Real Analysis Exchange
Vol. 22(1), 1996-97, pp. 382–389

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ON HENSTOCK INTEGRABILITY IN EUCLIDEAN SPACES

Abstract

In this paper, we give a necessary and sufficient condition in terms of Lebesgue integrable functions for Henstock integrability in Euclidean space.

By means of the Cauchy and Harnack extension theorems for the one-dimensional Henstock integral, Liu [5] proved that

Theorem 1 *If f is Henstock integrable on $[a,b]$, then there is a sequence $\{X_k\}$ of closed subsets of $[a,b]$ such that $X_k \subset X_{k+1}$ for all k, $\bigcup_{k=1}^{\infty} X_k = [a,b]$, f is Lebesgue integrable on each X_k and*

$$\lim_{k\to\infty} (L) \int_{X_k \cap [a,x]} f(t)\, dt = (H) \int_a^x f(t)\, dt$$

uniformly on $[a,b]$.

Liu's proof is real-line dependent, and so it is difficult to generalize Theorem 1 to higher dimensions. In this note, we shall give a direct proof of the multidimensional version of Liu's result. Consequently, we deduce a necessary and sufficient condition for Henstock integrability in higher dimensions (Theorem 7).

First, we give some preliminaries (see [3]).

Let $\mathbf{R}$ and $\mathbf{R}^+$ denote the real line and the positive real line respectively, m a fixed positive integer and $\mathbf{R}^m$ the m-dimensional euclidean space.

Mathematical Reviews subject classification: Primary: 26A39
Received by the editors January 1, 1996

Unless otherwise stated, an interval will always be a compact nondegenerate interval of the form $[s,t] = \prod_{i=1}^{m}[s_i, t_i]$ where $s = (s_1, s_2, \ldots, s_m)$ and $t = (t_1, t_2, \ldots, t_m)$.

Also, $E = \prod_{i=1}^{m}[a_i, b_i]$ will denote a fixed interval in $\mathbf{R}^m$, and $B(x,\delta)$ denotes an open ball in $\mathbf{R}^m$ with center x and radius δ. A finite collection of intervals whose interiors are disjoint is called a nonoverlapping collection. A partial division $D = \{(I,\xi)\}$ of E is a finite collection of interval-point pairs such that the collection of intervals are non-overlapping. If, in addition, the union of I from D gives E, we say that D is a division of E. Let $\delta : E \longrightarrow \mathbf{R}^+$ be given. A partial division $D = \{(I,\xi)\}$ is said to be *δ-fine* if for each $(I,\xi) \in D$ with ξ being a vertex of I, we have $I \subset B(\xi, \delta(\xi))$.

In this note, all functions will be assumed to be real-valued, and often the same letter is used to denote a function on E as well as its restriction to a set $Z \subset E$. A function $f : E \longrightarrow \mathbf{R}$ is said to be *Henstock integrable* to a real number A on E if for every $\varepsilon > 0$, there exists $\delta : E \longrightarrow \mathbf{R}^+$ such that for any δ-fine division $D = \{(I,\xi)\}$ of E, we have

$$\left|(D)\sum f(\xi)\,|I| - A\right| < \varepsilon.$$

We write $A = (H)\int_E f$. If g is Lebesgue integrable on E, we write the Lebesgue integral of g over E as $(L)\int_E g$. It is known that if g is Lebesgue integrable on E, then g is Henstock integrable there with the same integral value. For a proof, see [6, Proposition 4, Remark 6]. The words "measure", "measurable" and "almost everywhere" always refer to the m-dimensional Lebesgue measure. If X is measurable, we shall write $|X|$ as the m-dimensional Lebesgue measure of X. We next give Henstock's lemma.

Theorem 2 *If f is Henstock integrable on E, then for every $\varepsilon > 0$, there exists $\delta : E \longrightarrow \mathbf{R}^+$ such that for any δ-fine partial division $D = \{(I,\xi)\}$ of E, we have*

$$(D)\sum \left| f(\xi)\,|I| - (H)\int_I f\right| < \varepsilon.$$

As a consequence of Henstock's lemma, we shall prove the following two lemmas.

