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CARDINAL INVARIANTS CONCERNING EXTENDABLE AND PERIPHERALLY CONTINUOUS FUNCTIONS

Abstract

Let $\mathcal{F}$ be a family of real functions, $\mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$. In the paper we will examine the following question. For which families $F \subseteq \mathbb{R}^{\mathbb{R}}$ does there exist $g\colon \mathbb{R} \to \mathbb{R}$ such that $f + g \in \mathcal{F}$ for all $f \in F$? More precisely, we will study a cardinal function $\mathrm{A}(\mathcal{F})$ defined as the smallest cardinality of a family $F \subseteq \mathbb{R}^{\mathbb{R}}$ for which there is no such g. We will prove that $\mathrm{A}(\mathrm{Ext}) = \mathrm{A}(\mathrm{PR}) = \mathfrak{c}^+$ and $\mathrm{A}(\mathrm{PC}) = 2^{\mathfrak{c}}$, where Ext, PR and PC stand for the classes of extendable functions, functions with perfect road and peripherally continuous functions from $\mathbb{R}$ into $\mathbb{R}$, respectively. In particular, the equation $\mathrm{A}(\mathrm{Ext}) = \mathfrak{c}^+$ immediately implies that every real function is a sum of two extendable functions. This solves a problem of Gibson [6].

We will also study the multiplicative analogue $\mathrm{M}(\mathcal{F})$ of the function $\mathrm{A}(\mathcal{F})$ and we prove that $\mathrm{M}(\mathrm{Ext}) = \mathrm{M}(\mathrm{PR}) = 2$ and $\mathrm{A}(\mathrm{PC}) = \mathfrak{c}$.

This article is a continuation of papers [10, 3, 12] in which functions $\mathrm{A}(\mathcal{F})$ and $\mathrm{M}(\mathcal{F})$ has been studied for the classes of almost continuous, connectivity and Darboux functions.

Key Words: cardinal invariants, extendable functions, functions with prefect road, peripherally continuous functions

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1 Introduction

We will use the following terminology and notation. Functions will be identified with their graphs. The family of all functions from a set X into Y will be denoted by Y^X. The symbol $|X|$ will stand for the cardinality of a set X. The cardinality of the set $\mathbb{R}$ of real numbers is denoted by $\mathfrak{c}$. For a cardinal number κ we will write $\mathrm{cf}(\kappa)$ for the cofinality of κ. A cardinal number κ is regular, if $\kappa = \mathrm{cf}(\kappa)$. For $A \subseteq \mathbb{R}$ its characteristic function is denoted by χ_A. In particular, $\chi_\emptyset$ stands for the zero constant function.

In his study of the class D of Darboux functions (See definition below.) Fast [5] proved that for every family $F \subseteq \mathbb{R}^\mathbb{R}$ of cardinality at most that of the continuum there exists $g \colon \mathbb{R} \to \mathbb{R}$ such that $f+g$ is Darboux for every $f \in F$. Natkaniec [10] proved the similar result for the class AC of almost continuous functions and defined the following two cardinal invariants for every $\mathcal{F} \subseteq \mathbb{R}^\mathbb{R}$.

$$\begin{aligned}\mathrm{A}(\mathcal{F}) &= \min\{|F| \colon F \subseteq \mathbb{R}^\mathbb{R} \ \& \ \neg\exists\, g \in \mathbb{R}^\mathbb{R} \ \forall\, f \in F \ f+g \in \mathcal{F}\} \cup \{(2^\mathfrak{c})^+\} \\ &= \min\{|F| \colon F \subseteq \mathbb{R}^\mathbb{R} \ \& \ \forall\, g \in \mathbb{R}^\mathbb{R} \ \exists\, f \in F \ f+g \notin \mathcal{F}\} \cup \{(2^\mathfrak{c})^+\}\end{aligned}$$

and

$$\begin{aligned}\mathrm{M}(\mathcal{F}) &= \min\{|F| \colon F \subseteq \mathbb{R}^\mathbb{R} \ \& \ \neg\exists\, g \in \mathbb{R}^\mathbb{R} \setminus \{\chi_\emptyset\} \ \forall\, f \in F \ f \cdot g \in \mathcal{F}\} \cup \{(2^\mathfrak{c})^+\} \\ &= \min\{|F| \colon F \subseteq \mathbb{R}^\mathbb{R} \ \& \ \forall\, g \in \mathbb{R}^\mathbb{R} \setminus \{\chi_\emptyset\} \ \exists\, f \in F \ f \cdot g \notin \mathcal{F}\} \cup \{(2^\mathfrak{c})^+\}.\end{aligned}$$

Thus, Fast and Natkaniec effectively showed that $\mathrm{A}(\mathrm{D}) > \mathfrak{c}$ and $\mathrm{A}(\mathrm{AC}) > \mathfrak{c}$.

The extra assumption that $g \neq \chi_\emptyset$ is added in the definition of M since otherwise for every family $\mathcal{F} \subseteq \mathbb{R}^\mathbb{R}$ containing $\chi_\emptyset$ we would have $\mathrm{M}(\mathcal{F}) = (2^\mathfrak{c})^+$.

Notice the following basic properties of functions A and M.

Proposition 1.1 *Let $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathbb{R}^\mathbb{R}$.*

(1) $\mathrm{A}(\mathcal{F}) \leq \mathrm{A}(\mathcal{G})$.

(2) $\mathrm{A}(\mathcal{F}) \geq 2$ *if* $\mathcal{F} \neq \emptyset$.

(3) $\mathrm{A}(\mathcal{F}) \leq 2^\mathfrak{c}$ *if* $\mathcal{F} \neq \mathbb{R}^\mathbb{R}$.

Proof. (1) is obvious. To see (2) let $h \in \mathcal{F}$ and $F = \{f\}$ for some $f \in \mathbb{R}^\mathbb{R}$. Then $f+g \in \mathcal{F}$ for $g = h - f$. To see (3) note that for $F = \mathbb{R}^\mathbb{R}$ and every $g \in \mathbb{R}^\mathbb{R}$ there is $f \in F$ with $f+g \notin \mathcal{F}$, namely $f = h - g$, where $h \in \mathbb{R}^\mathbb{R} \setminus \mathcal{F}$. $\square$

Proposition 1.2 *Let $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathbb{R}^\mathbb{R}$.*

(1) $\mathrm{M}(\mathcal{F}) \leq \mathrm{M}(\mathcal{G})$.

