nu - r\mu)^-(E) = (\nu - r\mu)(E \cap C_r)$. (5.4.b) $\Longrightarrow$ (5.4.a)

It is easy to check that, if (5.4.b) holds, then (5.3.b.1.) and (5.3.b.2.) are satisfied. The assertion follows by Proposition 5.2. □

References

[1] C. D. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces*, Academic Press, 1978.

[2] S. J. Bernau, *Unique representation of Archimedean lattice groups and normal Archimedean lattice rings*, Proc. London Math. Soc., **15** (1965), 599–631.

[3] A. Boccuto, *Riesz spaces, integration and sandwich theorems*, Tatra Mountains Math. Publ., **3** (1993), 213–230.

[4] A. Boccuto, *Abstract integration in Riesz spaces*, Tatra Mountains Math. Publ., **5** (1995), 107–124.

[5] J. K. Brooks and A. Martellotti, *On the De Giorgi-Letta integral in infinite dimensions*, Atti Sem. Mat. Fis. Univ. Modena, **4** (1992), 285–302.

[6] G. Choquet, *Theory of capacities*, Annales de l'Istitut de Fourier, **5** (1953-54), 131–295.

[7] M. Congost Iglesias, *Medidas y probabilidades en estructures ordenadas*, Stochastica, **5** (1981), 45–68.

[8] R. Cristescu, *On integration in ordered vector spaces and on some linear operators*, Rend. Circ. Mat. Palermo, Suppl., **33** (1993), 289–299.

[9] E. De Giorgi and G. Letta, *Une notion générale de convergence faible des fonctions croissantes d'ensemble*, Ann. Scuola Sup. Pisa, **33** (1977), 61–99.

[10] D. Denneberg, *Non-additive measure and integral*, Kluwer Acad. Publ., 1994.

[11] M. Duchoň and B. Riečan, *On the Kurzweil-Stieltjes integral in ordered spaces*, to appear on Tatra Mountains Math. Publ.

[12] I. Gilboa and D. Schmeidler, *Additive representation of non-additive measures and the Choquet integral*, Ann. Oper. Res., **52** (1994), 43–65.

[13] G. H. Greco, *Integrale monotono*, Rend. Sem. Mat., Univ. Padova, **57** (1977), 149–166.

[14] G. H. Greco, *Un teorema di Radon-Nikodym per funzioni d'insieme subadditive*, Ann. Univ. Ferrara, **27** (1981), 13–19.

[15] J. Haluška, *On integration in complete vector lattices*, Tatra Mountains Math. Publ., **3** (1993), 201–212.

[16] M. C. Isidori, A. Martellotti and A. R. Sambucini, *The monotone integral*, preprint, 1995.

[17] W. A. J. Luxemburg and A. C. Zaanen, *Riesz Spaces I*, North-Holland Publishing Co., 1971.

[18] P. Maličký, *Random variables with values in a vector lattice (mean value and conditional mean value operators)*, Acta Math. Univ. Comenianae, **52-53** (1987), 249–263.

[19] P. Mcgill, *Integration in vector lattices*, J. Lond. Math. Soc., **11** (1975), 347–360.

[20] E. J. Mcshane, *A Riemann-type integral that includes Lebesgue-Stieltjes, Bochner and stochastic integrals*, Mem. Amer. Math. Soc., **88** (1969).

[21] B. Riečan, *On the Kurzweil integral for functions with values in ordered spaces I*, Acta Math. Univ. Comenian., **56-57** (1989), 411–424.

[22] B. Riečan and M. Vrábelová, *On the Kurzweil integral for functions with values in ordered spaces II*, Math. Slovaca, **43** (1993), 471–475.

[23] H. Scheffe, *A useful convergence theorem for probability distributions*, Ann. Math. Stat., **18** (1947) 434–438.

[24] J. Šipoš, *Integral with respect to a pre-measure*, Math. Slov., **29** (1979), 141–155.

[25] B. Z. Vulikh, *Introduction to the theory of partially ordered spaces*, Wolters-Noordhoff Sci. Publ., Groningen, 1967.

[26] J. D. M. Wright, *Stone-algebra-valued measures and integrals*, Proc. Lond. Math. Soc., **19** (1969), 107–122.

[27] J. D. M. Wright, *Measures with values in a partially ordered vector space*, Proc. London Math. Soc., **25** (1972), 675–688.

[28] A. C. Zaanen, *Riesz spaces II*, North-Holland Publishing Co., 1983.