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ANOTHER NOTE ON THE GRADIENT PROBLEM OF C. E. WEIL

Abstract

Assume that $G \subset \mathbb{R}^n$ is open and $f : G \to \mathbb{R}$ is a differentiable function. C. E. Weil raised the gradient problem. In this problem it is asked whether ∇f satisfies the natural multidimensional generalization of the Denjoy-Clarkson property. We verify that if there are two dimensional counterexample functions to the gradient problem then their range should satisfy certain paradoxical convexity properties and the inverse image of "many" values of ∇f is of positive linear measure.

1 Introduction

Assume that $G \subset \mathbb{R}^n$ is open and $f : G \to \mathbb{R}$ is a differentiable function. Then ∇f is a mapping from G to $\mathbb{R}^n$. Denote its inverse by Δ, that is, for $H \subset \mathbb{R}^n$ put $\Delta(H) = \{\mathbf{x} \in G : \nabla f(\mathbf{x}) \in H\}$. C. E. Weil raised the following question. Does $H \subset \mathbb{R}^n$ open, $\Delta(H) \neq \emptyset$ imply that $\Delta(H)$ is of positive n-dimensional measure?

In [Bu1] we verified that from $H \subset \mathbb{R}^n$ open, $\Delta(H) \neq \emptyset$ it follows that $\mu_1(\Delta(H)) > 0$, where μ_1 denotes the one-dimensional Hausdorff measure.

In [HMWZ] the authors verified that

(i) any one-dimensional projection of $\Delta(H)$ is of positive μ_1- measure.

(ii) $\Delta(H)$ is non σ-porous.

(iii) $\Delta(H)$ is porous at none of its points.

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In this note we work in dimension two and focus on the range of the gradient mapping. We denote the open unit disk in $\mathbb{R}^2$, by $B(\mathbf{0},1)$. If there is a counterexample to the gradient problem, then, as we shall show in this paper, it is not difficult to see that there exists a differentiable function defined on an open set $G \subset \mathbb{R}^2$ such that the closure of the set $\Delta(B(\mathbf{0},1))$ is non-empty and of zero two dimensional measure. Furthermore the closure of the range of the gradient, $\mathrm{cl}(\nabla f(G))$, does not contain $B(\mathbf{0},1)$. We verify that in this case $B(\mathbf{0},1) \setminus \mathrm{cl}(\nabla f(G))$ is a convex open set, and if $\mathbf{p}$ belongs to the interior of $B(\mathbf{0},1) \cap \mathrm{cl}(\nabla f(G))$, then $\mu_1(\Delta(\{\mathbf{p}\})) > 0$ should hold. This is a sort of paradoxical property saying that if there is an open set with nonempty but relatively small, that is of two-dimensional measure zero, Δ image, then there are many individual values, $\mathbf{p}$, such that their Δ images are relatively large, that is, of positive linear measure. Actually if we denote by $f^{\mathbf{p}}$ the mapping $f^{\mathbf{p}}(\mathbf{x}) = f(\mathbf{x}) - \mathbf{p} \cdot \mathbf{x}$ (where $\cdot$ denotes the dot product) and we denote its level sets by $f^{\mathbf{p}}_c$, that is, $f^{\mathbf{p}}_c = \{\mathbf{x} : f^{\mathbf{p}}(\mathbf{x}) = c\}$, then we prove that there is a set of positive μ_1-measure consisting of c's for which $f^{\mathbf{p}}_c$ contains a point, $\mathbf{x}_c$, where $\nabla f^{\mathbf{p}}(\mathbf{x}_c) = \mathbf{0}$, that is, $\nabla f(\mathbf{x}_c) = \mathbf{p}$. This property might look strange in itself and might suggest that f has too many tangent planes. In [Bu2] we showed that such strange looking functions exist. In fact we constructed a C^1 function f, and a set $E \subset \mathbb{R}^2$ of zero μ_2-measure such that "using the natural parameterization of tangent planes $z = ax + by + c$ to the surface $z = f(x,y)$ the (three dimensional) interior of the set of parameter values, (a,b,c), of tangent planes corresponding to points $(x,y) \in E$ is non-empty."

After mentioning the "many tangent planes" result of [Bu2] we now turn again to the level sets $f^{\mathbf{p}}_c$ and explain how this paper is organized. There are two lemmas and a Theorem stating the main result. The proof of Lemma 2 contains most of the mathematical difficulty. In that proof an auxiliary function g is introduced, and some parts of the level sets of g are studied. These level set parts coincide with the graphs of some one-dimensional functions φ_c. The one-dimensional argument of Lemma 1 applies to these functions. This argument is a variant of the demonstration of the one-dimensional "gradient theorem", which is the so called Denjoy-Clarkson property [C], [D]. In a remark following Lemma 1 we explain in detail how Lemma 1 implies the Denjoy-Clarkson property. After this remark we continue with the two dimensional part of the paper and explain why the assumptions of the Theorem would be satisfied if we had a counterexample function for the two-dimensional gradient problem. After the statement and the proof of the Theorem the main tool, Lemma 2, is stated and proved.