(2) $\mathrm{M}(\mathcal{F}) \geq 2$ *if* $\chi_\emptyset, \chi_\mathbb{R} \in \mathcal{F}$.

(3) $\mathrm{M}(\mathcal{F}) \leq \mathfrak{c}$ *if* $r\chi_{\{x\}} \notin \mathcal{F}$ *for every* $r, x \in \mathbb{R}$, $r \neq 0$.

Proof. (1) is obvious. To see (2) let $F = \{f\}$ for some $f \in \mathbb{R}^{\mathbb{R}}$. If there is $x \in \mathbb{R}$ such that $f(x) = 0$ take $g = \chi_{\{x\}}$. Otherwise, define $g(x) = 1/f(x)$ for every $x \in \mathbb{R}$. Then $f \cdot g \in \{\chi_\emptyset, \chi_\mathbb{R}\} \subseteq \mathcal{F}$. To see (3) note that for $F = \{\chi_{\{x\}} : x \in \mathbb{R}\}$ and every $g \in \mathbb{R}^{\mathbb{R}} \setminus \{\chi_\emptyset\}$ there is $f \in F$ with $f \cdot g \notin \mathcal{F}$, namely $f = \chi_{\{x\}}$, where x is such that $g(x) = r \neq 0$. □

Proposition 1.3 *Let $\chi_\emptyset \in \mathcal{F} \subseteq \mathbb{R}^{\mathbb{R}}$. Then $\mathrm{A}(\mathcal{F}) = 2$ if and only if $\mathcal{F} - \mathcal{F} = \{f_1 - f_2 : f_1, f_2 \in \mathcal{F}\} \neq \mathbb{R}^{\mathbb{R}}$.*

Proof. "⇒" Assume that $\mathcal{F} - \mathcal{F} = \mathbb{R}^{\mathbb{R}}$. We will show that $\mathrm{A}(\mathcal{F}) > 2$. So, pick arbitrary $f_1, f_2 \in \mathbb{R}^{\mathbb{R}}$ and put $F = \{f_1, f_2\}$. It is enough to find $g \in \mathbb{R}^{\mathbb{R}}$ such that $f_1 + g, f_2 + g \in \mathcal{F}$. But $f_1 - f_2 \in \mathbb{R}^{\mathbb{R}} = \mathcal{F} - \mathcal{F}$. So, there exist $h_1, h_2 \in \mathcal{F}$ such that $f_1 - f_2 = h_1 - h_2$. Put $g = h_1 - f_1 = h_2 - f_2$. Then $f_i + g = f_i + (h_i - f_i) = h_i \in \mathcal{F}$ for $i = 1, 2$.

"⇐" By Proposition 1.1(2) we have $\mathrm{A}(\mathcal{F}) \geq 2$. To see that $\mathrm{A}(\mathcal{F}) \leq 2$ let $h \in \mathbb{R}^{\mathbb{R}} \setminus (\mathcal{F} - \mathcal{F})$, take $F = \{\chi_\emptyset, h\}$ and choose an arbitrary $g \in \mathbb{R}^{\mathbb{R}}$. It is enough to show that $f + g \notin \mathcal{F}$ for some $f \in F$. But if $g = \chi_\emptyset + g \in \mathcal{F}$ and $h + g \in \mathcal{F}$, then $h \in \mathcal{F} - g \subset \mathcal{F} - \mathcal{F}$, contradicting the choice of h. □

Now, let X and Y be topological spaces. In what follows we will consider the following classes of functions from X into Y. (In fact, we will consider these classes mainly for $X = Y = \mathbb{R}$.)

$\mathrm{D}(X, Y)$ of *Darboux functions* $f : X \to Y$; i.e., such that $f[C]$ is connected in Y for every connected subset C of X.

$\mathrm{Conn}(X, Y)$ of *connectivity functions* $f : X \to Y$; i.e., such that the graph of f restricted to C (that is $f \cap [C \times Y]$) is connected in $X \times Y$ for every connected subset C of X.

$\mathrm{AC}(X, Y)$ of *almost continuous functions* $f : X \to Y$; i.e., such that every open subset U of $X \times Y$ containing the graph of f, there is a continuous function $g : X \to Y$ with $g \subset U$.

$\mathrm{Ext}(X, Y)$ of *extendable functions* $f : X \to Y$; i.e., such that there exists a connectivity function $g : X \times [0, 1] \to Y$ with $f(x) = g(x, 0)$ for every $x \in X$.

PR of *functions* $f : \mathbb{R} \to \mathbb{R}$ *with perfect road* ($X = Y = \mathbb{R}$); i.e., such that for every $x \in \mathbb{R}$ there exists a perfect set $P \subseteq \mathbb{R}$ having x as a bilateral limit point for which restriction $f|_P$ of f to P is continuous at x.

$\mathrm{PC}(X, Y)$ of *peripherally continuous functions* $f : X \to Y$; i.e., such that for every $x \in X$ and any pair $U \subseteq X$ and $V \in Y$ of open neighborhoods of x and $f(x)$, respectively, there exists an open neighborhood W of x with $\mathrm{cl}(W) \subseteq U$ and $f[\mathrm{bd}(W)] \subseteq V$, where $\mathrm{cl}(W)$ and $\mathrm{bd}(W)$ stand for the closure and the boundary of W, respectively.

We will write D, Conn, AC, Ext, and PC in place of $\mathrm{D}(X, Y)$, $\mathrm{Conn}(X, Y)$, $\mathrm{AC}(X, Y)$, $\mathrm{Ext}(X, Y)$, and $\mathrm{PC}(X, Y)$ if $X = Y = \mathbb{R}$.

Note also, that function $f\colon \mathbb{R} \to \mathbb{R}$ is peripherally continuous ($f \in \mathrm{PC}$) if and only if for every $x \in \mathbb{R}$ there are sequences $a_n \nearrow x$ and $b_n \searrow x$ such that $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(x)$. In particular, if graph of f is dense in $\mathbb{R}^2$, then f is peripherally continuous.