Lemma 3 *If f is Henstock integrable on E, then for every $\epsilon > 0$, there exists $\delta : E \longrightarrow \mathbf{R}^+$ such that for every δ-fine partial division $D = \{(I,\xi)\}$ of E, we have*

$$(D)\sum \left| f(\xi)|I \cap E_0| - (H)\int_{I\cap E_0} f\right| < \varepsilon$$

for every subinterval E_0 of E.

PROOF. By Theorem 2, for $\varepsilon > 0$, there exists $\delta : E \longrightarrow \mathbf{R}^+$ such that for any δ-fine partial division $D = \{(I, \xi)\}$ of E, we have

$$(D) \sum \left| f(\xi)\,|I| - (H)\int_I f \right| < \frac{\varepsilon}{2^{m+1}}. \tag{1}$$

Let E_0 be a subinterval of E. We let

$$D_0 = \{(I \cap E_0, \xi) : |I \cap E_0| > 0 \text{ and } (I, \xi) \in D\}.$$

Writing $S = (D) \sum \left| f(\xi)\,|I \cap E_0| - (H)\int_{I \cap E_0} f \right|$, we want to show that $S < \varepsilon$. Note that

$$\begin{aligned} S &= (D) \sum \left| f(\xi)\,|I \cap E_0| - (H)\int_{I \cap E_0} f \right| \\ &= (D_0) \sum \left| f(\xi)\,|I \cap E_0| - (H)\int_{I \cap E_0} f \right| \\ &= (D_0) \sum_{\xi \in E_0} \left| f(\xi)\,|I \cap E_0| - (H)\int_{I \cap E_0} f \right| \\ &\quad + (D_0) \sum_{\xi \notin E_0} \left| f(\xi)\,|I \cap E_0| - (H)\int_{I \cap E_0} f \right| \\ &< \frac{\varepsilon}{2^{m+1}} + (D_0) \sum_{\xi \notin E_0} \left| f(\xi)\,|I \cap E_0| - (H)\int_{I \cap E_0} f \right| \end{aligned}$$

as $\{(I \cap E_0, \xi) : \xi \in E_0 \text{ and } (I, \xi) \in D\}$ is a δ-fine partial division of E. Hence we have

$$S < \frac{\varepsilon}{2^{m+1}} + (D_0) \sum_{\xi \notin E_0} \left| f(\xi)\,|I \cap E_0| - (H)\int_{I \cap E_0} f \right|. \tag{2}$$

It remain to prove that the second term in (2) is less than $\frac{\varepsilon}{2}$. Note that when $\xi \notin E_0$, the interval $I \cap E_0$ does not contain ξ and therefore $(I \cap E_0, \xi)$ is no longer δ-fine.

Let $D_1 = \{(I \cap E_0, \xi) \in D_0 : \xi \notin E_0 \text{ and } (I, \xi) \in D\} = \{(I_j \cap E_0, \xi_j)\}_{j=1}^{p}$ and for each subinterval E_1 of E, we put

$$G_j(I_j \cap E_1, \xi_j) = f(\xi_j)\,|I_j \cap E_1| - (H)\int_{I_j \cap E_1} f \tag{3}$$

for each $j = 1, 2, \ldots, p$.