2 Dimension One

Notation. By $\mathbb{R}$ we denote the real numbers. It will be convenient to regard the open interval (a,a) equal to the empty set. The k-dimensional Hausdorff measure is denoted by μ_k.

Lemma 1. *Assume that $\varphi : [0,1] \to \mathbb{R}$ is continuous, and $F \subset (0,1)$ is a non-empty closed set of μ_1-measure zero and $\mu_1(\varphi(F)) = 0$ as well. Furthermore if (a,b) is contiguous to F, then there exists $c \in [a,b]$ such that $\varphi'(x) = 1$ on (a,c) and $\varphi'(x) = -1$ on (c,b). We also assume that φ has at least one local minimum in $(0,1)$, and the set $\Psi \subset \mathbb{R}^2$ satisfies the following two properties:*

(i) the graph of φ restricted to F is a subset of Ψ; that is, $\{(x,\varphi(x)) : x \in F\} \subset \Psi$,

(ii) Ψ is above the graph of φ; that is, $\Psi \subset \{(x,y) : x \in [0,1],\ \varphi(x) \le y\}$.

Then there exists a $y_0 \in F$ such that Ψ has no tangent at the point $(y_0, \varphi(y_0)) \in \Psi$.

Remark. In applications of this lemma Ψ will either be the graph of a differentiable function, or the level set of a differentiable function of two variables.

Proof. From our assumptions it follows that $|\varphi(x) - \varphi(y)| \le |x-y|$ holds for any $x, y \in [0,1]$, and $\varphi(x) - \varphi(y) = x - y$ (or $\varphi(x) - \varphi(y) = y - x$) implies $\varphi'(\alpha) = 1$ (or $\varphi'(\alpha) = -1$, respectively) for all α in the interval determined by x and y. It is also clear that if φ is monotone increasing on an interval (a,b), then $\varphi'(x) = 1$ on (a,b), and a similar statement holds for intervals where φ is monotone decreasing.

Proceeding towards a contradiction, assume that Ψ has a tangent at all of its points belonging to the graph of φ. For convenience, using $\{(x,\varphi(x)) : x \in F\} \subset \Psi$, we put $\psi(x) = \varphi(x)$ for $x \in F$. Later we will extend the definition of ψ to a set larger than F, in a way that the points $(x, \psi(x))$ still stay in Ψ.

Assume that $x_0 \in (0,1)$ is a local minimum of φ. Then $x_0 \in F$. Observe that x_0 cannot be an isolated local minimum, since if it were, then in a neighborhood of x_0 we would have $\varphi(x) = \varphi(x_0) + |x - x_0|$. But then Ψ, which is above φ, would not have a tangent at $(x_0, \varphi(x_0))$. Let $(a_0, b_0) \subset (0, x_0]$ be a maximal subinterval on which φ is monotone decreasing. Then $\varphi'(x) = -1$ for all $x \in (a_0, b_0)$. Clearly our assumptions imply $b_0 \in F$. Since Ψ is above the graph of the Lipschitz function φ and it has a tangent at $(b_0, \varphi(b_0))$, we can extend the definition of ψ to a point $d_0 \in (a_0, b_0)$ such that $(d_0, \psi(d_0)) \in \Psi$ and using that Ψ is above φ we also have $\varphi(b_0) - \psi(d_0) \le \varphi(b_0) - \varphi(d_0) = -(b_0 - d_0)$.

Using the continuity of φ choose $\delta_0 \in (0,1)$ such that

$$-\frac{1}{2} > \frac{\varphi(x) - \varphi(d_0)}{x - d_0} \geq \frac{\varphi(x) - \psi(d_0)}{x - d_0}$$

for $x \in (b_0, b_0 + \delta_0)$. Again, since (a_0, b_0) is a maximal interval on which φ is monotone decreasing there is an interval (a_1, b_1) such that $a_1 \in F$, $a_1 \in (b_0, b_0 + \delta_0)$, φ is monotone increasing on (a_1, b_1) and this interval is maximal with respect to this last property. Then $\varphi'(x) = 1$ for $x \in (a_1, b_1)$. From $a_1 = b_0$ it would follow that $a_1 = b_0$ was an isolated local minimum, which is impossible. Thus $b_0 < a_1$.