For the classes of functions (from $\mathbb{R}$ into $\mathbb{R}$) defined above we have the following proper inclusions $\subset$, marked by arrows $\longrightarrow$. (See [2].)

$$
\begin{tikzcd}
 & \mathrm{AC} \arrow[r] & \mathrm{Conn} \arrow[r] & \mathrm{D} \arrow[dr] & \\
\mathrm{Ext} \arrow[ur] \arrow[drr] & & & & \mathrm{PC} \\
 & & \mathrm{PR} \arrow[urr] & &
\end{tikzcd}
$$

In what follows we will also use the following theorem due to Hagan [9].

Theorem 1.4 *If* $n \geq 2$, *then* $\mathrm{Conn}(\mathbb{R}^n, \mathbb{R}) = \mathrm{PC}(\mathbb{R}^n, \mathbb{R})$.

The functions A and M for the classes AC, Conn and D were studied in [10, 3, 12]. In particular, the following is known.

Theorem 1.5 [12] $\mathrm{M}(\mathrm{AC}) = \mathrm{M}(\mathrm{Conn}) = \mathrm{M}(\mathrm{D}) = \mathrm{cf}(\mathfrak{c})$.

Theorem 1.6 [3] $\mathfrak{c}^+ \leq \mathrm{A}(\mathrm{AC}) = \mathrm{A}(\mathrm{Conn}) = \mathrm{A}(\mathrm{D}) \leq 2^{\mathfrak{c}}$, $\mathrm{cf}(\mathrm{A}(\mathrm{D})) > \mathfrak{c}$ *and it is pretty much all that can be shown in ZFC. More precisely, it is consistent with ZFC that* $\mathrm{A}(\mathrm{D})$ *can be equal to any regular cardinal between* $\mathfrak{c}^+$ *and* $2^{\mathfrak{c}}$ *and that it can be equal to* $2^{\mathfrak{c}}$ *independent of the cofinality of* $2^{\mathfrak{c}}$.

The goal of this paper is to prove the following theorem.

Theorem 1.7 (1) $\mathrm{M}(\mathrm{PC}) = \mathfrak{c}$.

(2) $\mathrm{M}(\mathrm{Ext}) = \mathrm{M}(\mathrm{PR}) = 2$.

(3) $\mathrm{A}(\mathrm{PC}) = 2^{\mathfrak{c}}$.

(4) $\mathrm{A}(\mathrm{Ext}) = \mathrm{A}(\mathrm{PR}) = \mathfrak{c}^+$.

This will be proved in the next sections. Notice only that the equation $\mathrm{A}(\mathrm{Ext}) = \mathfrak{c}^+$ and Proposition 1.3 immediately imply the following corollary, which gives a positive answer to a question of Gibson [6]. (Compare also [13] and [14].)

Corollary 1.8 *Every function* $f\colon \mathbb{R} \to \mathbb{R}$ *is the sum of two extendable functions.*

2 Proof of Theorem 1.7(1), (2) and (3)

Proof of M(Ext) = M(PR) = 2. The inequalities $2 \le \mathrm{M(Ext)} \le \mathrm{M(PR)}$ follow from Proposition 1.2. To see that $\mathrm{M(PR)} \le 2$ take $F = \{\chi_B, \chi_{\mathbb{R}\setminus B}\}$ where $B \subset \mathbb{R}$ is a Bernstein set. Then for every $g \in \mathbb{R}^{\mathbb{R}} \setminus \{\chi_\emptyset\}$ we have $f \cdot g \ne \mathrm{PR}$ for some $f \in F$. To see it, take $x \in \mathbb{R}$ such that $g(x) = r \ne 0$. If $x \in B$, then $\chi_B \cdot g$ does not have a perfect road at x, since $(\chi_B \cdot g)(x) = r \ne 0$ and $(\chi_B \cdot g)^{-1}(0) \cap P \ne \emptyset$ for every perfect set $P \subseteq \mathbb{R}$. Similarly, $\chi_{\mathbb{R}\setminus B} \cdot g$ does not have a perfect road at x if $x \in \mathbb{R} \setminus B$. □

Proof of M(PC) = $\mathfrak{c}$. The inequality $\mathrm{M(PC)} \le \mathfrak{c}$ follows from Proposition 1.2. So, it is enough to show that $\mathrm{M(PC)} \ge \mathfrak{c}$.

Let $F \subseteq \mathbb{R}^{\mathbb{R}}$ be a family of cardinality less than or equal to κ with $\omega \le \kappa < \mathfrak{c}$. We will find $g \in \mathbb{R}^{\mathbb{R}} \setminus \chi_\emptyset$ such that $f \cdot g \in \mathrm{PC}$ for every $f \in F$. For $f \colon \mathbb{R} \to \mathbb{R}$ let $[f \ne 0]$ denote $\{x \in \mathbb{R} \colon f(x) \ne 0\}$ and let

$$A_f = \{x \in \mathbb{R} \colon f(x) \ne 0 \ \&\ [f \ne 0] \ \text{ is not bilaterally } \kappa^+\text{-dense at } x\},$$

where set $S \subseteq \mathbb{R}$ is said to be bilaterally κ^+-dense at x if for every $\varepsilon > 0$ each of the sets $S \cap [x - \varepsilon, x]$ and $S \cap [x, x + \varepsilon]$ have cardinality at least κ^+. Note that $|A_f| \le \kappa$ for every $f \colon \mathbb{R} \to \mathbb{R}$. This is the case, since for every $x \in A_f$ there exists a closed interval J with non-empty interior such that $x \in J$ and $|[f \ne 0] \cap J| \le \kappa$. Now, if $\mathcal{J}$ is the family of all maximal intervals J with non-empty interior such that $|[f \ne 0] \cap J| \le \kappa$, then $|\mathcal{J}| \le \omega$, $A_f \subseteq \bigcup_{J \in \mathcal{J}} ([f \ne 0] \cap J)$ and $|A_f| \le |\bigcup_{J \in \mathcal{J}} ([f \ne 0] \cap J)| \le \kappa$.