We recall that if $x = (x_1, x_2, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_m)$ are two distinct vertices of an interval I, x and y are said to be opposite if $x_i \neq y_i$ for all $i = 1, 2, \ldots, m$. We shall denote an interval with ξ, x as opposite vertices by $< \xi, x >$. Then for each $j = 1, 2, \ldots, p$,

$$\begin{aligned}\left| f(\xi_j)\,|I_j \cap E_0| - (H)\int_{I_j \cap E_0} f \right| &= |G_j(I_j \cap E_0, \xi_j)| \quad \text{by (3)} \\ &= \left| \sum_{l=1}^{2^m} (-1)^{n(l,j)} G_j(< \xi_j, \gamma^{(l,j)} >, \xi_j) \right| \leq \sum_{l=1}^{2^m} \left| G_j(< \xi_j, \gamma^{(l,j)} >, \xi_j) \right|\end{aligned}$$

where $\gamma^{(l,j)} = (\gamma_1^{(l,j)}, \gamma_2^{(l,j)}, \ldots, \gamma_m^{(l,j)})$ represents a vertex of $I_j \cap E_0 = \prod_{i=1}^m [\alpha_i^{(j)}, \beta_i^{(j)}]$ and $n(l,j)$ is the cardinality of the set $\{i : \gamma_i^{(l,j)} = \alpha_i^{(j)}\}$.

Hence, for $j = 1, 2, \ldots, p$,

$$|G_j(I_j \cap E_0, \xi_j)| \leq \sum_{l=1}^{2^m} \left| G_j(< \xi_j, \gamma^{(l,j)} >, \xi_j) \right| \tag{4}$$

Consequently, by (4),

$$\begin{aligned}\sum_{j=1}^{p} \left| f(\xi_j)\,|I_j \cap E_0| - (H)\int_{I_j \cap E_0} f \right| &= \sum_{j=1}^{p} |G_j(I_j \cap E_0, \xi_j)| \\ \leq \sum_{j=1}^{p}\sum_{l=1}^{2^m} \left| G_j(< \xi_j, \gamma^{(l,j)} >, \xi_j) \right| &= \sum_{l=1}^{2^m}\sum_{j=1}^{p} \left| G_j(< \xi_j, \gamma^{(l,j)} >, \xi_j) \right|\end{aligned}$$

Recall that $D_0 = \{(I \cap E_0, \xi) : |I \cap E_0| > 0 \text{ and } (I, \xi) \in D\}$ and $D_1 = \{(I \cap E_0, \xi) \in D_0 : \xi \notin E_0 \text{ and } (I, \xi) \in D\} = \{(I_j \cap E_0, \xi_j)\}_{j=1}^p$. By our definition of D_0 and D_1, we see that $\{(< \xi_j, \gamma^{(l,j)} >, \xi_j)\}_{j=1}^p$ is a δ-fine partial division of E for each $l = 1, 2, \ldots, 2^m$. We have

$$\begin{aligned}\sum_{j=1}^{p} \left| f(\xi_j)\,|I_j \cap E_0| - (H)\int_{I_j \cap E_0} f \right| &\leq \sum_{l=1}^{2^m}\sum_{j=1}^{p} \left| G_j(< \xi_j, \gamma^{(l,j)} >, \xi_j) \right| \\ &< \sum_{l=1}^{2^m} \frac{\varepsilon}{2^{m+1}} \quad \text{by Theorem 2} \\ &= \frac{\varepsilon}{2}\end{aligned}$$

By (2), we get the required inequality. The proof is complete. □

Lemma 4 *Suppose f is Henstock integrable on E, and f is Lebesgue integrable on some closed subset Y of E. Then given $\epsilon > 0$, there exists $\delta : Y \longrightarrow \mathbf{R}^+$ such that for any δ-fine partial division $D = \{(I, \xi)\}$ with $\xi \in Y$, we have*

$$(D) \sum \left| (L) \int_{I \cap Y \cap E_0} f - (H) \int_{I \cap E_0} f \right| < \varepsilon$$

for every subinterval E_0 of E.

PROOF. By Lemma 3, there exists $\delta_1 : E \longrightarrow \mathbf{R}^+$ such that for any δ_1-fine partial division $D = \{(I, \xi)\}$ of E, we have

$$(D) \sum \left| f(\xi) \, |I \cap E_0| - (H) \int_{I \cap E_0} f \right| < \frac{\varepsilon}{2} \tag{5}$$

for every subinterval E_0 of E.

