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LIMITS OF TRANSFINITE CONVERGENT SEQUENCES OF DERIVATIVES

Abstract

The paper solves the question whether the limit of transfinite convergent sequence of derivatives is again the derivative. It shows that this problem cannot be solved in the Zermelo-Fraenkel axiomatic system and that this statement is equivalent to the covering number for Lebesgue null ideal being bigger that $\aleph_1$. In the second part of the paper author proved an analogue of Preiss's theorem [P] for the transfinite sequences of derivatives.

1 Introduction

The convergence of transfinite sequences of functions was introduced in the paper [Sie]. Let Ω be the first uncountable ordinal number, let I be a real interval and $f_\xi : I \to \mathbb{R}$, $1 \le \xi < \Omega$ be a sequence of real functions. We say that $f : I \to \mathbb{R}$ is the pointwise limit of this sequence if $f_\xi(x) \to f(x)$ holds for every $x \in I$, i.e.

$$\forall x \in T \quad \forall \varepsilon > 0 \quad \exists \eta < \Omega \quad \forall \xi \ge \eta : \quad |f(x) - f_\xi(x)| < \varepsilon$$

We shall denote this convergence by $f_\xi \to f$ or more precisely $\lim_{\xi<\Omega} f_\xi = f$. An important question is whether the pointwise transfinite convergence preserves some important properties of functions, e.g., continuity or first Baire class. These questions were solved positively in the paper [Š] or [Sie] respectively. In the present paper the question of preserving the property of "being a derivative" will be discussed. The results of this paper can be also used

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to solve the question of preserving the property of "being an approximately continuous function". This problem was mentioned in the paper [Š] as open.

A partial answer to previous questions gives us the following theorem proved by T. Šalát (oral communication).

Theorem. *Let $f_\xi : I \to \mathbb{R}$, $1 \le \xi < \Omega$, be functions differentiable at every point of an interval I. Let $f, g : I \to \mathbb{R}$ be such functions that*

$$f_\xi \to f \text{ and } f'_\xi \to g. \text{ Then } f' = g.$$

PROOF. Let $x_0 \in I$ be an arbitrary point. Then there exists an ordinal number $\eta < \Omega$ such that ${f'}_\xi(x_0) = g(x_0)$ holds for every $\eta \le \xi < \Omega$: (See [Sie]). It is sufficient to prove that $\dfrac{f(x_n) - f(x_0)}{x_n - x_0} \to g(x_0)$ holds for every sequence $x_n \to x_0$, $n \in \mathbb{N}$; $x_n \in I \setminus \{x_0\}$.

Since $f_\xi \to f$, there exists an ordinal number $\xi_0 < \Omega$ such that for all $\xi \ge \xi_0$ we have $f_\xi(x_k) = f(x_k) \quad (k = 0, 1, 2, 3, \dots)$. But then for any ordinal number $\xi \ge \max\{\xi_0, \eta\}$ we have

$$\frac{f(x_n) - f(x_0)}{x_n - x_0} = \frac{f_\xi(x_n) - f_\xi(x_0)}{x_n - x_0} \to {f'}_\xi(x_0) = g(x_0) \text{ for } n \to \infty. \qquad \square$$

Let Δ denote the set of all derivatives on the interval I; i.e., all functions $f : I \to \mathbb{R}$ having primitive functions $F : I \to \mathbb{R}$ such that $f(x) = F'(x)$ for each $x \in I$. Let us introduce the following notation.

$(\mathcal{CH})$ Continuum hypothesis: $\aleph_1 = 2^{\aleph_0}$.

$(\mathcal{MA})_{\aleph_1}$ Martin's axiom: For a nonempty poset (partially ordered set) P having the (CCC)[1] property and a family $\{D_j; j \in J\}$ of dense[2] sets in P ($\mathrm{card}(J) \le \kappa$) there exists a subnet[3] $Q \subset P$ such that $Q \cap D_j \ne \emptyset$. (See [Sch]). We shall use this axiom for $\kappa = \aleph_1$.