Again we can extend the definition of ψ to a point $d_1 \in (a_1, b_1)$ such that $(d_1, \psi(d_1)) \in \Psi$, and $\varphi(a_1) - \psi(d_1) \leq \varphi(a_1) - \varphi(d_1) = a_1 - d_1 < 0$. Choose $\delta_1 \in (0, 1/2)$ such that $b_0 < a_1 - \delta_1$ and

$$\frac{1}{2} < \frac{\varphi(x) - \varphi(d_1)}{x - d_1} \leq \frac{\varphi(x) - \psi(d_1)}{x - d_1}$$

for $x \in (a_1 - \delta_1, a_1)$. Repeating the above argument choose a maximal interval (a_2, b_2) on which φ is monotone decreasing, $b_2 \in (a_1 - \delta_1, a_1)$ and $b_2 \in F$. Observe that $b_0 < a_2$ should hold, since otherwise (a_0, b_0) would not be a maximal interval on which φ is decreasing. Since the local minimums of φ are not isolated, we have again $b_2 < a_1$.

Again choose $d_2 \in (a_2, b_2)$ and define $\psi(d_2)$ such that $(d_2, \psi(d_2)) \in \Psi$, and $\varphi(b_2) - \psi(d_2) \leq \varphi(b_2) - \varphi(d_2) = -(b_2 - d_2)$. Choose $\delta_2 \in (0, 1/3)$ such that $b_2 + \delta_2 < a_1$ and

$$-\frac{1}{2} > \frac{\varphi(x) - \varphi(d_2)}{x - d_2} \geq \frac{\varphi(x) - \psi(d_2)}{x - d_2}$$

for $x \in (b_2, b_2 + \delta_2)$.

Repeating the above arguments for even and odd indices with $\delta_n < \frac{1}{n+1}$ we can obtain intervals (a_n, b_n) and extend the definition of ψ to points $d_n \in (a_n, b_n)$. Put $y_0 = \bigcap_{n=0}^{\infty} [b_{2n}, a_{2n+1}]$. Then one can easily see that our construction implies

$$\frac{1}{2} < \frac{\varphi(y_0) - \psi(d_{2n+1})}{y_0 - d_{2n+1}} \quad \text{and} \quad -\frac{1}{2} > \frac{\varphi(y_0) - \psi(d_{2n})}{y_0 - d_{2n}}$$

holds for $n = 1, 2, \ldots$. Since F is closed, since $a_{2n+1}, b_{2n} \in F$ for $n = 1, 2, 3, \ldots$, and since $\delta_n \to 0$, it follows easily that $y_0 \in F$ and hence $\varphi(y_0) = \psi(y_0)$. Therefore Ψ has no tangent at $(y_0, \varphi(y_0))$. This contradiction concludes the proof of Lemma 1.

Remark. It is not difficult to see that Lemma 1 implies the Denjoy—Clarkson property in dimension one. Here we outline this proof. Assume that a differentiable function f does not have the Denjoy—Clarkson property. Then, after rescaling and adding constants, we can assume that $f : [0,1] \to \mathbb{R}$ is differentiable, $f(1/2) = 0$, $f'(1/2) = 0$, and $|f'(x)| > 1$ almost everywhere. Since $f'(1/2) = 0$, after another suitable rescaling we can also assume that, $|f(x)| < |x - 1/2|/2$ for $x \in [0,1]$.

Put

$$\varphi_1(x) = -x + \min\{f(t) + t : t \in [x,1]\}, \varphi_2(x) = x + \min\{\varphi_1(t) - t : t \in [0,x]\}.$$
$$F_1 = \{x : \varphi_1(x) = f(x)\},\ F_2 = \{x : \varphi_1(x) = \varphi_2(x)\} \text{ and}$$
$$F = \{x : \varphi_2(x) = f(x)\}.$$

It is easy to see that $f(x) \geq \varphi_1(x) \geq \varphi_2(x)$. First one can observe that φ_1 is linear and has slope -1 on intervals contiguous to F_1, and $\varphi_2(x)$ is linear and has slope 1 on intervals contiguous to F_2. It is not difficult to verify that if (a,b) is an interval contiguous to F, then there exists $c \in [a,b]$ such that (a,c) and (c,b) are contiguous to F_2 and to F_1, respectively. Our assumptions imply that $f(1/2) = \varphi_2(1/2) = 0$ and hence $1/2 \in F$. It is also clear that $\varphi_2(x) - x$ is monotone decreasing and $\varphi_2(x) + x$ is monotone increasing. Therefore if $x \in F$ is not an isolated point of F, then one can easily see that $|f'(x)| \leq 1$ and hence $\mu_1(F) = 0$. Since f is differentiable and $\varphi_2 = f$ on F, we also have $\mu_1(\varphi_2(F)) = 0$. If φ_2 has no local minimum, then either it is monotone increasing with slope 1 on $(0,1/2)$, or is monotone decreasing with slope -1 on $(1/2,1)$. This would imply $\min\{\varphi_2(x) : x \in [0,1]\} \leq -1/2$. On the other hand $|f(x)| < \dfrac{|x - 1/2|}{2}$ and the definition of φ_2 imply $\min\{f(x) : x \in [0,1]\} = \min\{\varphi_2(x) : x \in [0,1]\} \geq -1/4$. Lemma 1, used with Ψ being the graph of f, and $\varphi = \varphi_2$, implies that there is a point $y_0 \in F$ such that Ψ has no tangent at $(y_0, f(y_0))$ and hence f is not differentiable at y_0, which concludes our proof.