Let $A = \bigcup_{f \in F} A_f$. Then $|A| \le \kappa$. Notice that the set $[f \ne 0] \setminus A$ is bilaterally κ^+-dense at x for every $f \in F$ and x from $[f \ne 0] \setminus A$. To define g let $\langle \langle f_\alpha, q_\alpha, I_\alpha \rangle \colon \alpha < \kappa \rangle$ be the sequence of all triples with $f_\alpha \in F$, $q_\alpha \in \mathbb{Q}$ and I_α be an open interval with rational end points. By induction define on $\alpha < \kappa$ a one-to-one sequence $\langle x_\alpha \colon \alpha < \kappa \rangle$ by choosing

(i) $x_\alpha \in [f_\alpha \ne 0] \cap (I_\alpha \setminus A) \setminus \{x_\beta \colon \beta < \alpha\}$ if the choice can be made, and

(ii) $x_\alpha \in (I_\alpha \setminus A) \setminus \{x_\beta \colon \beta < \alpha\}$ otherwise.

Now, for $x \in \mathbb{R}$ we put

$$g(x) = \begin{cases} \frac{q_\alpha}{f_\alpha(x)} & \text{if there is } \alpha < \kappa \text{ with } x = x_\alpha \text{ and } f_\alpha(x) \ne 0 \\ 1 & \text{if there is } \alpha < \kappa \text{ with } x = x_\alpha \text{ and } f_\alpha(x) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then $g \ne \chi_\emptyset$ and for each $q \in \mathbb{Q}$ and $f \in F$ the set $(g \cdot f)^{-1}(q)$ is bilaterally dense at every x from $[f \ne 0] \setminus A$. Moreover, $(g \cdot f)(x) = 0$ outside of $[f \ne 0] \setminus A$

and $(g \cdot f)^{-1}(0)$ is bilaterally dense at every $x \in \mathbb{R}$. So, $g \cdot f \in \mathrm{PC}$ for every $f \in F$. $\square$

To prove $\mathrm{A}(\mathrm{PC}) = 2^{\mathfrak{c}}$ we will use the following result.

Theorem 2.1 *Let A and B be such that $|A| = \omega$ and $|B| = \mathfrak{c}$. Then there exists a family $\mathcal{C} \subseteq A^B$ of size $2^{\mathfrak{c}}$ such that for every one-to-one sequence $\langle g_a \in \mathcal{C} \colon a \in A\rangle$ there is $b \in B$ with $g_a(b) = a$ for every $a \in A$.*

Proof. The theorem is proved in [4, Corollary 3.17, p. 77] for $A = \omega$ and $B = \mathfrak{c}$. The generalization is obvious. $\square$

From this we will conclude the following lemma.

Lemma 2.2 *If $B \subseteq \mathbb{R}$ has cardinality $\mathfrak{c}$ and $H \subseteq \mathbb{Q}^B$ is such that $|H| < 2^{\mathfrak{c}}$, then there is $g \in \mathbb{Q}^B$ such that $h \cap g \neq \emptyset$ for every $h \in H$.*

Proof. Let $\mathcal{C}$ be as in Theorem 2.1 with $A = \mathbb{Q}$. For each $h \in H$ there only finitely many $g \in \mathcal{C}$ for which $h \cap g = \emptyset$, since any countable infinite subset of $\mathcal{C}$ can be enumerated as $\{g_a \in \mathcal{C} \colon a \in \mathbb{Q}\}$. So there is $g \in \mathcal{C}$ such that $h \cap g \neq \emptyset$ for every $h \in H$. $\square$

Proof of $\mathrm{A}(\mathrm{PC}) = 2^{\mathfrak{c}}$. By Proposition 1.1 to prove $\mathrm{A}(\mathrm{PC}) = 2^{\mathfrak{c}}$ it is enough to show that $\mathrm{A}(\mathrm{PC}) \geq 2^{\mathfrak{c}}$. So, let $F \subseteq \mathbb{R}^{\mathbb{R}}$ be such that $|F| < 2^{\mathfrak{c}}$. We will find $g \colon \mathbb{R} \to \mathbb{R}$ such that $f + g \in \mathrm{PC}$ for every $f \in F$. Let $\mathcal{G}$ be the family of all triples $\langle I, p, m\rangle$ where I is a non-empty open interval with rational end points, $p \in \mathbb{Q}$ and $m < \omega$. For each $\langle I, p, m\rangle \in \mathcal{G}$ define a set $B_{\langle I,p,m\rangle} \subseteq I$ of size $\mathfrak{c}$ such that $B_{\langle I,p,m\rangle} \cap B_{\langle J,q,n\rangle} = \emptyset$ for any distinct $\langle I, p, m\rangle$ and $\langle J, q, n\rangle$ from $\mathcal{G}$.

Next, fix $\langle I, p, m\rangle \in \mathcal{G}$ and for each $f \in F$ choose $h^f_{\langle I,p,m\rangle} \colon B_{\langle I,p,m\rangle} \to \mathbb{Q}$ such that $|p - (f(x) + h^f_{\langle I,p,m\rangle}(x))| < \frac{1}{m}$ for every $x \in B_{\langle I,p,m\rangle}$. Then, by Lemma 2.2 used with a set $H_{\langle I,p,m\rangle} = \{h^f_{\langle I,p,m\rangle} \colon f \in F\}$, there exists $g_{\langle I,p,m\rangle} \colon B_{\langle I,p,m\rangle} \to \mathbb{Q}$ such that

$$\forall\, f \in F\ \exists\, x \in B_{\langle I,p,m\rangle}\ h^f_{\langle I,p,m\rangle}(x) = g_{\langle I,p,m\rangle}(x).$$

In particular, if $g \colon \mathbb{R} \to \mathbb{Q}$ is a common extension of all functions $g_{\langle I,p,m\rangle}$, then for every $\langle I, p, m\rangle \in \mathcal{G}$ and every $f \in F$ there exists $x \in B_{\langle I,p,m\rangle} \subseteq I$ such that

$$|p - (f(x) + g(x))| < \frac{1}{m}.$$

So, for every $f \in F$ the graph of $f + g$ is dense in $\mathbb{R}^2$. Thus, $f + g \in \mathrm{PC}$. $\square$

3 Proof of Theorem 1.7(4): $\mathrm{A}(\mathrm{Ext}) = \mathrm{A}(\mathrm{PR}) = \mathfrak{c}^+$

By Proposition 1.1 we have $\mathrm{A}(\mathrm{Ext}) \leq \mathrm{A}(\mathrm{PR})$. Thus, it is enough to prove two inequalities: $\mathrm{A}(\mathrm{PR}) \leq \mathfrak{c}^+$ and $\mathrm{A}(\mathrm{Ext}) \geq \mathfrak{c}^+$.