$(\mathcal{ADD})_{\aleph_1}$ The union of $\aleph_1$ null sets (Lebesgue measure on $\mathbb{R}$) has (Lebesgue) measure zero. This statement can be written as $\mathrm{add}(L) > \aleph_1$ where $\mathrm{add}(L)$ is the usual notation for the smallest cardinal κ with the property that there are κ null sets such that their union is not null.

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[1]Poset $(P, \prec)$, briefly P has the (CCC) property if every set $Q \subset P$ whose elements are pairwise incompatible is at most denumerable. Two elements p, q ($p \ne q$) of an poset P are incompatible if there does not exist any element $r \in P$ such that $p \prec r$ and $q \prec r$.

[2]Set D is dense in the poset P if for an arbitrary $p \in P$ there exists $d \in D$ such that $p \prec d$.

[3]Q is subnet of P if $Q \subset P$ and Q is a net; i.e. for every elements $p, q \in Q$ there is an element $r \in Q$ such that $p \prec r$ and $q \prec r$.

$(\mathcal{COV})_{\aleph_1}$ There are $\aleph_1$ null sets (Lebesgue measure on $\mathbb{R}$) covering $\mathbb{R}$. This statement can be written as $\mathrm{cov}(L) = \aleph_1$ where $\mathrm{cov}(L)$ is the usual notation for the smallest cardinal κ such that the real line is the union of κ null sets.

$(\mathcal{D})$ If $f_\xi : I \to \mathbb{R}$, $1 \le \xi < \Omega$ is an arbitrary transfinite pointwise convergent sequence of derivatives, then the limit function $f = \lim\limits_{\xi<\Omega} f_\xi$ is also a derivative; i.e. $f \in \Delta$.

$(\mathcal{AC})$ If $f_\xi : I \to \mathbb{R}$; $1 \le \xi < \Omega$ is an arbitrary transfinite pointwise convergent sequence of approximately continuous functions, then the limit function $f = \lim\limits_{\xi<\Omega} f_\xi$ is also approximately continuous.

$(\mathcal{ZFC})$ Zermelo-Fraenkel set theory including the axiom of choice.

Both the continuum hypothesis $(\mathcal{CH})$ and Martin's axiom $(\mathcal{MA})_{\aleph_1}$ are statements that are independent with respect to Zermelo-Fraenkel set theory $(\mathcal{ZFC})$ and can be added as a new axiom (of course not both together). The following relations between previous statements were proved in paper [Sch] or they can be easily derived.

$$(\mathcal{ZFC}) + (\mathcal{MA})_{\aleph_1} \implies (\mathcal{ZFC}) + (\mathcal{ADD})_{\aleph_1} \implies (\mathcal{ZFC}) + \neg(\mathcal{COV})_{\aleph_1}$$
$$(\mathcal{ZFC}) + (\mathcal{CH}) \implies (\mathcal{ZFC}) + (\mathcal{COV})_{\aleph_1}$$

The main aim of this paper is to prove following implications.

$$(\mathcal{ZFC}) + \neg(\mathcal{COV})_{\aleph_1} \implies (\mathcal{ZFC}) + (\mathcal{D})$$
$$(\mathcal{ZFC}) + (\mathcal{COV})_{\aleph_1} \implies (\mathcal{ZFC}) + \neg(\mathcal{D})$$

which means that $(\mathcal{D})$ and $\neg(\mathcal{D})$ are statements that cannot be derived from $(\mathcal{ZFC})$ because both $(\mathcal{ZFC}) + (\mathcal{D})$ and $(\mathcal{ZFC}) + \neg(\mathcal{D})$ remain consistent if $(\mathcal{ZFC})$ is consistent. In addition following axiomatic systems are equivalent.

$$(\mathcal{ZFC}) + \neg(\mathcal{COV})_{\aleph_1} \iff (\mathcal{ZFC}) + (\mathcal{D})$$
$$(\mathcal{ZFC}) + (\mathcal{COV})_{\aleph_1} \iff (\mathcal{ZFC}) + \neg(\mathcal{D})$$

Remark 1. We also prove that the statements $(\mathcal{AC})$ and $\neg(\mathcal{AC})$ are independent with respect to $(\mathcal{ZFC})$ axioms because from results of this paper the following equivalences can also be derived.