3 Dimension Two

Notation. By $\mathrm{cl}(H)$, $\mathrm{int}(H)$ we denote the closure and the interior of the set $H \subset \mathbb{R}^2$. We put $Q((u,v),r) = \{(u',v') : |u' - u| \leq r, |v' - v| \leq r\}$. The closed triangle defined by the points $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ is denoted by $T_{\mathbf{pqr}}$. If $f : G \to \mathbb{R}$ is differentiable and $H \subset \mathbb{R}^2$, let $\Delta(H) = \{\mathbf{x} \in G : \nabla f(\mathbf{x}) \in H\}$ and $\overline{\Delta}(H) = \mathrm{cl}(\Delta(H))$.

Before stating our Theorem we first show that if there exists a counterexample to the gradient theorem, then there are functions satisfying the assumptions of the Theorem.

Assume that $G' \subset \mathbb{R}^2$ is open and $f : G' \to \mathbb{R}$ is a counterexample to the gradient problem, that is, it is differentiable and there exists $\kappa, \eta > 0$, $\mathbf{x'} \in G'$, $\mathbf{p'_0} \in \mathbb{R}^2$ such that $\nabla f(\mathbf{x'}) = \mathbf{p'_0}$ and $\mu_2(\{\mathbf{y} \in B(\mathbf{x'}, \kappa) : \nabla f(\mathbf{y}) \in B(\mathbf{p'_0}, \eta)\}) = 0$. Without limiting generality we can assume $\mathbf{p'_0} = \mathbf{0}$ and $\eta > 1$. We can also assume that we work with the restriction of f to $B(\mathbf{x'}, \kappa)$, that is $G' = B(\mathbf{x'}, \kappa)$. Then $\mu_2(\{\mathbf{y} \in B(\mathbf{x'}, \kappa) : \nabla f(\mathbf{y}) \in B(\mathbf{p'_0}, \eta)\}) = 0$ reduces to $\mu_2(\Delta(B(\mathbf{0}, \eta))) = 0$. Put $\Delta_1 = \Delta(B(\mathbf{0}, 1))$ and $\overline{\Delta}_1 = \overline{\Delta}(B(\mathbf{0}, 1))$.

Since ∇f is Baire 1 on $\overline{\Delta}_1$, choose $\mathbf{x} \in \overline{\Delta}_1$ which is a point of continuity of ∇f with respect to $\overline{\Delta}_1$. Then obviously $|\nabla f(\mathbf{x})| \leq 1$. Choose $\delta > 0$ such that $B(\mathbf{x}, \delta) \subset B(\mathbf{x'}, \kappa)$ and for $\mathbf{y} \in B(\mathbf{x}, \delta) \cap \overline{\Delta}_1$ we have

$$|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})| \leq \min\{\eta - 1, 1/2\}.$$

Then $B(\mathbf{x}, \delta) \cap \overline{\Delta}_1 \subset \Delta(B(\mathbf{0}, \eta))$ and hence $\mu_2(B(\mathbf{x}, \delta) \cap \overline{\Delta}_1) = 0$. On the other hand $\mathcal{R} = \nabla f(B(\mathbf{x}, \delta)) \cap B(\mathbf{0}, 1) \subset B(\nabla f(\mathbf{x}), 1/2)$ and hence $\emptyset \neq B(\mathbf{0}, 1) \setminus \mathrm{cl}(\mathcal{R}) = \mathcal{G}$. Therefore using $G = B(\mathbf{x}, \delta)$ and the restriction of f to this G, the assumptions of the following theorem are satisfied.

Theorem. *Assume that f is a differentiable function on $G \subset \mathbb{R}^2$, $\overline{\Delta}_1 \neq \emptyset$ and $\mu_2(\overline{\Delta}_1) = 0$. (Recall that $\overline{\Delta}_1 = \mathrm{cl}\{\mathbf{x} \in G : \nabla f(\mathbf{x}) \in B(\mathbf{0}, 1)\}$.) Put $\mathcal{R} = B(\mathbf{0}, 1) \cap \nabla f(G)$ and $\mathcal{G} = B(\mathbf{0}, 1) \setminus \mathrm{cl}(\mathcal{R})$. Then $\mathcal{G}$ is a convex open subset of the plane and $\mathcal{G} \neq \emptyset$ implies that for any $\mathbf{p} \in \mathrm{int}(\mathrm{cl}(\mathcal{R}))$ we have $\mu_1(\{\mathbf{y} : \nabla f(\mathbf{y}) = \mathbf{p}\}) > 0$.*

Remark. It follows easily from the above theorem that if $\mathcal{G} \neq \emptyset$ and $\mathbf{0} \in \mathcal{R}$, then $B(\mathbf{0}, 1)$ contains an open half-circle, HC, such that for any $\mathbf{p} \in HC$ we have $\mu_1(\{\mathbf{y} : \nabla f(\mathbf{y}) = \mathbf{p}\}) > 0$.