First we will prove $\mathrm{A}(\mathrm{PR}) \leq \mathfrak{c}^+$. For this we need the following lemma.

Lemma 3.1 *There is a family $F \subseteq \mathbb{R}^\mathbb{R}$ of size $\mathfrak{c}^+$ such that for every distinct $f, h \in F$, every perfect set P and every $n < \omega$ there exists an $x \in P$ with $|f(x) - h(x)| \geq n$.*

Proof. The family $F = \{f_\xi \colon \xi < \mathfrak{c}^+\}$ is constructed by induction using a standard diagonal argument. If for some $\xi < \mathfrak{c}^+$ the functions $\{f_\zeta \colon \zeta < \xi\}$ are already constructed, we construct f_ξ as follows. Let $\langle\langle P_\alpha, h_\alpha, n_\alpha\rangle \colon \alpha < \mathfrak{c}\rangle$ be an enumeration of all triples $\langle P, h, n\rangle$ where $P \subseteq \mathbb{R}$ is perfect, $h = f_\zeta$ for some $\zeta < \xi$ and $n < \omega$. By induction on $\alpha < \mathfrak{c}$ choose $x_\alpha \in P_\alpha \setminus \{x_\beta \colon \beta < \alpha\}$ and define $f_\xi(x_\alpha) = h_\alpha(x_\alpha) + n_\alpha$. Then any extension of f_ξ to $\mathbb{R}$ will have the desired properties. $\square$

Proof of $\mathrm{A}(\mathrm{PR}) \leq \mathfrak{c}^+$. Now let F be a family from Lemma 3.1. We will show that for every $g \colon \mathbb{R} \to \mathbb{R}$ there exists $f \in F$ such that $f + g \notin \mathrm{PR}$. By way of contradiction assume that there exists a function $g \colon \mathbb{R} \to \mathbb{R}$ such that $f + g \in \mathrm{PR}$ for every $f \in F$. Then, for every $f \in F$ there exists a perfect set P_f such that 0 is a bilateral limit point of P_f and $(f + g)|_{P_f}$ is continuous at 0. Since there are $\mathfrak{c}^+$-many functions in F and only $\mathfrak{c}$-many perfect sets, there are distinct $f, h \in F$ with $P_f = P_h$. Then the function $((f + g) - (h + g))|_{P_f} = (f - h)|_{P_f}$ is continuous at 0 contradicting the choice of the family F. $\square$

The proof of $\mathrm{A}(\mathrm{Ext}) \geq \mathfrak{c}^+$ is based on the following facts.

Lemma 3.2 *For every meager subset M of $\mathbb{R}$ there exists a family $\{h_\xi \in \mathbb{R}^\mathbb{R} \colon \xi < \mathfrak{c}\}$ of increasing homeomorphisms such that $h_\zeta[M] \cap h_\xi[M] = \emptyset$ for every $\zeta < \xi < \mathfrak{c}$.*

Proof. Let $\{D_\zeta : \zeta < \mathfrak{c}\}$ be a family of pairwise disjoint $\mathfrak{c}$-dense, meager F_σ-sets. Then by [8, Lemma 4] there are homeomorphisms $\{h_\zeta : \mathbb{R} \to \mathbb{R} : \zeta < \mathfrak{c}\}$ such that $h_\zeta[M] \subset D_\zeta$. $\square$

For $f \in \mathcal{F} \subset \mathbb{R}^\mathbb{R}$ we say that a set $G \subseteq \mathbb{R}$ is *f-negligible for the class $\mathcal{F}$* provided $g \in \mathcal{F}$ for every $g \colon \mathbb{R} \to \mathbb{R}$ such that $g|_{\mathbb{R}\setminus G} = f|_{\mathbb{R}\setminus G}$. Thus, $G \subseteq \mathbb{R}$ is f-negligible for $\mathcal{F}$ if we can modify f arbitrarily on G remaining in the class $\mathcal{F}$.

Theorem 3.3 *There exists a connectivity function $f \colon \mathbb{R}^2 \to \mathbb{R}$ with graph dense in $\mathbb{R}^3$ such that some dense G_δ subset G of $\mathbb{R}^2$ is f-negligible for the class* $\mathrm{Conn}(\mathbb{R}^2, \mathbb{R})$.

The proof of this theorem will be postponed to the end of this section.

Corollary 3.4 *There exists an extendable function $\hat{f}\colon \mathbb{R} \to \mathbb{R}$ with graph dense in $\mathbb{R}^2$ such that some dense G_δ subset $\hat{G}$ of $\mathbb{R}$ is $\hat{f}$-negligible for the class* Ext*.*

PROOF. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ and G be as in Theorem 3.3. Then there exists $y \in \mathbb{R}$ with $G^y = \{x\colon \langle x, y\rangle \in G\}$ being a dense G_δ subset of $\mathbb{R}$. Clearly the set $\hat{G} = G^y$ and the function $\hat{f}\colon \mathbb{R} \to \mathbb{R}$ defined by $\hat{f}(x) = f(x, y)$ for every $x \in \mathbb{R}$ satisfy the requirements. $\square$

The existence of a function as in Corollary 3.4 was first announced by H. Rosen at the 10th Auburn Miniconference in Real Analysis, April 1995. However, the construction presented at that time had a gap. This gap was removed later, as described in [14].

The construction presented in this paper is an independently discovered repair of the original Rosen's gap. It is also more general than that of [14], since [14] does not contain any example similar to that of Theorem 3.3.

Next, we will show how Lemma 3.2 and Corollary 3.4 imply $\mathrm{A}(\mathrm{Ext}) \geq \mathfrak{c}^+$. The argument is a modification of the proof of Corollary 1.8. (Compare also [11] and [14].)

