RESEARCH

Real Analysis Exchange
Vol. 22(2), 1996-97, pp. 740–759

P. S. Bullen, Department of Mathematics, University of British Columbia, Vanvouver BC, Canada V6T 1Z2,
e-mail address: `bullen@unixg.math.ubc.ca`

D. N. Sarkhel Department of Mathematics, University of Kalyani, Kalyani, W. B., India 741235

PROPERTIES OF DERIVATIVE-LIKE FUNCTIONS

Abstract

In this paper refinements and extensions of properties that give generalized derivatives the basic properties of ordinary derivatives are discussed; for instance the Darboux, Baire-1, Denjoy, Zahorski properties.

1 Introduction

It is known that a finite approximate derivative, (ap)$F' = f : I = [a, b] \to \mathbb{R}$, $a < b$, shares many of the interesting properties of ordinary derivatives; ([5], [7], [9], [16]). Properties such as:

(i) the Baire-1 property,

(ii) the Darboux or intermediate value property,

(iii) the mean value property,

(iv) the Denjoy or Denjoy-Clarkson property: and in addition,

(v) $f(x) = F'(x)$ on a dense open set in I.

Further: Weil [17] has strengthened (iv) to:

(vi) if $f^{-1}(]\alpha, \beta[) \neq \emptyset$, then $\{x;\, x \in f^{-1}(]\alpha, \beta[),\, f(x) = F'(x)\}$ has positive measure; and O'Malley [8] proved the surprisingly sharp property:

(vii) for every x in I there is an x_0 in I such that $f(x) = F'(x_0)$.

Weil [18] has also proved:

Key Words: Generalized derivatives, porosity, partitioning covers, Baire-1, interlocking, Zahorski, Darboux
Mathematical Reviews subject classification: Primary: 26A24
Received by the editors October 20, 1996

(viii) f has a certain property Z on I, a property stronger than the Zahorski property M_3 [19].

It might be remarked that the Baire-1 property often seems crucial in the deduction of the other properties.

In a welcome attempt to reveal how the various properties arise Bruckner, O'Malley & Thomson [2] have studied the unified notion of path derivatives, and Thomson [15] has studied the more general notion of derivatives relative to simple systems, proving various properties of derivatives under the external intersection condition, (E.I.C. [m]), the intersection condition, (I.C.), and the non-porosity condition. The Baire-1 property in particular is obtained in [2, 6.3, p. 113] under (E.I.C. [m]), and in [15] under a wider condition [15, Lemma 9.7, p. 358]; the Darboux property is proved by assuming the Baire-1 property [2, 6.4, p. 113], [15, 8.1, p. 352].

In this paper we show, amongst other things, that a *semi-Baire*-1 property and all the properties (ii) through (viii) in stronger and more revealing forms extend to functions like derivatives relative to *limiting systems*, (essentially two-sided simple systems [15, p. 280]), under an *interlocking condition* wider than (I.C.), and often a *generalized non-porosity condition.* The Baire-1 property may however fail even under stronger conditions. This and other critical aspects of the present theory are illustrated by appropriate examples.

2 Definitions and Lemmas

Throughout we suppose that $F, f : I = [a, b] \to \mathbb{R}$, $a < b$ and T is a fixed *limiting system on* $\mathbb{R}$, by which we mean an arbitrary collection $T = \{T(x); x \in \mathbb{R}\}$ where each $T(x)$ is a non-empty family of subsets of $\mathbb{R}$ such that, if $A \in T(x)$, then $x \in A$ and x is a two-sided limit point of A, and $A \cap]c, d[\in T(x)$ for all c, d, $c < x < d$.

For each $\tau(x) \in T(x)$, $x \in I$, we denote by $I_{\tau(x)}$ the family of all closed intervals $[u, v]$, $u \le v$ with u in $\tau(x) \cap [a, x]$ and v in $\tau(x) \cap [x, b]$. We define the *extreme T-derivates of F on I* by

$$T\overline{F}'(x) = \inf_{\tau(x) \in T(x)} \left[\sup \left\{ \frac{F(v) - F(u)}{v - u};\ [u, v] \in I_{\tau(x)},\ u < v \right\} \right],$$
$$T\underline{F}'(x) = \sup_{\tau(x) \in T(x)} \left[\inf \left\{ \frac{F(v) - F(u)}{v - u};\ [u, v] \in I_{\tau(x)},\ u < v \right\} \right].$$

Clearly $T(\overline{-F})'(x) = -T\underline{F}'(x)$. If $T\overline{F}'(x) \le T\underline{F}'(x)$, then F is said to be *T-derivable at the point x*. If $T\overline{F}'(x) = T\underline{F}'(x)$, then this common value is called the *T-derivative of F at x, $TF'(x)$.*

As well as finite or infinite T-derivatives we will also study *T-derivative-like functions*; that is, functions g that satisfy $T\overline{F}'(x) \le g(x) \le T\underline{F}'(x)$.

Remark 2.1 *In the absence of other conditions we can have that $T\overline{F}'(x) < T\underline{F}'(x)$. This cannot happen if T is filtering down; that is, if the intersection of two members of $T(x)$ is also a member of $T(x)$. However we do not assume this condition.*

Remark 2.2 *The definition of the T-derivative is unique, but this is not true for the simple-derivatives. The simple system naturally associated with T is $S = \{S(x); x \in \mathbb{R}\}$, where $S(x) = \{E; \mathbb{R} \supseteq E \supseteq \tau(x) \in T(x)\}\}$. Then, according to Thomson* [15, p. 281], *F has an S-derivative α at a point x in I if for every $\eta > 0$ there is a $\tau(x) \in T(x)$ such that*

$$|F(y) - F(x) - \alpha(y - x)| \leq \eta|y - x|$$

for all y in $\tau(x) \cap I$. Clearly then F is T-derivable at x and $T\overline{F}'(x) \leq \alpha \leq T\underline{F}'(x)$, but the converse is not true.

Remark 2.3 *The weakest limiting system is T_1, where each $T_1(x)$ consists of all sets containing x and having x as a two-sided limit point. Clearly $T_1\overline{F}'(x) = \max\{D_+F(x), D_-F(x)\}$, and $T_1\underline{F}'(x) = \min\{D^+F(x), D^-F(x)\}$, with obvious modifications if $x = a, b$. Darboux and mean value properties of T_1-derivative-like functions have been studied in Sarkhel & Seth* [12], *and Bullen & Sarkhel* [4].

Our main line of reasoning will be Baire's theorem [10, (9.2), p. 54] and the partitioning property, which has many other interesting applications. (See [3] and [14].)

Let $E \subseteq X \subseteq I$. Following Sarkhel & Kar [11, §2], a sequence of sets $\{E_n\}$ whose union is E is called an *E-form with parts E_n*; if further each part E_n is closed in E, then the E-form is said to be *closed*. An increasing E-form is called an *E-chain*. By a *perfect portion of E* we shall mean a *section* $E \cap [p, q]$ that is perfect and contains both p and q.

Every choice $\tau = \{\tau(x) \in T(x); x \in X\}$ is called a *T-full cover of E*. A finite, possibly empty, family $\varpi = \{([u_j, v_j], x_j); [u_j, v_j] \in I_{\tau(x_j)}\}$ is called a *τ-partition over E (in I)* if for all j the intervals $]u_j, v_j[$ are pairwise disjoint, and $x_j \in E$; if further $\bigcup_j [u_j, v_j] = E$, then ϖ is called a *τ-partition of E*. We put

$$|\varpi| = \sum_j (v_j - u_j), \qquad \sum(f, \varpi) = \sum_j f(x_j)(v_j - u_j),$$
$$\Delta(F, \varpi) = \sum_j (F(v_j) - F(u_j)),$$

where all these sums are to be zero if $\varpi = \emptyset$.

Given a neighborhood $]x_1, x_2[$ of each point $x \in E$ we will call $\{\tau(x) \cap]x_1, x_2[\}$ a *refinement of τ on E*.