PROOF. If $\mathcal{G} = \emptyset$, we are done. If $\mathcal{G} \neq \emptyset$, then the following claim holds.
Claim. Choose $\mathbf{q} \in \mathcal{G}$ and $\mathbf{p} \in B(\mathbf{0}, 1)$ such that the segment connecting $\mathbf{q}$ and $\mathbf{p}$ contains $\mathbf{p_0} \in \mathcal{R}$, $\mathbf{p_0} \neq \mathbf{p}$. Then $\mathbf{p} \in \mathcal{R}$, moreover $\mu_1(\Delta(\{\mathbf{p}\})) > 0$.

We prove the Claim later. Using this claim one can first see that if $\mathbf{p}$, $\mathbf{q} \in \mathcal{G}$, then the whole segment $\mathbf{pq}$ is disjoint from $\mathcal{R}$. Indeed, if there were a $\mathbf{p_0} \in \mathcal{R}$ on this segment, then the Claim would imply that $\mathbf{p} \in \mathcal{R}$. Therefore $\mathcal{G}$ is a convex open subset of the plane. Moreover if $\mathcal{G}$ is nonempty, then for any $\mathbf{p} \in \mathrm{int}(\mathrm{cl}(\mathcal{R}))$ one can easily choose a $\mathbf{q} \in \mathcal{G}$ such that the segment $\mathbf{pq}$ contains a point $\mathbf{p_0} \in \mathcal{R}$. Our claim applied to these three points shows that $\mathbf{p} \in \mathcal{R}$ and $\mu_1(\Delta(\{\mathbf{p}\})) > 0$.

Now we turn to the proof of the Claim. Assume that $\epsilon_0 > 0$ is chosen such that $B(\mathbf{q}, \epsilon_0) \subset B(\mathbf{0}, 1)$ and $B(\mathbf{q}, \epsilon_0) \cap \mathrm{cl}(\mathcal{R}) = \emptyset$. Denote by ℓ_0 the half line starting at $\mathbf{p}$ and going through $\mathbf{p_0}$. Choose an $\alpha \in (0, \pi/2)$ and denote by ℓ_1 the half line starting at $\mathbf{p}$ and making angle $-\alpha$ with ℓ_0. Using angle $+\alpha$ define the halfline ℓ_2 similarly. The closed convex sector defined by ℓ_1 and ℓ_2 is denoted by Ω. Denote by $\mathbf{q_1}$ the intersection point of ℓ_1 and the line which goes through $\mathbf{q}$ and is perpendicular to ℓ_0. Define the point $\mathbf{q_2}$ on ℓ_2 similarly. We can assume that α is chosen so small that the whole closed triangle $T_{\mathbf{pq_1q_2}}$ is in $B(\mathbf{0}, 1)$ and $\mathbf{q_1}, \mathbf{q_2} \in B(\mathbf{q}, \epsilon_0)$. Choose $\mathbf{q_1'}$, and $\mathbf{q_2'}$ on ℓ_1, and ℓ_2, respectively, such that they are symmetric about the halfline ℓ_0, the triangle $T_{\mathbf{pq_1'q_2'}}$ contains the triangle $T_{\mathbf{pq_1q_2}}$, $\mathbf{q_1'} \neq \mathbf{q_1}$ and $\mathbf{q_1'}, \mathbf{q_2'} \in B(\mathbf{q}, \epsilon_0)$. Denote by $\mathcal{T}$ the trapezoid determined by $\mathbf{q_1q_1'q_2'q_2}$. Then $\mathcal{T} \subset B(\mathbf{q}, \epsilon_0)$ and hence $\Delta(\mathcal{T}) = \emptyset$. It is also clear that $\mu_2(\Delta(T_{\mathbf{pq_1q_2}})) = 0$. After a suitable change of coordinates we can apply Lemma 2 and this completes the proof of the Theorem.