PROOF OF $\mathrm{A}(\mathrm{Ext}) \geq \mathfrak{c}^+$. Let $F = \{f_\xi \in \mathbb{R}^\mathbb{R}\colon \xi < \mathfrak{c}\}$. We will find $g\colon \mathbb{R} \to \mathbb{R}$ such that $f_\xi + g \in \mathrm{Ext}$ for every $\xi < \mathfrak{c}$. So, let $\hat{f}\colon \mathbb{R} \to \mathbb{R}$ and $\hat{G} \subseteq \mathbb{R}$ be as in Corollary 3.4. Put $M = \mathbb{R} \setminus \hat{G}$ and take $\{h_\xi \in \mathbb{R}^\mathbb{R}\colon \xi < \mathfrak{c}\}$ as in Lemma 3.2. For $\xi < \mathfrak{c}$ define g on $h_\xi[M]$ to be $(\hat{f} \circ h_\xi^{-1} - f_\xi)|_{h_\xi[M]}$ and extend it to $\mathbb{R}$ arbitrarily. To see that $f_\xi + g \in \mathrm{Ext}$ note that $f_\xi + g = \hat{f} \circ h_\xi^{-1}$ on $h_\xi[M]$. But the set $\mathbb{R} \setminus h_\xi[M] = h_\xi[\hat{G}]$ is $(\hat{f} \circ h_\xi^{-1})$-negligible for the class Ext. (See [11] for an easy proof.) So, each $f_\xi + g$ is extendable. $\square$

PROOF OF THEOREM 3.3. We will construct a peripherally continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ with dense G_δ subset G of $\mathbb{R}^2$ which is f-negligible for the class $\mathrm{PC}(\mathbb{R}^2, \mathbb{R})$. It is enough since, by Theorem 1.4, $\mathrm{Conn}(\mathbb{R}^2, \mathbb{R}) = \mathrm{PC}(\mathbb{R}^2, \mathbb{R})$. The construction is a modification of that from [7], where a similar example of a function from $[0, 1] \times [0, 1]$ onto $[0, 1]$ was constructed. (Compare also [1].) The additional difficulty in our construction is to make sure that some sequences of points in the range of f ($= \mathbb{R}$) have cluster points, which is obvious for all sequences in $[0, 1]$. Also, our basic construction step will be based on a triangle, while the construction in [7] was based on a square. Triangles work better, since for three arbitrary non-collinear points in $\mathbb{R}^3$ there is precisely one plane passing through them, while it is certainly false for four points.

Basic Idea: We will construct, by induction on $n < \omega$, a sequence $\langle S_n\colon n < \omega\rangle$ of triangular "grids" formed with equilateral triangles of side length $1/2^{k_n}$, as

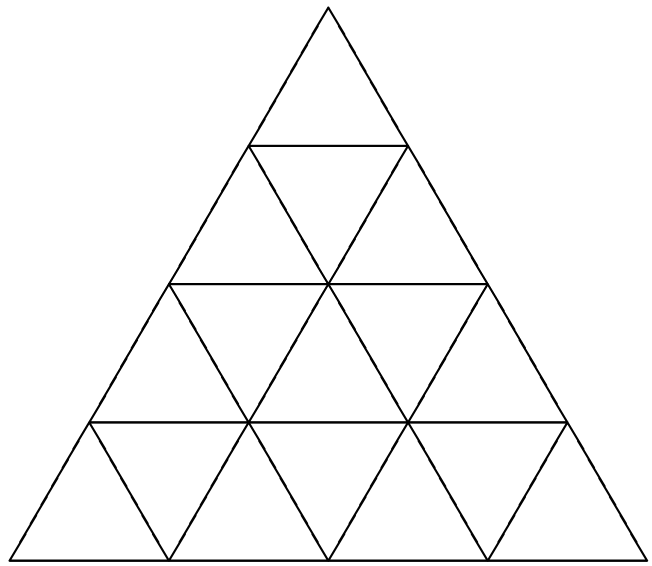

Figure 1: Grid S_n

in Figure 1. The grid S_n will be identified with the points on the edges of triangles forming it and we will be assuming that $S_n \subseteq S_{n+1}$ for all $n < \omega$. With each grid S_n we will associate a continuous function $f_n \colon S_n \to \mathbb{R}$ which is linear on each side of a triangle from S_n. Moreover, each f_{n+1} will be an extension of f_n. Function f will be defined as an extension of $\bigcup_{n<\omega} f_n$.

Terminology: In what follows a *triangle* will be identified with the set of points of its interior or its boundary.

For a grid S we say that a *triangle T is from S* if the interior of T is equal to a component of $\mathbb{R}^2 \setminus S$.

For an equilateral triangle T, its *basic partition* will be its division into seven equilateral triangles, as in Figure 2. The central triangle $\hat{T}$ of Figure 2 will be referred as *the middle quarter of* T. Thus, $\hat{T} \cap \mathrm{bd}(T) = \emptyset$ and the length of each side of $\hat{T}$ is equal to $1/4$ of the length of a side of T.

If a function F is defined on the three vertices of a triangle T, its *basic extension* is defined as the unique function $\hat{F} \colon T \to \mathbb{R}$ extending F whose graph is a subset of a plane. Notice, that $\hat{F}$ is linear on each side of the triangle T and that $\hat{F}$ extends F even if the function F has already been defined on some side of T as long as F is linear on this side.

Inductive Construction: We will define inductively three increasing sequences $\langle S_n \colon n < \omega \rangle$ of triangular grids as in Figure 1, $\langle f_n \in \mathbb{R}^{S_n} \colon n < \omega \rangle$ of

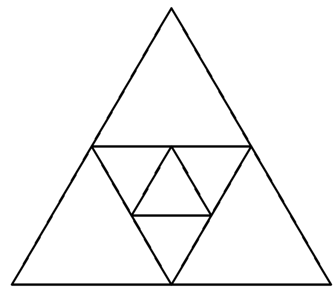

Figure 2: Basic partition

continuous functions and $\langle k_n < \omega \colon n < \omega \rangle$ of natural numbers such that the following inductive conditions are satisfied for every $n < \omega$.

(i) $f_n \colon S_n \to [-2^n, 2^n]$ and is linear on each side of a triangle T from S_n.

(ii) The side length of each triangle from S_n is equal to $1/2^{k_n}$.

(iii) The variation of f_n on each triangle from S_n is $\leq 1/2^n$.

(iv) If $n > 0$, then for every triangle T from S_{n-1} and every dyadic number $i/2^n \in [-2^n, 2^n]$ with $i \in \mathbb{Z}$ $(-4^n \leq i \leq 4^n)$ there is a triangle $T_i \subseteq \hat{T}$ such that $\mathrm{bd}(T_i) \subseteq S_n$ and $f_n(x) = i/2^n$ for every $x \in \mathrm{bd}(T_i)$.