Lemma 2.1 *Let τ be a T-full cover of a set $E \subseteq I$. If E is of measure zero, then for every $\epsilon > 0$ there is a refinement λ of τ on E such that for every λ-partition ϖ over E we have $\sum(|f|, \varpi) < \epsilon$ and $|\varpi| < \epsilon$.*

PROOF. Since $|E| = 0$, there are open sets $G_n \supseteq E$ with $|G_n| < n^{-1}2^{-n}\epsilon$, $n = 1, 2, 3, \ldots$. It then suffices to take a refinement λ on E such that $\lambda(x)$ is contained in a component of G_n if $n - 1 \leq |f(x)| < n$. $\square$

We say that T is *partitioning, or has the partitioning property* on I if every T-full cover τ of I induces a τ-partition of I.

Lemma 2.2 *If T is partitioning on I, then it is also partitioning on every subinterval $[c, d]$, $c < d$, of I.*

PROOF. Let λ be any T-full cover of $[c, d]$. For each $x \in I \setminus [c, d]$ select $\tau(x) \in T(x)$ such that $\tau(x) \subseteq]-\infty, c[$ or $\tau(x) \subseteq]d, \infty[$; also let $\tau(c) = \lambda(c)$, $\tau(d) = \lambda(d)$, and $\tau(x) = \lambda(x) \cap]c, d[$ for $c < x < d$. Then τ is a T-full cover of I, and hence there is a τ-partition of I which clearly induces a λ-partition of $[c, d]$. $\square$

The partitioning property arises from various intersection conditions, [2]. Following [2] and [15], we say that T satisfies the *intersection condition, (I.C.), on I* if for every T-full cover τ of I there are $\delta_x > 0$, $x \in I$, such that if $x, y \in I$ and $0 < y - x < \min\{\delta_x, \delta_y\}$, then $\tau(x) \cap \tau(y) \cap [x, y] \neq \emptyset$. If $A_n = \{x \in I;\, \delta_x > n^{-1}\}$, then $\{A_n\}$ is an *I-chain.* If $x, y \in A_n$ and $0 < y - x < n^{-1}$, then $\tau(x)$ and $\tau(y)$ intersect as above and so $[x, y]$ has a τ-partition of the form $\{([x, u], x),\ ([u, y], y)\}$. We generalize this to the *interlocking property, ILP*, in terms of the *interlocking condition, ILC*, as follows.

We say *τ satisfies ILC on a set* $A \subseteq I$ if for any two points $x < y$ in A there is a set $E \subset]x, y[$, of measure zero, such that for any refinement λ of τ on E there is a τ-partition of $[x, y]$ of the form $\{([x, u], x), ([v, y], y)\} \cup \varpi$, where ϖ is a λ-partition of $[u, v]$ over E.

Then T *is said to satisfy ILP on* I if for every T-full cover τ of I there is an I-chain $\{A_n\}$ such that for each n there is a $\delta_n > 0$ such that τ satisfies ILC on $A_n \cap J$ for every closed interval J with $|J| < \delta_n$. We then say that τ *satisfies ILP on I via* $\{(A_n, \delta_n)\}$.

The following result extends (4.7.3) of [2, p. 109].

Lemma 2.3 *The interlocking property implies the partitioning property.*

PROOF. Let an arbitrary T-full cover τ of I satisfy ILP on I via $\{(A_n, \delta_n)\}$. Let E denote the set of points x of I such that every neighborhood of x in I contains a closed interval admitting no τ-partition. Then E is clearly closed, and it is easily seen that the closure of every interval contained in $I \setminus E$ has a τ-partition. Hence E is perfect, and we need only show that E is empty.

Suppose that $E \neq \emptyset$. Then by Baire's theorem some A_m must be dense in some perfect portion, $E \cap [c,d]$ of E, with $d - c < \delta_m$.

Let $c \le u < v \le d$. If $E \cap \,]u,v[= \emptyset$, then as remarked above $[u,v]$ has a τ-partition. If $E \cap \,]u,v[\neq \emptyset$ put $u_1 = \inf\big(E \cap \,]u,v[\big)$ and $v_1 = \sup\big(E \cap \,]u,v[\big)$. Then $u_1, v_1 \in E$ and $u \le u_1 < v_1 \le v$. Since $E \cap \,]u,u_1[= \emptyset = E \cap \,]v_1,v[$, both $[u,u_1]$, $[v_1,v]$ have τ-partitions, and because E is perfect u_1, v_1 must be limit points of $E \cap \,]u_1,v_1[$, and hence of $A_m \cap \,]u_1,v_1[$. Since $\{A_n\}$ is an I-chain, there is an $A_n \supseteq A_m$ such that $u_1, v_1 \in A_n$. Then there are points $x, y \in A_m$, $u_1 < x < y < v_1$, such that $x - u_1 < \delta_n$ and $v_1 - y < \delta_n$. Since $u_1, x, y, v_1 \in A_n$, there are, by ILC, τ-partitions of both $[u_1,x]$, and $[y,v_1]$. Also since $x, y \in A_m$ and $0 < y - x < d - c < \delta_m$ we have, by ILC, a τ-partition of $[x,y]$. Hence it follows that every $[u,v] \subset]c,d[$ has a τ-partition, and this contradicts the condition $E \cap \,]c,d[\neq \emptyset$. □

As observed by Thomson [15, p. 420], set porosity is precisely the right notion to capture certain properties of generalized derivatives. The *porosity of a set A at a point x* is the number

$$p(A,x) = \limsup_{\delta \to 0+} \left[\sup\left\{ \frac{(v-u)}{\delta};\ ]u,v[\subseteq]x-\delta, x+\delta[\setminus (A \cup \{x\}) \right\} \right].$$

We set $p_T(x) = \sup\{p(A,x);\ A \in T(x)\}$, and if $p_T(x) = 0$ for all $x \in I$, then T is said to be *non-porous on I*.

We say that T is *non-porous in the generalized sense on I*, (NPG), if there is a closed I-form $\{E_n\}$ such that for each n we have that $\sup\big\{p_T(x);\ x \in E_n\big\} < 1$.

Note *The properties ILP and (NPG) are hereditary.*

The following lemma greatly extends Theorem 4.4 of [2, p. 106].

Lemma 2.4 *Hypotheses: $x \in E \subseteq I$, E is closed, $p_T(x) < 1$, F is monotone on the closure of each component of $I \setminus E$, $\alpha \in \mathbb{R}$ and $M = \max\{\alpha, T\overline{F}'(x)\}$; $m = \min\{\alpha, T\underline{F}'(x)\}$.*

Conclusions: (i) If $F(v) - F(u) \le \alpha(v-u)$ for all $u, v \in E$, $u \le x \le v$, then

$$\overline{F}'(x) \le \begin{cases} M/(1 - p_T(x)) & \text{if } M \ge 0, \\ M(1 - p_T(x)) & \text{if } M \le 0; \end{cases}$$

if further x is a limit point of E, then

$$T\underline{F}'(x) \le \begin{cases} \alpha/(1 - p_T(x)) & \text{if } T\underline{F}'(x) \ge 0, \\ \alpha(1 - p_T(x)) & \text{if } T\underline{F}'(x) \le 0. \end{cases}$$

(ii) If $F(v) - F(u) \ge \alpha(v-u)$ for all $u, v \in E$, $u \le x \le v$, then

$$\underline{F}'(x) \ge \begin{cases} m/(1 - p_T(x)) & \text{if } m \le 0, \\ m(1 - p_T(x)) & \text{if } m \ge 0; \end{cases}$$

if further x is a limit point of E, then

$$T\overline{F}'(x) \geq \begin{cases} \alpha/(1-p_T(x)) & \text{if } T\overline{F}'(x) \leq 0, \\ \alpha(1-p_T(x)) & \text{if } T\overline{F}'(x) \geq . \end{cases}$$

Note *If $E = \{x\}$, then we can take any α in (i) and (ii). Hence, clearly, if F is non-decreasing on I, then*

$$T\underline{F}'(x)(1-p_T(x)) \leq \underline{F}'(x) \text{ and } \overline{F}'(x) \leq \frac{T\overline{F}'(x)}{1-p_T(x)},$$

with equality holding if $p_T(x) = 0$.