To formulate Lemma 2 we need some notation. Let $0 < \alpha < \pi/2$, $\alpha_1 = \pi/2 - \alpha$, $\alpha_2 = \pi/2 + \alpha$, $\mathbf{e_1} = (\cos(\alpha_1), \sin(\alpha_1))$, $\mathbf{e_2} = (\cos(\alpha_2), \sin(\alpha_2))$, $\mathbf{e_1}' = (\cos(\alpha_1 - \pi/2), \sin(\alpha_1 - \pi/2))$, $\mathbf{e_2}' = (\cos(\alpha_2 + \pi/2), \sin(\alpha_2 + \pi/2))$. We need to define a few points on the plane: $\mathbf{x_0} = \mathbf{p} = (0, 0)$, $\mathbf{p_0} = (0, 1)$, $\mathbf{q} = (0, q)$ with $q > 1$, $\mathbf{q_1} = (q/\cos\alpha)\mathbf{e_1}$, $\mathbf{q_2} = (q/\cos\alpha)\mathbf{e_2}$. Denote the halfline starting at $\mathbf{p}$ and going in the direction of $\mathbf{e_1}$ by ℓ_1. Define ℓ_2 similarly. The convex closed sector defined by ℓ_1 and ℓ_2 is denoted by Ω. With a $q' > q > 1$ put $\mathbf{q_1}' = (q'/\cos\alpha)\mathbf{e_1}$, and $\mathbf{q_2'} = (q'/\cos\alpha)\mathbf{e_2}$. Denote by $\mathcal{T}$ the trapezoid determined by $\mathbf{q_1q_1'q_2'q_2}$.

Lemma 2. *Assume that, using the above notation,* $f : Q(\mathbf{x_0}, 1) \to \mathbb{R}$ *is differentiable,* $f(\mathbf{x_0}) = (0, 0)$, $\nabla f(\mathbf{x_0}) = \mathbf{p_0}$, $\mu_2(\Delta(T_{\mathbf{pq_1q_2}})) = 0$ *and* $\Delta(\mathcal{T}) = \emptyset$. *Then* $\mu_1(\Delta(\{\mathbf{p}\})) > 0$.

PROOF. First we introduce an important auxiliary function g. Since $\nabla f(\mathbf{x_0}) = \mathbf{p_0} = (0, 1)$ for any $0 < \epsilon_1 < 1/2$, we can choose $\delta > 0$ such that

$$|f(u, v) - v| < \epsilon_1(|u| + |v|) \tag{1}$$

for any $(u, v) \in Q(\mathbf{x_0}, \delta)$. We choose a constant $0 < \nu < 1/2$ and put $\delta_0 = \nu\delta$. For $\mathbf{w} = (u, v) \in Q(\mathbf{x_0}, \delta_0)$ denote by $T(u, v)$ the closed triangle which is bounded by the lines $\{\mathbf{w} + t\mathbf{e_1}' : t \in \mathbb{R}\}$, $\{\mathbf{w} + t\mathbf{e_2}' : t \in \mathbb{R}\}$ and $\{(t, -2\delta_0) : t \in \mathbb{R}\}$. We may assume that ν was chosen so small that for any $\mathbf{w} \in Q(\mathbf{x_0}, \delta_0)$ the triangle $T(\mathbf{w})$ is in $Q(\mathbf{x_0}, \delta)$ and hence (1) is applicable for all points of $T(\mathbf{w})$. Observe that this restriction on ν depends only on α and not on ϵ_1.

Let $g(u, v) = \max\{f(x, y) : (x, y) \in T(u, v)\}$, $M(u, v) = \{(x, y) \in T(u, v) : f(x, y) = g(u, v)\}$. Choosing sufficiently small ϵ_1 in (1) we can achieve (the details are left for the reader) that for any $(x, y) \in M(u, v)$ and $(u, v) \in$

$Q(\mathbf{x_0}, \delta_0)$ we have $y > -2\delta_0$; that is, $M(u,v)$ cannot contain points on the lower side of $T(u,v)$. Since f is uniformly continuous on $Q(\mathbf{x_0}, 1)$,one can easily verify that $g(u,v)$ is continuous.

Using a small $0 < \delta_1 < \delta_0$ (the actual value of δ_1 is to be determined later) we say that a point (u,v) is red whenever it is in $Q(\mathbf{x_0}, \delta_1)$ and $g(u,v) = f(u,v)$. At red points f takes its maximum on $T(u,v)$ at (u,v) and hence $\nabla f(u,v)$ is in the sector Ω. It is also clear that the red set (the set of red points) is closed.

Step 1. In this step we show that the red set is a subset of $\Delta(T_{\mathbf{pq_1q_2}})$ and hence of μ_2 measure zero.

The proof uses the fact that $\Delta(\mathcal{T}) = \emptyset$ and a Darboux type property of the upper and lower partial derivatives of g. Proceeding towards a contradiction, assume that $\mathbf{w} = (u,v)$ is red and $\nabla f(\mathbf{w}) \notin T_{\mathbf{pq_1q_2}}$. Then our assumption $\Delta(\mathcal{T}) = \emptyset$ and the property $\nabla f(\mathbf{w}) \in \Omega$ imply that $\partial_y f(\mathbf{w}) > q'$. Introduce the auxiliary function of $\gamma(t) = g((u, v+t))$ for t such that $(u, v+t) \in Q(\mathbf{x_0}, \delta_0)$. Clearly $\gamma(t)$ is continuous. From the definitions of g and γ it is not difficult to verify (the details are left to the reader) that

$$D^+\gamma(t) = \limsup_{h \to 0+} (\gamma(t+h) - \gamma(t))/h \geq \sup\{\partial_y f(\mathbf{x}) : \mathbf{x} \in M((u, v+t))\}$$

and

$$D_-\gamma(t) = \liminf_{h \to 0-} (\gamma(t+h) - \gamma(t))/h \leq \inf\{\partial_y f(\mathbf{x}) : \mathbf{x} \in M((u, v+t))\}.$$