(v) If $n > 0$, T is a triangle from S_{n-1} and T' is a triangle from S_n such that $T' \subseteq T$ and $T' \not\subseteq \hat{T}$, then $f_n[\mathrm{bd}(T')] \subseteq [-M, M]$, where $M = \max\{|f_{n-1}(x)| \colon x \in \mathrm{bd}(T)\}$.

To start the induction, take $k_0 = 0$, define grid S_0 as in Figure 1 with all sides of length $1 = 1/2^0$ and choose $f_0 \colon S_0 \to \mathbb{R}$ as constantly equal 0. It is easy to see that the conditions (i)–(v) are satisfied with such a choice.

Next, assume that for some $n > 0$ we already have S_{n-1}, f_{n-1} and k_{n-1} satisfying (i)–(v). We will define S_n, find k_n and extend f_{n-1} to $f_n \colon S_n \to \mathbb{R}$ such that (i)–(v) will still hold. Put $F_n = f_{n-1}$.

Step 1. Let T be a triangle from S_{n-1} and extend F_n into each vertex of its middle quarter $\hat{T}$ by assigning it the value 0. Notice, that F_n is defined on all vertices of the basic partition of T.

Partition $\hat{T}$ into a grid S such that the size of each triangle from S is equal $1/2^{\hat{k}_n}$. The number $\hat{k}_n < \omega$ is chosen as a minimal number such that there are $2 \cdot 8^n + 1$ disjoint triangles $\{T_i \colon i \in \mathbb{Z}, -4^n \leq i \leq 4^n\}$ from S none of

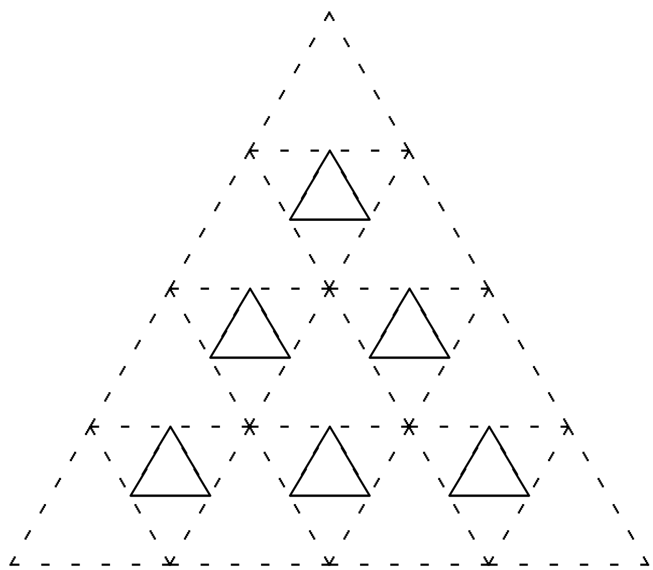

Figure 3: Some triangles of the grid of $\hat{T}$

which intersects the boundary of $\hat{T}$. (See Figure 3.) Notice that the value of $\hat{k}_n$ does not depend on T. On the vertices of each triangle T_i define F_n to be equal $i/2^n$. On the remaining undefined vertices of S define F_n to be equal 0. Notice that F_n is defined on all vertices of each triangle defined so far.

Step 2. Extend F_n into $\mathbb{R}^2$, by defining it on every triangle T constructed so far as the basic extension of $F_n|_{\mathrm{bd}(T)}$.

Notice that if we extend grid S_{n-1} to the grid $\hat{S}_n$ with side length of each triangle from $\hat{S}_n$ equal $1/2^{\hat{k}_n}$ and put $\hat{f}_n = F_n|_{\hat{S}_n}$, then the triple $\langle \hat{S}_n, \hat{f}_n, \hat{k}_n \rangle$ satisfies conditions (i), (ii), (iv) and (v).

Step 3. We have to modify $\hat{S}_n$, $\hat{f}_n$ and $\hat{k}_n$ to also get condition (iii), while keeping the other properties. First notice that for every triangle T from $\hat{S}_n$ and any interval J inside T the slope of F_n on J does not exceed the number $\frac{\text{length of the range of } F_n}{\text{length of a side of } T} \leq \frac{2\ 2^n}{2^{-\hat{k}_n}} = 2^{n\hat{k}_n+1}$. So, let $k_n \geq n\hat{k}_n + n + 1$, in which case

$$\frac{1}{2^{k_n}} 2^{n\hat{k}_n+1} \leq \frac{1}{2^n},$$

let S_n be a refinement of the grid $\hat{S}_n$ with triangles with side size $1/2^{k_n}$ and put $f_n = F_n|_{S_n}$. It is easy to see that this gives us (iii) while preserving the other conditions. This finishes the inductive construction.

Let $S=\bigcup_{n<\omega} S_n$ and define f on S by $f=\bigcup_{n<\omega} f_n$. To extend it to $\mathbb{R}^2\setminus S$ notice that

($\star$) for every $x\in\mathbb{R}^2\setminus S$ there exists a number $f(x)\in\mathbb{R}$ and a sequence $\langle T_k\colon k<\omega\rangle$ of triangles with x being an interior point of each T_k such that $\lim_{k\to\infty}\operatorname{diam}(T_k)\to 0$ and

$$f(x)=\lim_{k\to\infty}\min f[\operatorname{bd}(T_k)]=\lim_{k\to\infty}\max f[\operatorname{bd}(T_k)].$$

The proof of ($\star$) will finish the construction of f.

To see ($\star$) fix $x\in\mathbb{R}^2\setminus S$ and let T^0_n be the triangle from S_n such that x belongs to the interior of T^0_n. Let $N=\{n<\omega\colon n>0\ \&\ T^0_n\subseteq\hat{T}^0_{n-1}\}$. There are two cases to consider.

Case 1. The set N is infinite. Then, let $\langle n_k\colon k<\omega\rangle$ be a one-to-one enumeration of N and define $T_k=\hat{T}^0_{n_k-1}$. It is easy to see that this sequence satisfies ($\star$) with $f(x)=0$.