$$(\mathcal{ZFC}) + \neg(\mathcal{COV})_{\aleph_1} \iff (\mathcal{ZFC}) + (\mathcal{AC})$$
$$(\mathcal{ZFC}) + (\mathcal{COV})_{\aleph_1} \iff (\mathcal{ZFC}) + \neg(\mathcal{AC})$$

2 Limits of Pointwise Convergent Transfinite Sequences of Functions When $\neg(\mathcal{COV})_{\aleph_1}$ Holds.

In what follows we shall suppose that $\neg(\mathcal{COV})_{\aleph_1}$ or a stronger assumption $(\mathcal{ADD})_{\aleph_1}$ holds. First we introduce an auxiliary lemma which we need for the proof of the following Theorem 2.

Lemma 1 *Suppose $\neg(\mathcal{COV})_{\aleph_1}$. Then the inner Lebesgue measure of a union of $\aleph_1$ null sets is zero.*

Proof. Suppose that this statement does not hold. Then there exist sets A_j with $\lambda(A_j) = 0$, $j \in J$ and $\mathrm{card}(J) = \aleph_1$ such that

$$\lambda_*(A) > 0 \text{ where } A = \bigcup_{j \in J} A_j$$

(λ_* means the inner Lebesgue measure). Then according to a well-known fact $(\mathcal{COV})_{\aleph_1}$ holds; i.e. there exist $\aleph_1$ null sets which cover the entire real line which is contrary to the assumption $\neg(\mathcal{COV})_{\aleph_1}$. These sets can be chosen as

$$B_j = \left(C \cup \bigcup_{q \in Q} (A_j + q) \right) \text{ for } j \in J \text{ where } C = \mathbb{R} \setminus \bigcup_{q \in Q} (A + q). \qquad \square$$

Theorem 2 *Let $f_\xi : I \to \mathbb{R}$, $1 \le \xi < \Omega$, be a pointwise convergent transfinite sequence of measurable functions. Let $f = \lim_{\xi < \Omega} f_\xi$. Let (i) or (ii) hold.*

(i) Axiom $(\mathcal{ADD})_{\aleph_1}$ holds.

(ii) The function f is measurable and axiom $\neg(\mathcal{COV})_{\aleph_1}$ holds.

Then the function f is measurable and there exists an ordinal number $\eta < \Omega$ such that for every $\eta \le \xi < \Omega$ $f_\xi(x) = f(x)$ holds almost everywhere on I.

Proof. We will prove $(i) \Longrightarrow (ii)$. It is sufficient to prove that function f is measurable, because $(\mathcal{ADD})_{\aleph_1} \Longrightarrow \neg(\mathcal{COV})_{\aleph_1}$.

In paper [Sch] it was demonstrated that the union and the intersection of at most $\aleph_1$ Lebesgue measurable sets is a measurable set provided (i) holds. This fact will be used now.

Since for every $x \in I$ $f_\xi(x) \to f(x)$, there exist an ordinal number $\eta_x < \Omega$ such that $f_\xi(x) = f(x)$ for every $\eta_x \le \xi < \Omega$. Obviously we have

$$\{x \in I; f(x) > \alpha\} = \bigcup_{\eta < \Omega} \bigcap_{\eta < \xi} \{x \in I; f_\xi(x) > \alpha\}$$

and therefore the function f is measurable.

Now let (ii) hold. For every ordinal number $\xi < \Omega$ we define

$$E_\xi = \{x \in I; f_\eta(x) = f(x) \quad \forall \eta \geq \xi\}$$

The union of E_ξ is the interval I and $E_\xi \subset E_\zeta$ whenever $\xi \leq \zeta$. Without loss of generality we can suppose that I is a bounded interval. Let $c = \sup_{\xi<\Omega} \lambda^*(E_\xi)$, where λ^* is Lebesgue outer measure on I. According to the definition of supremum for every $n \in \mathbb{N}$ there exist an ordinal number ξ_n such that

$$\lambda^*(E_{\xi_n}) \geq c - \frac{1}{n}$$

There exists an ordinal number $\eta < \Omega$ such that $\xi_n \leq \eta$ for every $n \in \mathbb{N}$. Therefore for every $n \in \mathbb{N}$ we have

$$c \geq \lambda^*(E_\eta) \geq \lambda^*(E_{\xi_n}) \geq c - \frac{1}{n}$$

Hence $\lambda^*(E_\zeta) = c$ for all $\eta \leq \zeta < \Omega$. Let G be a measurable set such that $E_\eta \subset G \subset I$ and $\lambda^*(E_\eta) = \lambda(G)$.