It is clear that our assumptions imply $\nabla f(\mathbf{x}) \in \Omega$ and $\partial_y f(\mathbf{x}) \notin (q, q')$ for any $\mathbf{x} \in M((u, v+t))$. Let $q'' = (q+q')/2$. Clearly $q' > q'' > q > 1$. From $\nabla f(\mathbf{w}) \notin T_{\mathbf{pq_1q_2}}$ it follows that $D^+\gamma(0) > q'$. We will verify that this implies $0 \leq \gamma_0(t) = \gamma(t) - \gamma(0) - q''t$ for all $t > 0$ in the domain of γ. Indeed, we have $\gamma_0(0) = 0$, $D^+\gamma_0(0) > 0$ and if for a $t_1 > 0$ we have $\gamma_0(t_1) < 0$, then there is a $t_2 \in (0, t_1)$ where γ_0 takes its maximum on $[0, t_1]$. This implies $D^+\gamma_0(t_2) \leq 0$; that is, $D^+\gamma(t_2) \leq q''$. Therefore $\partial_y f(\mathbf{x}) \leq q''$ for all $\mathbf{x} \in M((u, v+t_2))$, and hence, by our assumptions, $\partial_y f(\mathbf{x}) \leq q$. Thus $\inf\{\partial_y f(\mathbf{x}) : \mathbf{x} \in M((u, v+t_2))\} \leq q$, and hence $D_-\gamma(t_2) \leq q$; that is, $D_-\gamma_0(t_2) < 0$ which shows that γ_0 cannot have a local maximum at t_2. Therefore $\gamma(t) \geq \gamma(0) + q''t$ for all $t > 0$ in the domain of γ.

Now we need upper estimations of γ. Since $\mathbf{w} = (u,v)$ is red, we have $\mathbf{w} \in Q(\mathbf{x_0}, \delta_1) \subset Q(\mathbf{x_0}, \delta_0)$. Thus $|u|, |v| < \delta_1$ and hence using that $\epsilon_1 < 1/2$ in (1) we have $|\gamma(0)| = |g(u,v)| = |f(u,v)| \leq |f(u,v) - v| + |v| < (2\epsilon_1 + 1)\delta_1 < 2\delta_1$. Let $\delta_1 = \min\{\delta_0/4, (q''-1)\delta_0/10\}$. Put $c = 5/(q''-1)$. Then $(u, v + c\delta_1) \in Q(\mathbf{x_0}, \delta_0)$. Therefore $\gamma(c\delta_1)$ is well defined. By our result in the previous paragraph $\gamma(c\delta_1) \geq \gamma(0) + q''c\delta_1$. On the other hand if $(x,y) \in$

$T((u, v + c\delta_1)) \subset Q(\mathbf{x_0}, \delta)$, then $y \leq v + c\delta_1$ and from (1) it follows that $|f(x, y) - y| < \epsilon_1(|x| + |y|) < \epsilon_1 2\delta$, and hence $v + c\delta_1 + \epsilon_1 2\delta > f(x, y)$ for all $(x, y) \in T(u, v + c\delta_1)$; that is, $v + c\delta_1 + \epsilon_1 2\delta > g(u, v + c\delta_1)$. The above inequalities imply $-2\delta_1 + q'' c\delta_1 < \gamma(0) + q'' c\delta_1 \leq \gamma(c\delta_1) = g(u, v + c\delta_1) \leq v + c\delta_1 + \epsilon_1 2\delta < (c + 1)\delta_1 + \epsilon_1 2\delta$. Observe that δ_1/δ is depending only on α but not on ϵ_1 and hence we can assume that ϵ_1 was so small that $\epsilon_1 2\delta < \delta_1$ and, then from the above we obtain $c(1 - q'') + 4 > 0$ which contradicts $c = 5/(q'' - 1)$. Therefore the red set is of μ_2 measure zero.