Proof. (i) First, ignoring the trivial case $M = \infty$, let $p_T(x) < q < 1$ and $M < \beta$, where $\beta < 0$ if $M < 0$. Suppose $a \leq x < b$. Then there are $\tau(x) \in T(x)$ and $t \in \tau(x) \cap]x, b[$ such that

$$F(y) - F(x) \leq \beta(y-x) \quad \text{if} \quad y \in \big(E \cup \tau(x)\big) \cap [x, t], \tag{1}$$

$$d - c < q(d-x) \quad \text{if} \quad]c, d[\subseteq]x, t[\setminus \tau(x),\ c > x. \tag{2}$$

Consider now any $y \in]x, t[\setminus \big(E \cup \tau(x)\big)$, if there are any. Let $]r, s[$ be the component of $]x, t[\setminus E$ that contains y. Put

$$c = \sup\Big(\big(E \cup \tau(x)\big) \cap [r, y[\Big) \text{ and } d = \inf\Big(\big(E \cup \tau(x)\big) \cap]y, s]\Big).$$

Then $x \leq r \leq c \leq y \leq d \leq s \leq t,\ c > x$. Since, by (2), $d - y < q(d-x)$, there is a $v \in \big(E \cup \tau(x)\big) \cap [d, s]$ such that $v - y < q(v-x)$; and so $(1-q)(v-x) < y - x$. Again by (2), $y - c < q(y-x)$ and hence there is a $u \in \big(E \cup \tau(x)\big) \cap [r, c]$ such that $y - u < q(y-x)$, whence $(1-q)(y-x) < u - x$.

Now by hypothesis, F is either non-decreasing or non-increasing on $[r, s]$. In the first case, by the above and (1) $F(y) - F(x) \leq F(v) - F(x) \leq \beta(v-x)$, whence

$$F(y) - F(x) \begin{cases} < \beta(y-x)/(1-q) & \text{if } \beta > 0, \\ \leq \beta(y-x) & \text{if } \beta < 0. \end{cases}$$

In the second case, by the above and (1), $F(y) - F(x) \leq F(u) - F(x) \leq \beta(u-x)$, whence

$$F(y) - F(x) \begin{cases} \leq \beta(y-x) & \text{if } \beta > 0, \\ < \beta(1-q)(y-x) & \text{if } \beta < 0. \end{cases}$$

These, in conjunction with (1), clearly prove

$$D^+F(x) \le \begin{cases} \beta/(1-q) & \text{if } \beta > 0, \\ \beta(1-q) & \text{if } \beta < 0. \end{cases}$$

By symmetry, if $a < x \le b$, then

$$D^-F(x) \le \begin{cases} \beta/(1-q) & \text{if } \beta > 0, \\ \beta(1-q) & \text{if } \beta < 0. \end{cases}$$

So

$$\overline{F}'(x) \le \begin{cases} \beta/(1-q) & \text{if } \beta > 0, \\ \beta(1-q) & \text{if } \beta < 0. \end{cases}$$

Letting $q \to p_T(x)+$ and $\beta \to M+$, the first two results follow. Next, let x be a limit point of E on the right. Ignoring the trivial case when $T\underline{F}'(x) = -\infty$, take $\gamma < T\underline{F}'(x)$, where $\gamma > 0$ if $T\underline{F}'(x) > 0$. Then there is $\lambda(x) \in T(x)$ and $e \in E \cap \,]x, b[$ such that

$$F(y) - F(x) \ge \gamma(y - x) \text{ if } y \in \lambda(x) \cap [x, e], \tag{3}$$

$$s - r < q(s - x) \text{ if }]r, s[\subseteq]x, e[\setminus \lambda(x),\ r > x. \tag{4}$$

Now, fix $t \in \lambda(x) \cap \,]x, e[$. If $t \in E$, then by hypothesis and (3) $\alpha(t-x) \ge F(t)-F(x) \ge \gamma(t-x)$, from which $\alpha \ge \gamma$.

If $t \notin E$, let $]r, s[$ be the component of $]x, e[\setminus E$ containing t. Then $r, s \in E$ and $x < r < t < s \le e$. Let $c = \inf\bigl(\lambda(x) \cap \,]r, t]\bigr)$ and $d = \sup\bigl(\lambda(x) \cap [t, s[\bigr)$. Since by (4), $c - r < q(c - x)$, there is a $u \in \lambda(x) \cap [c, t]$ such that $u - r < q(u - x)$, whence $(1 - q)(u - x) < r - x$. Again by (4) $s - d < q(s - x)$, and hence there is a $v \in \lambda(x) \cap [t, d]$ such that $s - v < q(s - x)$, whence $(1 - q)(s - x) < v - x$.

Now by hypothesis, F is either non-increasing or non-decreasing on $[r, s]$. In the first case, by hypothesis and by the above and (3)

$$\alpha(r - x) \ge F(r) - F(x) \ge F(u) - F(x) \ge \gamma(u - x),$$

whence $\alpha(r-x) > \gamma(r-x)/(1-q)$ or $\alpha > \gamma/(1-q)$ if $\gamma < 0$, and $\alpha(r-x) \ge \gamma(r-x)$ or $\alpha \ge \gamma$ if $\gamma > 0$. In the second case, by hypothesis and by the above and (3)

$$\alpha(s - x) \ge F(s) - F(x) \ge F(v) - F(x) \ge \gamma(v - x),$$

whence $\alpha(s - x) \ge \gamma(s - x)$ or $\alpha \ge \gamma$ if $\gamma < 0$, and $\alpha(s - x) > \gamma(s - x)(1 - q)$ or $\alpha > \gamma(1 - q)$ if $\gamma > 0$.

Thus we always have $\alpha \geq \gamma/(1-q)$ if $\gamma < 0$ and $\alpha \geq \gamma(1-q)$ if $\gamma > 0$. Letting $\gamma \to T\underline{F}'(x)$ and $q \to p_T(x)+$, the second two results follow.

A similar proof holds if x is a limit point of E on the left.

(ii) This follows form (i) applied to $-F$. □

Generalizing a well-known property of Baire-1 functions, we shall say that the function f is *semi-Baire*-1 *on* I if for every $\alpha \in f(I)$ the level set $f^{-1}(\alpha)$ contains a point of continuity of f relative to the closure of $f^{-1}(\alpha)$.

Finally we recall that a point $x \in I$ is termed a *point of absolute continuity of the function* F, an AC-point, if x has a neighborhood in I on which F is AC. Also F is called *(ACG) on* I if F is AC on each part of some closed I-form.

The following examples illustrate some of our ideas.

Example 2.5 *Every bilateral system of paths* [2, p. 100] $P = \{P_x;\, x \in \mathbb{R}\}$, $x \in P_x \subseteq \mathbb{R}$ *and* x *a two-sided limit point of* P_x, *generates a limiting system* T_P *which is defined by* $T_P(x) = \{P_x \cap\,]c, d\,[\,;\, c < x < d\}$, *which is filtering down. The notions of* P*-derivative and* T_P*-derivative coincide. If* P *satisfies (I.C.), then* T_P *satisfies (I.C.), and hence also ILP. If* P *is non-porous, then so is* T_P.