We want to prove $\lambda(I \setminus G) = 0$. Suppose not. We can write

$$I \setminus G = I \cap G^c = G^c \cap \bigcup_{\eta \leq \zeta} E_\zeta = \bigcup_{\eta < \zeta} (E_\zeta \setminus G).$$

If $\lambda(I \setminus G) = \lambda_*(I \setminus G) > 0$, then according to Lemma 1 there exist $\eta < \zeta$ such that $\lambda^*(E_\zeta \setminus G) > 0$. The set G is measurable and therefore according to Caratheodory's definition of measurability

$$\lambda^*(E_\zeta) \geq \lambda^*(E_\zeta \setminus G) + \lambda^*(E_\zeta \cap G) > \lambda^*(E_\zeta \cap G) \geq \lambda^*(E_\eta) = c$$

contrary to $\lambda^*(E_\zeta) > c$. Therefore $\lambda(I \setminus G) = 0$ i.e. $\lambda(E_\eta) = \lambda(I)$. Hence f and f_ζ for fixed $\zeta \geq \eta$ are two measurable functions which disagree on a set of inner measure zero ($\{x \in I; f(x) \neq f_\zeta(x)\} \subset I \setminus E_\zeta$) and therefore $f_\xi(x) = f(x)$ holds almost everywhere on I. □

Remark 2. The previous proof shows that the assumption $(\mathcal{ADD})_{\aleph_1}$ guarantees measurability of a transfinite limit of measurable functions. The converse of this statement is also true; i.e. the assumption that every transfinite limit of measurable functions is measurable give us that $(\mathcal{ADD})_{\aleph_1}$ holds.

Proof. Assume that $\neg(\mathcal{ADD})_{\aleph_1}$ holds. Then there exist sets A_ξ with $\lambda(A_\xi) = 0$, $\xi < \Omega$ such that

$$\lambda^*(A) > 0 \text{ where } A = \bigcup_{\xi<\Omega} A_\xi.$$

(λ^* means the outer Lebesgue measure.) If the set A is non-measurable, we define $B_\xi = A_\xi, \xi < \Omega$. Otherwise let B be a non-measurable subset of A and define $B_\xi = B \cap A_\xi, \xi < \Omega$. Then the sets B_ξ have measure zero and their union is a non-measurable set. Define

$$f_\xi = \chi_{\bigcup_{\zeta<\xi} B_\zeta}$$

(where χ_C means the characteristic function of the set C). Then $f_\xi \to \chi_B$; i.e. χ_B is a non-measurable function which is the transfinite limit of measurable functions. That is a contradiction and therefore the assumption $\neg(\mathcal{ADD})_{\aleph_1}$ cannot hold. □

The main theorem of this section is the following.

Theorem 3 *$(\mathcal{ZFC}) + \neg(\mathcal{COV})_{\aleph_1} \Longrightarrow (\mathcal{ZFC}) + (\mathcal{D})$.*
(In fact we prove that every pointwise convergent transfinite sequence of derivatives $(f_\xi)_{\xi<\Omega}$ is eventually constant; i.e. there exists an ordinal number $\eta < \Omega$ such that for every $\eta \le \xi < \Omega$ $f_\xi = f\eta$.)

PROOF. Let $f_\xi : I \to \mathbb{R};\ \ 1 \le \xi < \Omega$ be a pointwise convergent transfinite sequence of derivatives ($f_\xi \in \Delta$). The function $f = \lim_{\xi<\Omega} f_\xi$ is Baire 1 and hence measurable. According to Theorem 2 there exists an ordinal number $\eta < \Omega$ such that for every $\eta \le \xi < \Omega$ $f_\xi(x) = f(x)$ almost everywhere on the interval I.