Since f is Lebesgue integrable on Y, $f\chi_Y$ is Henstock integrable on E, where χ_Y denotes the characteristic function of Y. So there exists $\delta_2 : E \longrightarrow \mathbf{R}^+$ such that for any δ_2-fine partial division $D = \{(I, \xi)\}$ of E with $\xi \in Y$, we have

$$(D) \sum \left| f(\xi) \chi_Y(\xi) \, |I \cap E_0| - (L) \int_{I \cap E_0} f \chi_Y \right| < \frac{\varepsilon}{2} \tag{6}$$

for every subinterval E_0 of E.

Define $\delta : Y \longrightarrow \mathbf{R}^+$ as follows: $\delta(\xi) = \min\{\delta_1(\xi), \delta_2(\xi)\}$. Then for any δ-fine partial division $D = \{(I, \xi)\}$ of E with $\xi \in Y$, it is both δ_1-fine and δ_2-fine. Thus

$$\begin{aligned}
(D) \sum & \left| (L) \int_{I \cap Y \cap E_0} f - (H) \int_{I \cap E_0} f \right| \\
& \leq (D) \sum \left| (L) \int_{I \cap E_0} f \chi_Y - f(\xi) \, |I \cap E_0| \right| \\
& \qquad + (D) \sum \left| f(\xi) \, |I \cap E_0| - (H) \int_{I \cap E_0} f \right| \\
& < \varepsilon, \quad \text{by (5) and (6).}
\end{aligned}$$

The proof is complete. □

The next theorem due to Kurzweil and Jarnik [2, Theorem 2.10] is also an important tool. For convenience, in what follows, we shall write $X_k \uparrow E$ to mean $X_k \subset X_{k+1}$ for all k and $\bigcup_{k=1}^{\infty} X_k = E$.

Theorem 5 *Let f be Henstock integrable on E. Then there exists a sequence $\{Y_k\}$ of closed sets with $Y_k \uparrow E$ and f is Lebesgue integrable on each Y_k.*

We shall now state and prove the multidimensional version of Liu's result.

Theorem 6 *If f is Henstock integrable on the interval E, then there exists a sequence $\{X_k\}$ of closed subsets of E such that $X_k \uparrow E$, f is Lebesgue integrable on each X_k and*

$$\sup \left| (L) \int_{X_k \cap E_1} f - (H) \int_{E_1} f \right| \leq \frac{1}{k}$$

for each k, and the above supremum is over all subintervals E_1 of E.

PROOF. In view of Theorem 5, there exists a sequence $\{Y_k\}$ of closed sets with $Y_k \uparrow E$ and f is Lebesgue integrable on each Y_k. By Lemma 4, for every positive integer n and for each k there exists $\delta_k : Y_k \longrightarrow \mathbf{R}^+$ such that for any δ_k-fine partial division $D = \{(I, \xi)\}$ of E with $\xi \in Y_k$, we have

$$(D) \sum \left| (L) \int_{I \cap Y_k \cap E_0} f - (H) \int_{I \cap E_0} f \right| < \frac{1}{2^k n} \tag{7}$$

for every subinterval E_0 of E. Next we want to choose $\{X_n\}$ from $\{Y_k\}$ so that the required inequality holds. By our assumption, Y_k is closed, so dist(ξ,Y_k) > 0 if and only if $\xi \notin Y_k$ where $\text{dist}(\xi, Y_k)$ denotes the distance between ξ and Y_k. Define $\delta : E \longrightarrow \mathbf{R}^+$ as follows :

$$\delta(\xi) = \begin{cases} \delta_n(\xi), & \text{if } \xi \in Y_n \\ \\ \min\{\delta_k(\xi),\ \text{dist}(\xi, Y_{k-1})\}, & \text{if } \xi \in Y_k - Y_{k-1} \quad \text{for each } k > n. \end{cases}$$

Since a δ-fine division of E exists, see for example [3, p. 128], we may fix a δ-fine division $D_0 = \{(I, \xi)\}$ of E. For simplicity, we put $P_n = Y_n$ and $P_k = Y_k - Y_{k-1}$ for $k > n$. Next, we put

$$X_n = \bigcup_{j=n}^{\infty} \big\{ I \cap Y_j : (I, \xi) \in D_0 \ \text{ with } \ \xi \in P_j \big\} \tag{8}$$

The above union is a finite one because D_0 only has finitely many terms. Thus X_n is closed as each Y_j is closed.

