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CHAOTIC MAPS IN HYPERSPACES

Abstract

The dynamical system $(\mathcal{F}(X), T)$ which arises from an iterated function system $(X; w_1, \ldots, w_m)$, where X is a compact metric space identified with the attractor of the system and the w_i's are contractive invertible maps, is chaotic provided that the iterated function system satisfies the open set condition. The map T on the hyperspace $\mathcal{F}(X)$ of the closed subsets of X is defined for a closed subset E as

$$T(E) = w_1^{-1}(E) \cup \ldots \cup w_m^{-1}(E).$$

This extends results about the shift dynamical system for the non-overlapping case [1].

1 Notation

Let $(X; w_1, \ldots, w_m)$ be an iterated function system. X denotes a compact metric space with some metric d. The w_i for $i = 1, \ldots, m$ are invertible contractive maps $w_i : X \to X$ such that $d(w_i(x), w_i(y)) \leq r_i d(x, y)$ for all $x, y \in X$ and some $0 < r_i < 1$ with $i = 1, \ldots, m$. Note that $w_i^{-1} : w_i(X) \to X$ is a continuous map for all i. For simplicity we assume that X is also the attractor of the given iterated function system which means

$$X = w_1(X) \cup w_2(X) \cup \ldots \cup w_m(X).$$

We always assume that $w_i(X) \cap w_j(X) = \emptyset$ for $i \neq j$, $i, j = 1, \ldots, m$. This implies that X is totally disconnected. If this property holds, a map $T : X \to X$ can be uniquely defined by

$$T(x) = w_i^{-1}(x) \text{ provided that } x \in w_i(X).$$

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The dynamical system (X,T) is called the shift dynamical system associated with a totally disconnected hyperbolic IFS. It can be proved that it is chaotic; that is

1. (X,T) is sensitive to initial conditions; i.e. there exists some $\delta > 0$ such that for any $x \in X$ and any ball $B(x,\varepsilon)$ with radius $\varepsilon > 0$ there is some $y \in B(x,\varepsilon)$ and an integer $n \geq 0$ such that $d(T^n(x), T^n(y)) > \delta$;

2. (X,T) is transitive, i.e. if, whenever U and V are open subsets of X, there exists an integer n such that $U \cap T^n(V) \neq \emptyset$;

3. the set of periodic points of T is dense in X.

If the subsets $w_i(X)$ overlap, T cannot be defined in this way. It may happen that more than one w_i^{-1} can be applied to x. In [1] the construction of a so called lifted IFS is recommended. This ensures that the lifted map $\tilde{T}$ can again be defined in a unique way. To this end, let $\Sigma = \prod_{i=1}^{\infty}\{1,\ldots,m\}$ and

$$d_C(\omega,\sigma) = \sum_{n=1}^{\infty} \frac{|\omega_n - \sigma_n|}{(m+1)^n}.$$

The space (Σ, d_C) is called the code space on the m symbols $\{1,\ldots,m\}$. The following is well-known [1]. For each $\sigma \in \Sigma, n \in \mathbb{N}$, and $x \in X$ let

$$\phi(\sigma,n,x) = w_{\sigma_1} \circ w_{\sigma_2} \circ \cdots \circ w_{\sigma_n}(x).$$

Then the limit $\phi(\sigma) = \lim_{n\to\infty} \phi(\sigma,n,x)$ exists, belongs to the attractor of the IFS, and is independent of $x \in X$. $\phi : \Sigma \to X$ is a continuous function from the code space onto the attractor X of the IFS. An address of $x \in X$ is any member of the set

$$\phi^{-1}(x) = \big\{\omega \in \Sigma; \phi(\omega) = x\big\}.$$

The lifted IFS associated with an IFS $(X; w_1,\ldots,w_m)$ is the IFS $(X \times \Sigma; \tilde{w}_1,\ldots,\tilde{w}_m)$ where $\tilde{w}_i(x,\sigma) = (w_i(x), i\sigma)$ for all $(x,\sigma) \in X \times \Sigma$ and all $i = 1,\ldots,m$. Its attractor becomes totally disconnected and $\tilde{T}$ can be uniquely defined in the same way as T before.

The IFS is said to be totally disconnected if each point of X possesses a unique address. The IFS is said to be just touching if it is not totally disconnected yet X contains an open set O such that

(i) $w_i(O) \cap w_j(O) = \emptyset$ for $i \neq j$,

(ii) $\bigcup_{i=1}^{m} w_i(O) \subset O$.

An IFS whose attractor obeys (i) and (ii) is said to obey the open set condition. For the open set O we have $X = \bar{O}$ [2]. The IFS is said to be overlapping if it is neither just touching nor disconnected.

2 The Main Result

We give a sequence of lemmas.