Example 2.6 *If* $T_{ap}(x)$ *is the family of all measurable sets containing* x, *and having density* 1 *at* x, *then* T_{ap} *is a filtering down, non-porous limiting system satisfying (I.C.),* [2, p. 102]. *The notions of approximate derivative and* T_{ap}*-derivative coincide.*

Example 2.7 *For each* $\alpha > 0$, *we construct a limiting system* T_α *which is non-porous, and satisfies ILP but not (I.C.), and is or is not filtering down according as* α *is rational or not.*

Define

$$T_\alpha(x) = \begin{cases} \{]c, d\,[\,;\, c < x < d\} & \text{if } x \in \mathbb{Q}, \text{ and} \\ \{]c, d[\, \cap (x + t\mathbb{Q});\, c < x < d,\, t = 1, \alpha\} & \text{if } x \text{ is irrational.} \end{cases}$$

Clearly, T_α *is a non-porous limiting system.*

Let now τ *be any* T_α*-full cover of* I. *Let* $A_n = \{x \in I;\, \tau(x) \text{ is dense in }]x - n^{-1}, x + n^{-1}[\}$, $n = 1, 2, \ldots$. *Clearly* $\{A_n\}$ *is an* I*-chain. Let* $x, y \in A_n$ *and* $0 < y - x < n^{-1}$. *Fix a rational* $r \in]x, y[$. *If* λ *is any refinement of* τ *on* $\{r\}$, *then* $\lambda(r)$ *is a neighborhood of* r. *Also,* $\tau(x)$ *is dense in* $]x, r[$ *and* $\tau(y)$ *is dense in* $]r, y[$. *So* $\lambda(r)$ *intersects both* $\tau(x) \cap]x, r[$ *and* $\tau(y) \cap]r, y[$. *Hence, plainly,* τ *satisfies ILP on* I *via* $\{(A_n, n^{-1})\}$. *Thus* T_α *satisfies ILP.*

But, consider any I*-chain* $\{E_n\}$. *Obviously some* E_n *must contain an uncountable set* B *of irrationals. Then some* $\xi \in B$ *must be such that every neighborhood of* ξ *contains uncountably many points of* B*;* [6, p. 129]. *Since* $\xi + \mathbb{Q} + \alpha\mathbb{Q}$ *is countable, every neighborhood of* ξ *contains points* $\eta \in B$ *such that* $\eta \notin \xi + \mathbb{Q} + \alpha\mathbb{Q}$. *This means* *that* $\big[(\xi+\mathbb{Q})\cup(\xi+\alpha\mathbb{Q})\big] \cap \big[(\eta+\mathbb{Q})\cup(\eta+\alpha\mathbb{Q})\big] = \emptyset$. *Since* $\xi, \eta \in B \subseteq E_n$, *it follows that* T_α *cannot satisfy (I.C.) on* I.

Lastly, if α *is rational, then* $x + t\mathbb{Q} = x + \mathbb{Q}$ *for* $t = 1, \alpha$; *but if* α *is irrational, then* $(x+\mathbb{Q}) \cap (x+\alpha\mathbb{Q}) = \{x\}$. *Hence clearly* T_α *is filtering down if* α *is rational, but not if* α *is irrational.*

3 Main Results

We begin with a monotonicity theorem.

Theorem 3.1 *Suppose that* T *is partitioning on* I.

(i) If $T\overline{F}'(x) < \infty$ *on* I *and* $T\overline{F}' \leq \alpha$ *a.e. on* I, *then* $F(x) - \alpha x$ *is non-increasing on* I.

(ii) If $T\underline{F}'(x) > -\infty$ *on* I *and* $T\underline{F}' \geq \beta$ *a.e. on* I, *then* $F(x) - \beta x$ *is non-decreasing on* I.

Proof. (i) Let $A = \{x \in I; T\overline{F}'(x) \leq \alpha\}$, $E = I \setminus A$, and $\epsilon > 0$. Assuming that $f(x) > T\overline{F}'(x)$ for all $x \in I$, with $f(x) = \alpha + \epsilon$ for $x \in A$, there is a T-full cover τ of I such that

$$F(v) - F(u) \leq f(x)(v-u) \quad \text{for all} \quad [u,v] \in I_{\tau(x)},\, x \in I.$$

Again, since $|E| = 0$, by (2.1), τ has a refinement λ on I such that for every λ-partition ϖ_0 over E we have $\sum(|f|, \varpi_0) < \epsilon$ and $|\varpi_0| < \epsilon$.

Now let $a \leq c < d \leq b$. By (2.2) $[c,d]$ has a λ-partition, say ϖ. Then $\varpi = \varpi_1 \cup \varpi_0$ where ϖ_1 and ϖ_0 are λ-partitions, (and so τ-partitions), over A and E respectively. Hence, recalling the choice of f, τ and λ, we have

$$\begin{aligned} F(d) - F(c) = \Delta(F, \varpi) &\leq \sum(f, \varpi) = \sum(f, \varpi_1) + \sum(f, \varpi_0) \\ &\leq (\alpha+\epsilon)|\varpi_1| + \sum(|f|, \varpi_0) \\ &< \alpha(d-c) - \alpha|\varpi_0| + \epsilon|\varpi_1| + \epsilon \\ &\leq \alpha(d-c) + |\alpha|\epsilon + \epsilon|I| + \epsilon. \end{aligned}$$

Letting $\epsilon \to 0+$ we get $F(d) - F(c) \leq \alpha(d-c)$, which proves (i).

(ii) This follows by applying (i) to $-F$. □

Next we prove the fundamental theorem of this paper.

Theorem 3.2 *Suppose* T *satisfies ILP on* I, *and* $T\overline{F}'(x) < \infty$ *and* $T\underline{F}'(x) > -\infty$ *for all* $x \in I$. *Then:*

(i) *F lies between its one-sided extreme limits on either side, everywhere in I,*

(ii) *F is (ACG) on I, it is Darboux on I, $(\mathrm{ap})F'(x)$ exists finitely a.e. on I, and $F'(x)$ exists finitely a.e. on a dense open set in I,*

(iii) *if F is of bounded variation, VB, on a closed set $X \subseteq I$ and $F|X$ is continuous, then F is AC on X,*

(iv) *either F is strictly monotonic and AC on I, or F has a local extremum at some AC-point of F on $]a, b[$.*

Note *Other variants of (iv) appear in* [12, p. 14] *and* [4, p. 227].

PROOF. Part (i) is obvious. For (ii), assuming $f(x) > T\overline{F}'(x)$ and $-f(x) < T\underline{F}'(x)$, there are T-full covers τ, λ of I such that

$$F(v) - F(u) \le f(x)(v-u) \text{ for all } [u,v] \in I_{\tau(x)},\ x \in I,$$
$$F(v) - F(u) \ge -f(x)(v-u) \text{ for all } [u,v] \in I_{\lambda(x)},\ x \in I.$$

Let τ, λ satisfy ILP on I via $\{(A_n, \delta_n)\}$, $\{(B_n, \eta_n)\}$ respectively. Put

$$E_n = \{x \in A_n \cap B_n;\ |f(x)| \le n\}, \quad \rho_n = \min\{\delta_n, \eta_n\}.$$

Then $\{E_n\}$ is an I-chain and both τ, λ satisfy ILP on I via $\{(E_n, \rho_n)\}$.

Let $x, y \in E_n$, $0 < y - x < \rho_n$, and $Z \subset\ ]x, y[$ be the zero measure set as required by ILC of τ. By (2.1), given $\epsilon > 0$ there is a refinement μ of τ on Z such that for every μ-partition ϖ over Z we have $\sum(|f|, \varpi) < \epsilon$. By ILC of τ, there is a τ-partition $\{([x,u],x),\ ([v,y],y)\} \cup \varpi$ of $[x,y]$ where ϖ is a μ-partition of $[u,v]$. Then by the choice of f and τ we have

$$\begin{aligned} F(y) - F(x) &= \big(F(u) - F(x)\big) + \Delta(F, \varpi) + \big(F(y) - F(v)\big) \\ &\le f(x)(u-x) + \sum(f, \varpi) + f(y)(y-v) \\ &< n(u-x) + \epsilon + n(y-v). \end{aligned}$$

Hence $F(y) - F(x) \le n(y-x)$. Similarly, from ILC of λ, $F(y) - F(x) \ge -n(y-x)$. Hence $|F(y) - F(x)| \le n(y-x)$, and $F|E_n$ is continuous.

Now, for any n, let J_n be any closed interval with $0 < |J_n| < \rho_n$, and let $x_1 < y_1$ be any two points in the closure of $E_n \cap J_n$. There is $E_m \supseteq E_n$ such that $x_1, y_1 \in E_m$. Since $0 < y_1 - x_1 < \rho_n$ and $F|E_m$ is continuous, choosing points $x, y \in E_n$ close to x_1, y_1 respectively it follows at once from above that $|F(y_1) - F(x_1)| \le n(y_1 - x_1)$.