Case 2. The set N is finite. Let $m<\omega$ be such that $T^0_n\not\subseteq\hat{T}^0_{n-1}$ for every $n\geq m$ and let $M=\max\{|f_{m-1}(x)|\colon x\in\operatorname{bd}(T^0_{m-1})\}$. Then, by condition (v), $f_n[\operatorname{bd}(T^0_n)]\subseteq[-M,M]$ for every $n\geq m$. So, there exists an increasing sequence $\langle n_k\geq m\colon k<\omega\rangle$ such that $L=\lim_{k\to\infty}\max f[\operatorname{bd}(T^0_{n_k})]$ exists. It is easy to see that the sequence $\langle T_k\rangle=\langle T^0_{n_k}\rangle$ satisfies ($\star$) with $f(x)=L$, since the variation of f on $\operatorname{bd}(T^0_{n_k})$ tends to 0 as $k\to\infty$.

This finishes the construction of function f. It remains to show that f has the desired properties.

Clearly ($\star$) implies that f is peripherally continuous at every point $x\in\mathbb{R}^2\setminus S$. To see that f is peripherally continuous on S take $x\in S$. Then, there exists $k<\omega$ such that $x\in S_n$ for every $n\geq k$. For any such n let $\mathcal{T}_n$ be the set of all triangles from S_n to which x belongs. Notice that $\mathcal{T}_n$ has at most six elements and that x belongs to the interior of the polygon $P_n=\bigcup\mathcal{T}_n$. Hence, the variation on the boundary of P_n is at most $6/2^n$ and the diameter of P_n is at most $1/2^{n-1}$. So, the sequence $\langle P_n\rangle$ guarantees that f is peripherally continuous at x.

To finish the proof it is enough to find a dense G_δ set G which is f-negligible for $\mathrm{PC}(\mathbb{R}^2,\mathbb{R})$. For any dyadic number d and any $k\in\omega$ let $\mathcal{F}^k_d$ denote the family of all triangles T for which there exists $n\geq k$ such that T is from S_n and $f_n(x)=d$ for every $x\in\operatorname{bd}(T)$. Let G^k_d be the union of the interiors of all triangles $T\in\mathcal{F}^k_d$. Then, by condition (iv), each set G^k_d is open and dense. Therefore, $G=\bigcap\{G^k_d\colon k\in\omega\ \&\ d \text{ is dyadic}\}$ is a dense G_δ set. It is easy to see, that f is peripherally continuous if we redefine it on the set G in an arbitrary way.

This finishes the proof of Theorem 3.3. $\square$

4 Compositions of Lebesgue Measurable Functions

We can consider similar problems for compositions of functions. For example, we know that every function is a composition of Lebesgue measurable functions [15]. (See also *Problem 6378*, American Mathematical Monthly, **90**, 573.) It is easy to make every function in $\mathbb{R}^{\mathbb{R}}$ measurable (in a sense of definition of A) using composition with just one function. We simply take a composition with a constant function. So we need to define cardinal invariants in a different way.

The next definition will represent one of the ways the problem can be approached. Instead of "forcing the family H to be in $\mathcal{F}$" we will try to recover all elements of H with one "coding" function $\hat{f} \in \mathcal{F}$ and the class $\mathcal{F}$ of all codes. This leads to the following definitions.

$$C_r(\mathcal{F}) = \min\{|H|\colon H \subseteq \mathbb{R}^{\mathbb{R}} \ \& \ \neg\exists\, \hat{f} \in \mathcal{F}\, \forall\, h \in H \exists\, f \in \mathcal{F}\ f \circ \hat{f} = h\} \cup \{(2^{\mathfrak{c}})^+\}$$

and

$$C_l(\mathcal{F}) = \min\{|H|\colon H \subseteq \mathbb{R}^{\mathbb{R}} \ \& \ \neg\exists\, \hat{f} \in \mathcal{F}\, \forall\, h \in H \exists\, f \in \mathcal{F}\ \hat{f} \circ f = h\} \cup \{(2^{\mathfrak{c}})^+\}.$$

Let $\mathcal{L}$ be the family of all Lebesgue measurable functions from $\mathbb{R}$ into $\mathbb{R}$.

Theorem 4.1 *$C_r(\mathcal{L}) = (2^{\mathfrak{c}})^+$ and $C_l(\mathcal{L}) = \mathfrak{c}^+$.*

Proof. To see $C_r(\mathcal{L}) \geq (2^{\mathfrak{c}})^+$ we will show that the family $H = \mathbb{R}^{\mathbb{R}}$ of all functions can be "coded" by one function $\hat{f} \in \mathcal{L}$. Simply, let $\hat{f}$ be a Borel isomorphism from $\mathbb{R}$ onto the Cantor ternary set C. For any function $h \in \mathbb{R}^{\mathbb{R}}$ we define $f_h \in \mathcal{L}$ by putting $f_h \equiv 0$ on the complement of C and $f_h = h \circ \hat{f}^{-1}$ on C. Then $h = f_h \circ \hat{f}$.

To see that $C_l(\mathcal{L}) \geq \mathfrak{c}^+$ let $H = \{h_\xi \colon \xi < \mathfrak{c}\} \subseteq \mathbb{R}^{\mathbb{R}}$ and let $\{C_\xi \colon \xi < \mathfrak{c}\}$ be a partition of the Cantor ternary set C into perfect sets. Then for every $\xi < \mathfrak{c}$ take a Borel isomorphism $f_\xi \colon \mathbb{R} \to C_\xi$ and define $\hat{f}(x) = (h_\xi \circ f_\xi^{-1})(x)$ for $x \in C_\xi$ and $\hat{f}(x) = 0$ otherwise. It is easy to see that $\hat{f} \in \mathcal{L}$ and $\hat{f} \circ f_\xi = h_\xi$ for every $\xi < \mathfrak{c}$.

To prove $C_l(\mathcal{L}) \leq \mathfrak{c}^+$ take $\{h_\xi \colon \xi < \mathfrak{c}^+\}$ from Lemma 3.1. Assume that there exists a sequence $\{f_\xi \colon \xi < \mathfrak{c}^+\}$ of measurable functions and a measurable function $\hat{f}$ such that $h_\xi = \hat{f} \circ f_\xi$ for every $\xi < \mathfrak{c}^+$. For each $\xi < \mathfrak{c}^+$ let P_ξ be a perfect set such that $f_\xi|P_\xi$ is continuous. Then, by a cardinality argument, there are $\zeta < \xi < \mathfrak{c}^+$ such that $P_\zeta = P_\xi$ and $f_\zeta|P_\zeta = f_\xi|P_\xi$. So, $h_\zeta|P_\zeta = h_\xi|P_\xi$. Contradiction. $\square$

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