Lemma 1 *If the open set condition is satisfied with the open set O and*

$$A_u = \bigcap_{n=1}^{\infty} \Big(\bigcup \big\{ w_{\sigma_1} \circ \ldots \circ w_{\sigma_n}(O) \,|\, \sigma_1, \ldots, \sigma_n \in \{1, \ldots, m\} \big\} \Big),$$

then A_u is a dense subset of X which consists of points with a unique address.

PROOF. This follows immediately by Baire's Category Theorem and the properties of the open set O. □

Example 1 *Let $a \in [0,1]$ and define $w_1(x) = ax$ and $w_2(x) = ax + (1-a)$ on $\mathbb{R}$. Then the attractor X of the IFS $\{\mathbb{R}; w_1, w_2\}$ is equal to $[0,1]$ for $a \geq \frac{1}{2}$ and equal to some Cantor set for $a < \frac{1}{2}$. If A_u denotes the set of points with a unique address, then $A_u = X$ whenever $a < \frac{1}{2}$, but $A_u = \{0,1\}$ for $a > \frac{1}{2}$. At $a = \frac{1}{2}$ we obtain that $A_u = [0,1] \setminus \{k/2^n \,|\, 1 \leq k < 2^n, n \in \mathbb{N}\}$.*

We extend the definition of the map T to the hyperspace $(\mathcal{F}(X), d_H)$ as follows:

$$T(E) = \bigcup_{i=1}^{m} w_i^{-1}(E).$$

This definition includes the totally disconnected, just touching case as well the overlapping case of an IFS. Remember that $\mathcal{F}(X)$ is the set of all non-empty compact subsets of X and d_H is the Hausdorff metric, which is defined as

$$d_H(E,F) = \inf\big\{\varepsilon > 0; E \subseteq U_\varepsilon(F) \text{ and } F \subseteq U_\varepsilon(E)\big\}$$

for $E, F \in \mathcal{F}(X)$, where $U_\varepsilon(E)$ stands for the parallel body of E at distance ε. The ε-parallel body will be defined with the help of the distance function of the set E $\mathrm{d}(x,E) = \inf\big\{d(x,y) \mid y \in E\big\}$. Then $U_\varepsilon(E) = \big\{x \mid \mathrm{d}(x,E) \leq \varepsilon\big\}$.

Lemma 2 *The extended map $T : \mathcal{F}(X) \to \mathcal{F}(X)$ is sensitive with respect to initial conditions provided that the IFS $(X; w_1, \ldots, w_m)$ satisfies the open set condition.*

We need some further lemmas. For this purpose we use $d(E)$ as the notation for the diameter of the set $E \subseteq X$, i.e. $d(E) = \sup\big\{d(x,y) \mid x, y \in E\big\}$.

Lemma 3 *Let Y be a dense subset of X. For all $E \in \mathcal{F}(X)$*

$$\sup_{y \in Y} d_H(\{y\}, E) \geq \frac{1}{4} d(X).$$

Proof. First assume that $d(E) \geq \frac{1}{2}d(X)$. Let $B(x,r)$ be a ball such that $E \subseteq B(x,r)$. This implies $d(E) \leq 2r$. Hence $r \geq \frac{1}{4}d(X)$. As $\overline{Y} = X$ we can conclude that the desired inequality holds.

But if $d(E) < \frac{1}{2}d(X)$, we can choose $a, b \in X$ such that $d(a,b) = d(X)$ by the compactness of X. For arbitrary $u, v \in E$ the triangle inequality and the above assumption implies $\frac{1}{2}d(X) \leq d(a,u) + d(v,b)$. This gives

$$\frac{1}{2}d(X) \leq d(a, X) + d(b, E).$$

Hence for at least one of these points a or b we have, say $d(a,X) \geq \frac{1}{4}d(X)$. This proves the inequality of the lemma for the second case. □

We also use the following Blaschke's selection theorem.

Lemma 4 $(\mathcal{F}(X), d_H)$ *is a compact metric space provided that* (X, d) *is a compact metric space; i.e. every sequence of compact sets contains a* d_H*-convergent subsequence.*

We now give the proof of Lemma 2.

Proof. Let $\delta = \frac{1}{6}d(X)$ and $E_n = T^{-n}(E)$ for an arbitrary $E \in \mathcal{F}(X)$. According to Lemma 4 we can assume that $E_n \to K$ w.r.t. the metric d_H and some $K \in \mathcal{F}(X)$. Take any y in a set O, which fulfills the open set condition, such that $d_H(\{y\}, K) \geq \frac{1}{4}d(X)$. Now for a given $\varepsilon > 0$ we define a finite set F and $n \geq 0$ such that $d_H(E, F) \leq \varepsilon$, but $d(T^n(E), T^n(F)) > \delta$.