These Lipschitz conditions evidently imply that F is (ACG) on I. Hence $(\mathrm{ap})F'(x)$ exists finitely a.e. on I, [10, p. 223 infra], and $F'(x)$ exists finitely a.e. on a dense open set in I (since by Baire's theorem the AC-points of F are dense in I). Again, since F is (ACG) on I it is Baire-1 on I, which by (i) implies that F is Darboux on I, [13, Theorem III, p. 21], [1, Theorem 6.1, p. 103].

(iii) This follows from [10, (6.7), p. 227], since being (ACG) F satisfies Lusin's condition (N) on I, [10, (6.1), p. 225].

(iv) Let E denote the set of points of I having no neighborhood in I on which F is strictly monotonic. Clearly E is closed. Routine arguments shows that F is strictly monotone on every component $]u, v[$ of $]a, b[\setminus E$, and then (i) implies that F is strictly monotone and continuous on [u,v], and hence by (iii) F is AC on $[u, v]$. It follows at once that, if E has an isolated point, c say, then $a < c < b$, c is an AC-point of F, and F has a strict local extremum at c, but if E is empty, then F is strictly monotonic and AC on I.

Suppose now E is non-empty and perfect. Then, since by (ii) F is (ACG) on I, by Baire's theorem F must be AC on some perfect portion $E \cap [p, q]$ of E.

Since, as shown above, F is monotone and AC on each closed interval contiguous to E in $[p, q]$, it easily follows that F is VB and continuous on $[p, q]$. Hence by (iii) F is AC on $[p, q]$. Also, since $E\cap]p, q[\neq \emptyset$, F is not strictly monotone on $[p, q]$ and hence F must have a local extremum at some point $c \in]p, q[$. □

The above proof of (3.2)(ii) contains the germ of

Theorem 3.3 *Suppose T satisfies ILP on I and is non-porous on I. If $\infty \neq T\overline{F}'(x) \leq T\underline{F}'(x) \neq -\infty$ for all x in I, then $F'(x)$ exists finitely for all x in a dense open set in I.*

PROOF. From the proof of (3.2)(ii), F is Lipschitz on each part of some closed I-form. So by Baire's theorem, for every $[c, d] \subseteq I$ with $c < d$, there is a perfect portion $[p, q]$ of $[c, d]$ on which F is Lipschitz, say $|F(y) - F(x)| \leq N(y - x)$ for $p \leq x < y \leq q$. Then $G(x) = F(x) + Nx$ is non-decreasing on $[p, q]$. So for all $x \in]p, q[$, since $p_T(x) = 0$, using Note 2.4 for G we clearly have $\overline{G}'(x) = T\overline{G}'(x) = T\overline{F}'(x) + N$ and $\underline{G}'(x) = T\underline{G}'(x) = T\underline{F}'(x) + N$. Since $\infty \neq T\overline{F}'(x) \leq T\underline{F}'(x) \neq -\infty$, it follows that $G'(x)$ exists finitely for all x in $]p, q[$. Hence $F'(x) = G'(x) - N$ exists finitely for all x in $]p, q[$. □

We are now ready to analyze the Darboux property of derivatives.

Theorem 3.4 *Hypotheses: T satisfies ILP on I, $T\overline{F}'(x) < \infty$ and $T\underline{F}'(x) > -\infty$ for all x in I, the set D of points of I where F is T-derivable contains at least the AC-points of F in $]a, b[$, $D_* = \{x \in D; TF'(x)\,\text{exists}\}$, $D_{ac} = \{x \in D_*\,and\,x\,\text{is an AC-point of}\ F\ \text{in}\ ]a, b[\}$.*
Conclusions:

(i) *If $T\overline{F}'(p) < \alpha < T\underline{F}'(q)$ for some $\alpha \in \mathbb{R}$ and $p, q \in I$, with possibly $p = q$, then for every $]u, v[\subset I$ with $p, q \in [u, v]$, F has an AC-point $d \in]u, v[$ where $TF'(d)$ exists, so $d \in D_{ac}$, and equals α.*

(ii) *If* $D_{ac} \subseteq E \subseteq D$, *then every extended real-valued function* g, *satisfying* $T\overline{F}'(x) \leq g(x) \leq T\underline{F}'(x)$ *for all* x *in* E *is Darboux on* E*; in particular both* $T\overline{F}'$ *and* $T\underline{F}'$ *are Darboux on* E.

(iii) *If* $D_{ac} \subseteq E \subseteq D_*$, *then* TF' *is Darboux on* E.

PROOF. Clearly (i) implies both (ii) and (iii).

To prove (i) let $G(x) = F(x) - \alpha x$ for all $x \in I$. Then $T\overline{G}'(p) = T\overline{F}'(p) - \alpha < 0$ and $T\underline{G}'(q) = T\underline{F}'(q) - \alpha > 0$, which together imply that G is not monotone on $[u, v]$. Since $T\overline{G}'(x) = T\overline{F}'(x) - \alpha < \infty$ and $T\underline{G}'(x) = T\underline{F}'(x) - \alpha > -\infty$ for all $x \in [u, v]$, it follows from (3.2)(iv) that there is a $d \in\,]u, v[$, an AC-point of G, (hence also of F), such that $G(d)$ is a local extremum of G; so $T\overline{G}'(d) \geq 0$ and $T\underline{G}'(d) \leq 0$. Since by hypothesis F is T-derivable at d, we get

$$0 \leq T\overline{G}'(d) = T\overline{F}'(d) - \alpha \leq T\underline{F}'(d) - \alpha = T\underline{G}'(d) \leq 0.$$

Hence it follows that $TF'(d)$ exists with value α. □

A similar result to this is the desired mean value property.

Theorem 3.5 *Under the hypotheses of* (3.4), F *has an AC-point* $c \in]a, b[$ *where* $TF'(c)$ *exists and equals* $r = \big(F(b) - F(a)\big)/(b - a)$.

PROOF. This follows from the preceding proof, since now $G(x) = F(x) - rx$ is not strictly monotone on I because $G(a) = G(b)$. □

Corollary 3.6 *If in* (3.4) *it is assumed further that* $TF'(x) \geq 0$ *for all* x *in* D_{ac}, *then* F *is non-decreasing on* I.

Next we prove the semi-Baire-1 property, in a form which also gives a strengthened version of the O'Malley property [8].

Theorem 3.7 *Suppose* T *satisfies ILP on* I *and is (NPG) in* I, *and* $T\overline{F}'(x) \leq f(x) \leq T\underline{F}'(x)$ *for all* x *in* I. *Then* f *is semi-Baire-*1 *on* I.

In fact, if E *is the closure of* $f^{-1}\big(f(t)\big)$ *for any* $t \in I$, *then* F *has an AC-point* c *in* I, $a \leq t \leq c < b$ *or* $a < c \leq t \leq b$, *such that* $f(c) = f(t)$ *and* $F'(c)$ *exists, (and so does* $TF'(c)$*), with the value* $f(c) = f(t)$, *and all of the functions* $f, T\overline{F}', T\underline{F}', \overline{F}', \underline{F}'$ *are continuous at* c *relative to* E.

PROOF. We can assume without loss in generality that $f(t) = 0$, for otherwise we could consider the functions $F(x) - f(t)x$ and $f(x) - f(t)$. Now F is T-derivable on I, and $T\overline{F}'(x) < \infty$ and $T\underline{F}'(x) > -\infty$ for all x in I since $f(x)$ is a finite function. So by (3.4)(ii) f is Darboux on I.

First suppose E has no perfect portion. Then E must have an isolated point c, such that either $a \le t \le c < b$ or $a < c \le t \le b$, and then $f(c) = f(t) = 0$. Continuity at c relative to E is then trivial. Also, let $a \le c < d \le b$ where $E \cap]c, d] = \emptyset$. Since $f^{-1}(0) \subseteq E$, the Darboux property of f implies that for all $x \in]c, d[$, either $T\overline{F}'(x) \le f(x) < 0$ or $0 < f(x) \le T\underline{F}'(x)$. Hence by (2.3) and (3.1) F is monotone on $[c, d]$, and hence by (3.2)(i), (iii) F is AC on $[c, d]$. Besides, $p_T(c) < 1$ since T is (NPG) on I, and $T\overline{F}'(c) \le f(c) = 0 \le T\underline{F}'(c)$. Hence from (2.4) we get $F'_+(c) = 0$. By symmetry, if $a < c \le b$, then F is AC on some $[d, c], d < c$, and $F'_-(c) = 0$. Thus c is an AC-point of F on I and $F'(c) = 0 = f(t)$.