Step 2. In this step we investigate the level sets of g. It is not difficult to deduce from (1) that we can choose $0 < \epsilon_2 < \delta_1$ such that if $|c| < \epsilon_2$ and $|x| < \epsilon_2$, then $g(x, t) = c$ has a solution with $|t| < \delta_1$. Assume that there are $t_1, t_2 \in (-\delta_1, \delta_1)$ such that $g(x, t_1) = g(x, t_2) = c$ and $t_1 < t_2$. We also assume that $(x', y') \in M((x, t_1))$. Then (x', y') is a local maximum of f lying in the interior of $T(x, t_2)$. Hence $\nabla f(x', y') = \mathbf{0} = \mathbf{p}$ and $f(x', y') = c$. Since f has only countably many local maximum values for all but countable c's with $|c| < \epsilon_2$, we have only one solution of $g(x, t) = c$ for $|x| < \epsilon_2$. Denote the set of these "regular" c's by $\mathcal{C}$. Therefore for each $c \in \mathcal{C}$ there is a uniquely defined function $\varphi_c : (-\epsilon_2, \epsilon_2) \to (-\delta_1, \delta_1)$ for which $g(x, \varphi_c(x)) = c$. Since g is continuous, it is not difficult to see that φ_c is also continuous. Put $F_c = \{x : (x, \varphi_c(x)) \textit{ is red}\}$. Assume that (a, b) is an interval contiguous to F_c and $x \in (a, b) \cap (-\epsilon_2, \epsilon_2)$. Then choose an $(x', y') \in M((x, \varphi_c(x)))$. Since $c \in \mathcal{C}$, the point (x', y') cannot be in the interior of $T(x, \varphi_c(x))$. It is also clear that $f(x', y') = g(x', y') = c$. Then for any (x'', y'') on the segment connecting $(x, \varphi_c(x))$ to (x', y') the maximum of f on $T((x'', y''))$ will equal c; that is, $g(x'', y'') = c$. Therefore the intersection of this segment and $(-\epsilon_2, \epsilon_2) \times (-\delta_1, \delta_1)$ is a part of the graph of φ_c. One can easily verify from this that there exists an $e \in [a, b]$ such that the graph of φ_c is parallel to $\mathbf{e_1}'$ on (e, b) and parallel to $\mathbf{e_2}'$ on (a, e), that is, $\varphi_c' = \tan(-\alpha)$ on (e, b) and $\varphi_c' = \tan(\alpha)$ on (a, e). It is also clear that if ϵ_1 in (1) is sufficiently small, then φ_c is close to a horizontal line segment and hence φ_c has local minimums (and local maximums as well).

We recall the following consequence of the Coarea Formula [Fe, 3.2.11]. If $f : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz function and $H \subset \mathbb{R}^2$ is a measurable set, then from $\mu_2(H) = 0$ it follows that $\mu_1(H \cap f^{-1}(\{c\})) = 0$ for almost every $c \in \mathbb{R}$. Since the Coarea formula works only for Lipschitz functions we also recall that from [Fe, 3.1.8] it follows that if $f : G \to \mathbb{R}$ is differentiable, then G is the union of a countable family of μ_2-measurable sets such that the restriction of f to each member of the family is Lipschitzian.

Since f is differentiable and $\mu_2(\Delta(T_{\mathbf{pq_1q_2}})) = 0$, the above two theorems imply that $\mu_1(f^{-1}(\{c\}) \cap \Delta(T_{\mathbf{pq_1q_2}})) = 0$ for almost every $c \in \mathcal{C}$ and hence, for these c's, by the result at Step 1 the linear measure of the intersection of the red set and $f^{-1}(\{c\})$ equals zero. This implies $\mu_1(F_c) = 0$ and $\mu_1(\varphi_c(F_c)) = 0$. Therefore, after rescaling φ_c (for ease of notation this rescaled function is still denoted by φ_c), Lemma 1 is applicable with $\varphi = \varphi_c$. Furthermore F, and Ψ are suitable portions of F_c and of $f^{-1}(\{c\})$ (rescaled as φ_c), respectively. To apply Lemma 1 also observe that Ψ is above φ_c. We can find a $y_0 \in F_c$ such that Ψ has no tangent at $(y_0, \varphi_c(y_0))$. This is possible only when $\nabla f((y_0, \varphi_c(y_0))) = \mathbf{0}$, since otherwise the level set $f^{-1}(c)$ should have a tangent at $(y_0, \varphi_c(y_0))$. Thus we verified that $(y_0, \varphi_c(y_0)) \in \Delta(\{\mathbf{p}\}) = \Delta(\{\mathbf{0}\})$. This shows that $\Delta(\{\mathbf{p}\}) \cap f^{-1}(\{c\})$ is not empty for almost every $c \in (-\epsilon_2, \epsilon_2)$, and hence $\mu_1(f(\Delta(\{\mathbf{p}\}))) > 0$. Recall that if $H \subset \mathbb{R}^2$ and the map $f : H \to \mathbb{R}$ is differentiable and $\mu_1(H) = 0$, then $\mu_1(f(H)) = 0$ holds as well. (This is a well-known theorem. See, for example, [Sa, Lemma 1].) Applying this result to the differentiable function f from $\mu_1(f(\Delta(\{\mathbf{p}\}))) > 0$ it follows $\mu_1(\Delta(\{\mathbf{p}\})) > 0$. This completes the proof of Lemma 2. $\square$

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