Since for any address $\sigma = \sigma_1\sigma_2\ldots$ we get $d(w_{\sigma_1} \circ w_{\sigma_2} \circ \ldots \circ w_{\sigma_n}(X)) \downarrow 0$ provided that $n \to \infty$, we can find some $n_\varepsilon \in \mathbb{N}$ such that for $n \geq n_\varepsilon$ $n \in \mathbb{N}$ we get $d(w_{\sigma_1} \circ w_{\sigma_2} \circ \ldots \circ w_{\sigma_n}(X)) \leq \varepsilon$ for any choice of the $\sigma_1, \ldots, \sigma_n$ for a fixed n and, secondly $d_H(T^n(E), K) \leq \frac{1}{12}d(X)$. Now we define the finite set F by

$$F = \left\{ w_{\sigma_1} \circ \ldots \circ w_{\sigma_n}(y) \mid w_{\sigma_1} \circ \ldots \circ w_{\sigma_n}(X) \cap E \neq \emptyset \right\}$$

where $\sigma_1, \sigma_2, \ldots, \sigma_n$ run through all choices up to the fixed $n > n_\varepsilon$. This implies that $F \subseteq U_\varepsilon(E)$ as well as $E \subseteq U_\varepsilon(F)$. Hence $d_H(E, F) \leq \varepsilon$.

Note that for arbitrary n

$$d_H(\{y\}, K) \leq d_H(\{y\}, T^n(F)) + d_H(T^n(F), T^n(E)) + d_H(T^n(E), K).$$

The first term on the right hand side of the last inequality vanishes as n is chosen as in the definition of F since $T^n(F) = \{y\}$. To see this note that for $y \in O$

$$w_{\sigma_1} \circ w_{\sigma_2} \circ \ldots \circ w_{\sigma_n}(y) = w_\tau \circ w_{\sigma_2} \circ \ldots \circ w_{\sigma_n}(y)$$

implies $\tau = \sigma_1$. Moreover since the last term of the right hand side is smaller than $\frac{1}{12}d(X)$, the inequality $d_H(T^n(F), T^n(E)) > \delta$ follows. □

Lemma 5 *If $(X; w_1, \ldots, w_n)$ satisfies the open set condition, then the dynamical system $(\mathcal{F}(X), T)$ is transitive.*

PROOF. Since d_H generates the Vietoris topology on $\mathcal{F}(X)$, we can restrict our attention to open sets

$$\mathcal{U} = \big\{E \in \mathcal{F}(X) \mid E \subseteq U_1 \cup \ldots \cup U_l,\ E \cap U_i \neq \emptyset \text{ for } i = 1, \ldots, l\big\}$$

and

$$\mathcal{V} = \big\{E \in \mathcal{F}(X) \mid E \subseteq V_1 \cup \ldots \cup V_k,\ E \cap V_i \neq \emptyset \text{ for } i = 1, \ldots, k\big\},$$

where the U_i and V_i are given non-empty open subsets of X. If U is the open set which belongs to the open set condition, we fix some $x_i \in U \cap U_i$ for $i = 1, \ldots, l$ and some n sufficiently large such that for all pairs x_i and V_j, where $i = 1, \ldots, l$ and $j = 1, \ldots, k$, there is a finite sequence $\sigma_1, \ldots, \sigma_n \in \{1, \ldots, m\}$ such that

$$y_{ij} = w_{\sigma_1} \circ w_{\sigma_2} \circ \ldots \circ w_{\sigma_n}(x_i) \in V_j.$$

Define then $F = \{y_{ij} \mid i = 1, \ldots, l,\ j = 1, \ldots, k\}$. It follows that $F \in \mathcal{V}$ and $T^n(F) \in \mathcal{U}$. Hence $\mathcal{U} \cap T^n(\mathcal{U}) \neq \emptyset$. □

The last step is now to consider the periodic points of T.

Lemma 6 *If the IFS $(X; w_1, \ldots, w_m)$ satisfies the open set condition, then the set of periodic points of T is dense in $\mathcal{F}(X)$ w.r.t. to the Hausdorff metric (or Vietoris topology).*

PROOF. Let $\sigma_1, \ldots, \sigma_n \in \{1, \ldots, m\}$. The map $f_{\sigma_1,\ldots,\sigma_n} = f_{\sigma_1} \circ \ldots \circ f_{\sigma_n}$ is contractive and let $x_{\sigma_1,\ldots,\sigma_n}$ be its unique fix point within X. We define

$$F_n = \big\{x_{\sigma_1,\ldots,\sigma_n} \mid \sigma_1, \ldots, \sigma_n \in \{1, \ldots, m\}\big\}$$

and $F = \bigcup_{n\in\mathbb{N}} F_n$. Then $\overline{F} = X$. To see this let $x \in X$ and $\varepsilon > 0$. We may choose $n_\varepsilon \in \mathbb{N}$ such that for $n \geq n_\varepsilon$ $d(f_{\sigma_1,\ldots,\sigma_n}(X)) \leq \varepsilon$, the diameter of the set $f_{\sigma_1,\ldots,\sigma_n}$ is less than ε. We may find $\sigma_1, \ldots, \sigma_n \in \{1, \ldots, m\}$ such that $x \in$