Next suppose that E has a perfect portion. Then since T is (NPG) on I and, by (3.2)(ii), F is (ACG) on I, so by Baire's theorem there must exist a perfect portion E_0 of E and $0 < q < 1$, such that $p_T(x) < q$ for all $x \in E_0$, and F is AC on E_0.

Now, given an $\epsilon > 0$ let $\eta = \epsilon(1-q)^2/3$. Then there are T-full covers τ, λ of I such that for all $x \in I$ we have

$$F(v) - F(u) \le (f(x) + \eta)(v - u) \quad \text{for all} \quad [u, v] \in I_{\tau(x)},$$
$$F(v) - F(u) \ge (f(x) - \eta)(v - u) \quad \text{for all} \quad [u, v] \in I_{\lambda(x)}.$$

Let τ and λ satisfy ILP on I via $\{(A_n, \delta_n)\}$ and $\{(B_n, \rho_n)\}$ respectively. Then I is the union of the sets

$$E_{n,i} = A_n \cap B_n \cap f^{-1}\big([i\eta, (i+1)\eta]\big),\ n = 1, 2, \ldots,\ i = 0, \pm1, \pm2, \ldots.$$

So by Baire's theorem some $E_{n,i}$ must be dense in some perfect portion of E_0 which we can take as $E_0 \cap [r, s] = E \cap [r, s]$ with $0 < s - r < \min\{\delta_n, \rho_n\}$. Since $f^{-1}(0) \subseteq E$, it follows as before that F is monotone and AC on each closed interval contiguous to E in $[r, s]$, and hence by (3.2)(iii) F is AC on $[r, s]$.

Again, let $x, y \in E_{n,i} \cap [r, s]$, $x < y$. By ILC of τ and (2.1), given $\delta > 0$ there is a τ-partition $\{([x, u], x), ([v, y], y)\} \cup \varpi$ of $[x, y]$, where ϖ is a τ-partition of $[u, v]$ with $\sum(|f|, \varpi) < \delta$ and $v - u < \delta$. Then

$$\begin{aligned} F(y) - F(x) &= \big(F(u) - F(x)\big) + \Delta(F, \varpi) + \big(F(y) - F(v)\big) \\ &\le (f(x) + \eta)(u - x) + \sum(f, \varpi) + \eta(v - u) + (f(y) + \eta)(y - v) \\ &< (i+2)\eta(u - x) + \delta + \eta\delta + (i+2)\eta(y - v) \\ &\le (i+2)\eta(y - x) + |i+2|\eta\delta + (1+\eta)\delta. \end{aligned}$$

Letting $\delta \to 0+$ we get that $F(y) - F(x) \le (i+2)\eta(y - x)$. Similarly from ILC of λ we get that $F(y) - F(x) \ge (i-1)\eta(y - x)$.

Hence, since the set $E_{n,i}$ is dense in $E \cap [r, s]$ and F is continuous on $[r, s]$, for all distinct $x, y \in E \cap [r, s]$ we have

$$(i-1)\eta \le \frac{F(y) - F(x)}{y - x} \le (i+2)\eta. \tag{5}$$

Now, since E is the closure of $f^{-1}(0)$ and $E \cap]r,s[\neq \emptyset$, there is an $e \in E \cap]r,s[$ such that $f(e) = 0$. Then $T\overline{F}'(e) \leq 0 \leq T\underline{F}'(e)$ and $p_T(e) < 1$. Hence by (5) and (2.4) we get

$$\frac{(i-1)\eta}{1-p_T(e)} \leq 0 \leq \frac{(i+2)\eta}{1-p_T(e)}.$$

So $-2 \leq i \leq 1$. Hence by (1), for all distinct $x, y \in E \cap [r,s]$ we have

$$-3\eta \leq \frac{F(y)-F(x)}{y-x} \leq 3\eta.$$

Hence by (2.4), for all $x \in E \cap [r,s]$, if $T\underline{F}'(x) \geq 0$, then $T\overline{F}'(x) \leq T\underline{F}'(x) < 3\eta/(1-q)$, and if $T\overline{F}'(x) \leq 0$, then $T\underline{F}'(x) \geq T\overline{F}'(x) > -3\eta/(1-q)$, but always

$$\frac{\min\{-3\eta, T\underline{F}'(x)\}}{1-p_T(x)} \leq \underline{F}'(x) \leq \overline{F}'(x) \leq \frac{\max\{3\eta, T\overline{F}'(x)\}}{1-p_T(x)}.$$

Hence for all $x \in E \cap [r,s]$, considering all possible signs of $T\underline{F}'(x)$ and $T\overline{F}'(x)$ we get $-3\eta/(1-q)^2 < \underline{F}'(x) \leq \overline{F}'(x) < 3\eta/(1-q)^2$. Thus $-\epsilon < \underline{F}'(x) \leq \overline{F}'(x) < \epsilon$ for all $x \in E \cap [r,s]$.

Consequently, taking ϵ to be $1, 1/2, 1/3, \ldots$ in succession, we can find intervals $[r_n, s_n]$ with end points in E, such that $r_n < r_{n+1} < s_{n+1} < s_n < r_n + n^{-1}$, F is AC on $[r_1, s_1]$, and $-n^{-1} < \underline{F}'(x) \leq T\overline{F}'(x) \leq f(x) \leq T\underline{F}'(x) \leq \overline{F}'(x) < n^{-1}$ for all $x \in E \cap [r_n, s_n]$. Then the point $c = \lim r_n = \lim s_n$ evidently fulfills all the required conditions . □

This permits sharper versions of the Darboux and mean value properties.

Corollary 3.8 *Hypotheses: T satisfies ILP and is (NPG) on I, $\infty \neq T\overline{F}'(x) \leq T\underline{F}'(x) \neq -\infty$ for all x in I, $I_* = \{x \in I,\ TF'(x)$ exists$\}$ and $I_{ac} = \{x \in I_*,\ x$ is an AC-point of F in $]a,b[$ and $F'(x)$ exists$\}$. Conclusions:*

(i) If $T\overline{F}'(p) < \alpha < T\underline{F}'(q)$ for some $\alpha \in \mathbb{R}$ and $p, q \in I$, with possibly $p = q$, then for every $]u,v[\subset I$ with $p, q \in [u,v]$, F has an AC-point $d \in]u,v[$ where $F'(d)$ exists, so $d \in I_{ac}$, and equals α.

(ii) If $I_{ac} \subseteq E \subseteq I$, then every extended real-valued function g satisfying $T\overline{F}'(x) \leq g(x) \leq T\underline{F}'(x)$ for all $x \in E$, is Darboux on E.

(iii) If $I_{ac} \subseteq E \subseteq I_*$, then TF' is Darboux on E.

(iv) There is a $d \in I_{ac}$ such that $F(b) - F(a) = (b-a)F'(d)$.

PROOF. Clearly (i) implies (ii) and (iii). Now, assume that $T\overline{F}'(x) \leq f(x) \leq T\underline{F}'(x)$ for all $x \in I$. Then by (3.7), for every $c \in]a,b[$ there is a $d \in I_{ac}$ such that $F'(d) = f(c)$.

Since $f(c) = TF'(c)$ whenever $TF'(c)$ exists, (i) and (iv) follow from (3.4)(i) and (3.5), respectively. □

We may show by example that the function f in (3.7) may fail to be Baire-1 even under stronger conditions.