Let $\xi \ge \eta$ be an arbitrary ordinal number. The function

$$h_\xi(x) = f_\xi(x) - f_\eta(x)$$

is a derivative and equals zero almost everywhere; so the function h_ξ is a Lebesgue integrable derivative. Let H_ξ be its primitive function. According to [R] for Lebesgue integrable derivatives the Newton-Leibnitz formula holds.

$$H_\xi(x) - H_\xi(y) = \int_y^x h_\xi(t)dt = 0$$

Hence the function H_ξ is constant on the interval I; i.e. $h_\xi(x) = 0$ everywhere. Then for every ordinal number $\xi \ge \eta$ $f_\xi = f_\eta$, and therefore

$$f = \lim_{\xi<\Omega} f_\xi = f_\eta \in \Delta. \qquad \square$$

Remark 3. The implication $(\mathcal{ZFC}) + \neg(\mathcal{COV})_{\aleph_1} \Longrightarrow (\mathcal{ZFC}) + (\mathcal{AC})$ is an easy consequence of Theorem 2, because two approximately continuous functions which agree on a set of full measure have to be equal everywhere.

3 Limits of Pointwise Convergent Transfinite Sequences of Functions When $(\mathcal{COV})_{\aleph_1}$ Holds.

First we introduce a theorem of Petruska and Laczkovich [P-L] that will be used later.

Theorem 4 (Petruska and Laczkovich) *Let H be a subset of I. The restriction of each Baire 1 function on I to H can be extended to a derivative on I if and only if $\lambda(H) = 0$. This derivative can be chosen bounded if the restriction of the Baire 1 function to H is bounded on H.*

Remark 4. The analogue of this theorem obtained by replacing the word "derivative" with "approximately continuous function" is also valid.

The main theorem of this section follows.

Theorem 5 *Let $(\mathcal{COV})_{\aleph_1}$. The function $f : I \to \mathbb{R}$ is Baire 1 if and only if there exists a transfinite sequence of derivatives $(f_\xi)_{\xi<\Omega}$ such that $\lim\limits_{\xi<\Omega} f_\xi = f$.*

PROOF. The implication '$\Longleftarrow$' was proved by W. Sierpinski in [Sie]. He showed there that a transfinite limit of Baire 1 functions (i.e. also derivatives) is a Baire 1 function.

We prove the implication '$\Longrightarrow$'. Let $f : I \to \mathbb{R}$ be an arbitrary Baire 1 function. Then there are sets C_ξ,$1 \le \xi < \Omega$, all of measure zero such that $\mathbb{R} = \bigcup\limits_{\xi<\Omega} C_\xi$. Let $D_\xi = \bigcup\limits_{\eta\le\xi} C_\eta$. Then sets D_ξ have measure zero and $D_\eta \subset D_\xi$ whenever $\eta \le \xi$.

According to Theorem 4 there exist derivatives f_ξ such that $f\,|_{D_\xi} = f_\xi\,|_{D_\xi}$. Hence Theorem 5 is proved. □

Remark 5. This is a stronger version of a theorem published in [L] where only the implication '$\Longrightarrow$' was proved with the assumption of semi-continuity of function f instead of the assumption of being a Baire 1 function. Theorem 5 gives an affirmative answer to the question asked by the author of [L].

This theorem is also an analogue of Preiss's theorem [P]. He proved that each Baire 2 function is a limit of sequence of derivatives. The assumption $(\mathcal{COV})_{\aleph_1}$ provides us a similar theorem for transfinite sequences.

Corollary 6 *$(\mathcal{ZFC}) + (\mathcal{COV})_{\aleph_1} \Longrightarrow (\mathcal{ZFC}) + \neg(\mathcal{D})$.*

PROOF. Apply Theorem 5 to an arbitrary Baire 1 function which is not derivative. □

Remark 6. Previous proofs can be reformulated to approximately continuous functions instead of derivatives because of Remark 4. Hence also following statement is true.

$$(\mathcal{ZFC}) + (\mathcal{COV})_{\aleph_1} \Longrightarrow (\mathcal{ZFC}) + \neg(\mathcal{AC}) \ .$$

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