$f_{\sigma_1,\ldots,\sigma_n}(X)$. Also since $x_{\sigma_1,\ldots,\sigma_n} \in f_{\sigma_1,\ldots,\sigma_n}(X)$, we obtain $d(X, x_{\sigma_1,\ldots,\sigma_n}) \le \varepsilon$, which proves the density of F within X. Let U be the open set of the open set condition. We set $\mathcal{E}_n = \mathcal{P}_0(U \cap F_n)$ the non-empty (finite) subsets of $U \cap F_n$ and $\mathcal{E} = \bigcup_{n\in\mathbb{N}} \mathcal{E}_n$. We show that

a) $\mathcal{E}$ consists of periodic points of the map T;

b) $\mathcal{E}$ is dense in $(\mathcal{F}(X), d_H)$.

a) Take any $E \in \mathcal{E}_n$. Since for $x \in E$ we always have a unique preimage for n steps, it follows that $T^n(E) = E$.

b) For an arbitrary closed $F \subseteq X$ and $\varepsilon > 0$ we select some $E \in \mathcal{E}$ such that $d_H(F, E) < \varepsilon$. First, we cover F by a finite number of closed balls $B(x_k, \varepsilon/2)$ for $k = 1, \ldots, l$ such that $B(x_k, \varepsilon/2) \cap U \neq \emptyset$ since U is dense in X. Because U is open, we can find a common n such that for some finite sequence $\sigma_1, \ldots, \sigma_n$

$$f_{\sigma_1,\ldots,\sigma_n}(X) \subseteq B(x_k, \varepsilon/2) \cap U.$$

Hence for the fix point $x_{\sigma_1,\ldots,\sigma_n}$ of the map $f_{\sigma_1,\ldots,\sigma_n}$ we have

$$x_{\sigma_1,\ldots,\sigma_n} \in B(x_k, \varepsilon/2) \cap U.$$

This implies $F \subseteq \bigcup B(x_{\sigma_1,\ldots,\sigma_n}, \varepsilon)$. If we now take as E all the points $x_{\sigma_1,\ldots,\sigma_n}$, we clearly have $d_H(F, E) \le \varepsilon$ and E is also a periodic point of T. □

Hence, we have proved the following assertion.

Theorem 1 $(\mathcal{F}(X), T)$ *is a chaotic dynamical system provided that for the initial IFS the open set condition is satisfied.*

Finally, we discuss the overlapping case of Example 1. We have that $w_1^{-1}(x) = \frac{x}{a}$ and $w_2^{-1}(x) = \frac{x}{a} + \frac{a-1}{a}$ with the domains $[0, a]$ and $[1-a, 1]$. To verify sensitivity with respect to initial conditions it seems to be the best to start with $E = [0, 1]$ since for all n, we have $T^n(E) = E$. Is it possible to find some $\delta > 0$ such that for all $\varepsilon > 0$ there is some $n \in \mathbb{N}$ and some $F \in \mathcal{F}([0, 1])$ such that $d_H(T^n(E), T^n(F)) = d_H(E, T^n(F)) > \delta$? The first idea is now to use a finite set F of equidistant points

$$F = \left\{ \frac{i}{m}; i = 0, 1, \ldots, m \right\}$$

for some $n \in \mathbb{N}$. The image $T(F)$ consists of points of the kind $\frac{i}{am}$ or $\frac{i}{ma} + \frac{a-1}{a}$. If for some integer i the condition $\frac{i}{m} = a - 1$ is satisfied, then the minimal distance between points in $T(F)$ is at least $\frac{1}{ma}$. Hence

$$d_H(E, T(F)) = \frac{1}{a} d_H(E, F).$$

All points of $T(F)$ have then the form $\frac{i}{am}$ again. If we would iterate this idea, some times we obtain a sequence $(i_k)_{k\in\mathbb{N}}$ of integers such that $\frac{i_k}{m} = a^{k-1}(a-1)$. We conclude that $i_1(i_1+m)^{k-1} \equiv 0(\mathrm{mod}(m^{k-1})$. Since $i_1 < m$, this is impossible for $k \geq 3$. Hence, we can increase the distance between E and $T^n(F)$ only twice by the factor $\frac{1}{a}$. This motivates the following question.

Question *Is it true that the dynamical system* $(\mathcal{F}([0,1]), T_a)$ *arising from Example* 1 *for* $a > \frac{1}{2}$ *is never chaotic?*

References

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[2] K. J. Falconer, *The geometry of fractal sets*, Cambridge University Press, 1985