Example 3.9 *We shall construct a bounded F having a non-Baire-1 finite path derivative F'_P on I, relative to a bilateral non-porous system of paths P satisfying (I.C.), recall (2.5).*

Let E be a non-dense perfect set with bounds a, b and let $\{]a_n, b_n[\}$ be the sequence of the distinct components of $I \setminus E$. By induction we define distinct sequences of positive integers $\{1_k\}, \{2_k\}, \{3_k\}, \dots$ such that

$$b_n < \cdots < b_{n_2} < b_{n_1},\ b_{n_k} \to b_n \text{ as } k \to \infty, \tag{6}$$

$$\frac{a_{n_k} - b_{n_{k+1}}}{a_{n_k} - b_n} \to 0 \text{ as } k \to \infty; \tag{7}$$

and since they are distinct

$$m \neq p \implies m_j \neq p_k,\ j, k = 1, 2, \dots. \tag{8}$$

First observe the following construction: for any n and any $d_n > b_n$ there is $c_1 \in E \cap\,]b_n, d_n[$. Let $c_j = b_n + (c_1 - b_n)/j$, $j = 2, 3, \dots$. Then $b_n < \cdots < c_2 < c_1$, and $c_j \to b_n$ as $j \to \infty$. We can select n_1 such that $b_{n_1} \in\,]c_2, c_1]$, and then select the integers n_{k+1} successively such that $]a_{n_{k+1}}, b_{n_{k+1}}[$ intersects $]c_{2+j_k}, c_{1+j_k}[$, where j_k is the unique index such that $a_{n_k} \in]c_{1+j_k}, c_{j_k}]$. Clearly $\{n_k\}$ satisfies (6), and it also satisfies (7) because

$$0 < \frac{a_{n_k} - b_{n_{k+1}}}{a_{n_k} - b_n} < \frac{c_{j_k} - c_{2+j_k}}{c_{1+j_k} - b_n} < \frac{2}{j_k} \to 0 \quad \text{as} \quad k \to \infty.$$

Now, taking $n = 1$, so $d_1 > b_1$, take $d_1 = b$ say, we construct a sequence as above and call it $\{1_k\}$. Suppose then for some $n \geq 2$ and $p = 1, 2, \dots, n-1$ the sequences $\{p_k\}$ have been defined so as to satisfy (6), (7) and (8) among them. Evidently we can find $d_n > b_n$ such that $]b_n, d_n[$ does not contain any of the points b_{p_k} for $p = 1, 2, \dots, n-1$ and $k = 1, 2, \dots$. Then we define $\{n_k\}$ as above with $\{b_{n_k}\} \subset\,]b_n, d_n[$. This completes the induction.

Now, for each n we define a strictly increasing two-way sequence $\{t_{n,i}\}_{i=-\infty}^{\infty}$ in $]a_n, b_n[$ as follows:

$$t_{n,i} = a_n + \frac{n(b_n - a_n)}{2(n-i)} \quad \text{for} \quad i = 0, -1, -2, \dots,$$

$$t_{n,i} = b_n - \frac{n(b_n - a_n)}{2(n+i)} \quad \text{for} \quad i = 1, 2, \dots.$$

So $t_{n,-j} \to a_n$, and $t_{n,j} \to b_n$ as $j \to \infty$. Also we have

$$\max\left\{\frac{t_{n,i+5}-t_{n,i}}{t_{n,i+5}-a_n}\,,\ \frac{t_{n,i+5}-t_{n,i}}{b_n-t_{n,i}}\right\} \le \frac{5}{n+|i|}. \tag{9}$$

Let $I_{n,r} = \cup_{i=-\infty}^{\infty}\,]t_{n,6i+r}, t_{n,6i+r+1}]$ for $r = 0, 1, \ldots, 5$ only. Then $I_{n,r} \cap I_{n,s} = \emptyset$ for $r \neq s$. We define

$$P_x = \begin{cases} (\mathbb{R}\setminus I)\cup E\cup\bigcup_{n=1}^{\infty} I_{n,0} & \text{if } x\in E,\ x\notin\{b_n \\ \{x\}\cup I_{n,2}\cup\bigcup_{k=1}^{\infty} I_{n_k,4} & \text{if } x=b_n, n=1,2,\ldots, \\ \mathbb{R} & \text{if } x\in\mathbb{R}\setminus E. \end{cases}$$

Using (6), (7) and (9) we readily verify that $P = \{P_x;\, x \in \mathbb{R}\}$ is a bilateral, non-porous system of paths.

Also, if $\delta : \mathbb{R} \to]0,1[$ is such that $\delta(b_n) < \min\{\,|b_1 - b_n|, |b_2 - b_n|, \ldots, |b_{n-1} - b_n|\,\}$ for $n \geq 2$, then $|b_m - b_n| > \min\{\delta(b_m), \delta(b_n)\}$ for all $m \neq n$ and hence, clearly P satisfies (I.C.) with respect to δ.

Now, recalling (3), for each n we define $n^* = m$ if $n = m_k$ for some m and k, and $n^* = n$ otherwise. Then we define

$$F(x) = \begin{cases} 0 & \text{if } x\in E\cup\bigcup_{n=1}^{\infty} I_{n,0}, \\ x-b_n & \text{if } x\in I_{n,2}, n=1,2,\ldots, \\ x-b_{n^*} & \text{if } x\in I_{n,4}, n=1,2,\ldots. \end{cases}$$

Also we define $F(x)$ on each of $I_{n,1}$, $I_{n,3}$, $I_{n,5}$ in such a way that F becomes differentiable on each of the intervals $]a_n, b_n[$, and remains bounded on I. Then we see at once that $F'_P(x)$ exists finitely for all x in I. (Note that $(n_k)^* = n$.) But $F'_P(b_n) = 1$ for all n and $F'_P(x) = 0$ for all $x \in E, x \notin \{b_n\}$, so F'_P has no points of continuity in E relative to E since $\{a_n\}$ and $\{b_n\}$ are disjoint dense subsets of E. Hence F'_P is not Baire-1 on I, though by (3.7) it is semi-Baire-1 on I.

Our next example shows that the O'Malley property is non-trivial insofar as it may fail even for strictly increasing absolutely continuous F and for T satisfying (IC), in the absence of the condition (NPG).

Example 3.10 *It is not difficult to find a strictly increasing absolutely continuous F, such that $0 < F'(x) < \infty$ for all x in $]a,b]$ but with $0 = D_+F(a) < D^+F(a) < \infty$. Then there is a strictly decreasing sequence $\{a_n\}$ in $]a,b[$ converging to a such that $\big(F(a_n) - F(a)\big)/(a_n - a) \to 0$. We define a bilateral system of paths P by setting $P_x = \mathbb{R}$ for $x \neq a$, and $P_a =]-\infty, a] \cup \{a_n\}$. Obviously P satisfies (I.C.), and $F'_P(x)$ exists finitely for all x in I with $F'_P(a) = 0$. But the O'Malley property fails at a, since $F'(a)$ does not exist and $F'(x) > 0$ for $a < x \leq b$.*

Next, we give a proof of a stronger version of the Weil property [17], inclusive of the Denjoy property.

Theorem 3.11 *Suppose T satisfies ILP on I and is (NPG) on I, and $T\overline{F}'(x) \le f(x) \le T\underline{F}'(x)$ for all x in I. Then for every $t \in I$ and $\alpha < f(t) < \beta$, every one-sided neighborhood of t in I contains an interval J on which F is AC, (so $f(x) = F'(x)$ a.e. on J), such that $|J \cap f^{-1}(]\alpha, \beta[)| > 0$.*

Proof. Let E denote the closure of $f^{-1}\big(f(t)\big)$, and suppose that $a \le t < v < b$. By (3.7), F has an AC-point c in $[t, v[$ such that $f(c) = f(t)$, (so $\alpha < f(c) < \beta$) and both $T\overline{F}', T\underline{F}'$ are continuous at c relative to $E \cap [t, v[$. Then let $c < d < v$ be such that F is AC on $J_0 = [c, d]$, and

$$\alpha < T\overline{F}'(x) \le f(x) \le T\underline{F}'(x) < \beta \text{ for all } x \in E \cap J_0. \tag{10}$$

If $|E \cap J_0| > 0$, then by (10) $|J \cap f^{-1}(]\alpha, \beta[)| > 0$ with $J = J_0$.