Define

$$k(n) = \max\big\{ j : (I, \xi) \in D_0 \ \text{ and } \ \xi \in P_j \big\} + 1 \tag{9}$$

Since $Y_k \uparrow E$, we have

$$Y_{k(n)} \supseteq X_n \tag{10}$$

By the definition of δ and the compactness of Y_n, any δ-fine division $D = \{(I,\xi)\}$ must cover Y_n. Hence

$$Y_n \subseteq X_n. \tag{11}$$

By (10) and (11), we have

$$Y_n \subseteq X_n \subseteq Y_{k(n)}. \tag{12}$$

By (12), we note that f is Lebesgue integrable on X_n.

Claim. $\left|(L)\int_{E_1\cap X_n} f - (H)\int_{E_1} f\right| \leq \frac{1}{n}$ for every closed subinterval E_1 of E.

Observe that if $(I,\xi) \in D_0$ with $\xi \in P_l$ for some positive integer l, then by (8),

$$I \cap X_n = I \cap Y_l. \tag{13}$$

Note that D_0 may have its associated points belonging to P_n only. Without loss of generality, we may suppose D_0 has its associated points belonging to

$$P_{s_1}, P_{s_2}, \ldots, P_{s_l}$$

for some positive integers $s_1 < s_2 < \cdots < s_l$ with $s_1 = n$. Now, we have for all closed subintervals E_1 of E,

$$\begin{aligned}\left|(L)\int_{E_1\cap X_n} f - (H)\int_{E_1} f\right| &\leq (D_0)\sum\left|(L)\int_{E_1\cap I\cap X_n} f - (H)\int_{E_1\cap I} f\right| \\ &\leq \sum_{i=1}^{l}(D_0)\sum_{\xi\in P_{s_i}}\left|(L)\int_{E_1\cap I\cap X_n} f - (H)\int_{E_1\cap I} f\right| \\ &= \sum_{i=1}^{l}(D_0)\sum_{\xi\in P_{s_i}}\left|(L)\int_{E_1\cap I\cap Y_{s_i}} f - (H)\int_{E_1\cap I} f\right| \quad \text{by (13)} \\ &< \frac{1}{n} \quad \text{by (7)}\ .\end{aligned}$$

It is easy to see from (12) that there exists a subsequence of $\{X_n\}$, denoted again by $\{X_n\}$, such that $X_n \uparrow E$. The proof is complete. □

We remark that Theorem 6 is indeed a generalization of Liu's result. Furthermore, Theorem 7 below is also an improvement of the results in Liu [5] and Lee [4].

Theorem 7 *A function f is Henstock integrable on E if and only if there exists a sequence $\{X_k\}$ of closed subsets of E such that $X_k \uparrow E$, f is Lebesgue integrable on each X_k and the following condition holds: for every $\epsilon > 0$ there exists an integer N such that if $k \geq N$ then there exists $\delta_k : E \longrightarrow \mathbf{R}^+$ such that for every δ_k-fine division $D = \{(I, \xi)\}$ of E we have*

$$\left| (D) \sum_{\xi \notin X_k} f(\xi) |I \cap E_1| \right| < \epsilon$$

for every subinterval E_1 of E.

Proof. The proof follows easily as in Bartle [1], by taking note that

$$(D) \sum_{\xi \notin X_k} = (D) \sum - (D) \sum_{\xi \in X_k},$$

and using Theorem 6. □

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