Suppose that $|E \cap J_0| = 0$. Let $J = [r, s]$ be the closure of a component of $J_0 \setminus E$. Since $c \in E$, clearly $r \in E \cap J_0$. So by (10) $D^+F(r) \ge T\underline{F}'(r) > \alpha$, and $D_+F(r) \le T\overline{F}'(r) < \beta$, which by (3.1)(i), (ii) imply, respectively

$$|\{x \in J;\, T\overline{F}'(x) > \alpha\}| > 0 \text{ and } |\{x \in J;\, T\underline{F}'(x) < \beta\}| > 0.$$

Since $T\overline{F}'(x) \le f(x) \le T\underline{F}'(x)$ for all x in $J = [r, s]$ we get

$$|\{x \in J;\, f(x) > \alpha\}| > 0 \text{ and } |\{x \in J;\, f(x) < \beta\}| > 0. \tag{11}$$

But, since $]r, s[\cap f^{-1}\big(f(t)\big) = \emptyset$, the Darboux property of f, (3.4)(ii), implies that either $f(x) < f(t) < \beta$ or $f(x) > f(t) > \alpha$ for $x \in]r, s[$. Hence, by (11), in either case $|J \cap f^{-1}(\,]\alpha, \beta[\,)| > 0$.

Similarly, if $a < u < t \le b$, we can find a J in some $[d, c] \subset]u, t]$, and this completes the proof of the theorem. □

Finally we will say that a function f *has the property* Z^* *on* I, if for every $c \in I$ and $\epsilon > 0, \eta > 0$ there is a neighborhood I_c of c in I such that the following conditions Z^+, Z^- hold.

Z^+: If $f(x) \ge f(c) - \epsilon$ a.e. on a closed interval $J \subset I_c$ then $|A| - |B| \le \eta\rho(c, J)$, where $A = \{x \in J;\, f(x) \ge f(c) + \epsilon\}$ and $B = \{x \in J;\, f(c) - \epsilon \le f(x) < f(c)\}$, and $\rho(c, J) = \max\{|x - c|; x \in J\}$.

Z^-: If $f(x) \le f(c) + \epsilon$ a.e. on a closed interval $J \subset I_c$ then $|A| - |B| \le \eta\rho(c, J)$, where $A = \{x \in J;\, f(x) \le f(c) - \epsilon\}$ and $B = \{x \in J;\, f(c) < f(x) \le f(c) + \epsilon\}$.

We remark that f satisfies Z^- if and only if $-f$ satisfies Z^+. In another paper it will be shown that property Z^* is strictly stronger than the Zahorski-Weil property Z; ([18, p. 528], has a misprint of $\le \epsilon$ for $\ge \epsilon$), and that every approximate Peano derivative has the property Z^*.

Theorem 3.12 *If T satisfies ILP on I and is non-porous on I and $T\overline{F}'(x) \le f(x) \le T\underline{F}'(x)$ for all x in I, then f has the property Z^* on I.*

PROOF. Considering $-f$ and $-F$ clearly we need only prove Z^+ for f.

Fix $c \in I$, $\epsilon > 0$, $\eta > 0$, put $G(x) = F(x) - F(c) - (x-c)f(c)$ and $g(x) = f(x) - f(c)$ for x in I. Then $G(c) = g(c) = 0$, and

$$T\overline{G}'(x) = T\overline{F}'(x) - f(c) \le g(x) \le T\underline{F}'(x) - f(c) = T\underline{G}'(x), \quad x \in I.$$

Since $T\overline{G}'(c) \le 0 \le T\underline{G}'(c)$, there are $\tau(c), \lambda(c) \in T(c)$ such that

$$\frac{G(q)}{q-c} \begin{cases} < \dfrac{\epsilon\eta}{4} & \text{if } c \neq q \in I \cap \tau(c), \\ > -\dfrac{\epsilon\eta}{4} & \text{if } c \neq q \in I \cap \lambda(c). \end{cases} \tag{12}$$

Since $p_T(c) = 0$, there is a neighborhood I_c of c in I such that

$$s - r \le \frac{\eta}{4}\rho(c, [r,s]) \text{ if }]r,s[\subset I_c \setminus \tau(c) \quad \text{or} \quad]r,s[\subset I_c \setminus \lambda(c). \tag{13}$$

We show that for every $J = [x,y] \subset I_c$ for which $G|J$ is continuous, there is a $[u,v] \subseteq J$ such that

$$u - x + y - v \le \frac{\eta}{2}\rho(c,J) \text{ and } G(v) - G(u) \le \frac{\epsilon\eta}{2}\rho(c,J). \tag{14}$$

If $c < x \le y$, take $v = \inf\{t \in [x,y];\ ]t,y[\cap \tau(c) = \emptyset\}$ and $u = \sup\{t \in [x,v];\ ]x,t[\cap \lambda(c) = \emptyset\}$. Note that, if $u \neq v$, then v and u belong to the closures of $\lambda(c) \cap [u,v]$ and $\tau(c) \cap [u,v]$, respectively.

If $x \le y < c$ take $v = \inf\{t \in [x,y];\ ]t,y[\cap \lambda(c) = \emptyset\}$ and $u = \sup\{t \in [x,v];\ ]x,t[\cap \tau(c) = \emptyset\}$. Note that, if here $u \neq v$, then v and u belong to the closures of $\lambda(c) \cap [u,v]$ and $\tau \cap [u,v]$ respectively.

If $x \le c \le y$ take $v = \sup\big([c,y] \cap \tau(c)\big)$ and $u = \inf\big([x,c] \cap \tau(c)\big)$. Note that, now both of u, v belong to the closure of $\tau(c) \cap [u,v]$.

In all cases, the first part of (14) follows at once from (13), and the second part follows by noting that, if $u \neq v$, then continuity of $G\big|[u,v]$ implies by (12) that $G(v) \le \big(\epsilon\eta/4\big)|v-c|$ and $-G(u) \le \big(\epsilon\eta/4\big)|u-c|$.

Now, let $f(x) \ge f(c) - \epsilon$ a.e. on some $J = [x,y] \subset I_c$. With A, B as in Z^+, we have $A = \{x \in J;\ g(x) \ge \epsilon\}$, and $B = \{x \in J;\ -\epsilon \le g(x) < 0\}$.

Since $T\underline{G}'(x) \ge g(x) \ge -\epsilon$ a.e. on J, by (3.1) $G(x) + \epsilon x$ is non-decreasing on J. Hence, clearly, by (3.2)(i), (iii) G is AC on J. So there is a $[u,v] \subseteq J$ satisfying (14), and since $g(x) = G'(x)$ a.e. on J and $g(x) \ge 0$ a.e. on $J \setminus A \cup B$, we have, further

$$G(v) - G(u) = \int_u^v g \ge \epsilon\big|A \cap [u,v]\big| - \epsilon\big|B \cap [u,v]\big|. \tag{15}$$

Since $|A|-|B| \leq u-x+y-v+\big|A\cap[u,v]\big|-\big|B\cap[u,v]\big|$, from (14) and (15) it follows that $|A|-|B| \leq \eta\rho(c,J)$.

Thus f has the property Z^+ on I, and we are finished. $\square$

References

[1] A. M. Bruckner and J. G. Ceder, *Darboux continuity*, Jber. Deutsch. Math. Verein, **67** (1965), 93–117.

[2] A. M. Bruckner, R. J. O'Malley and B. S. Thomson, *Path derivatives: A unified view of certain generalized derivatives*, Trans. Amer. Math. Soc., **283** (1984), 97–125.

[3] P. S. Bullen, *Some applications of partitioning covers*, Real Analysis Exch., **9** (1983-1984), 539–557.

[4] P. S. Bullen and D. N. Sarkhel, *On Darboux and mean value properties*, Canad. Math. Bull., **30** (1987), 223–230.

[5] C. Goffman and C. J. Neugebauer, *On approximate derivatives*, Proc. Amer. Math. Soc., **11** (1960), 962–966.

[6] E. W. Hobson, *The Theory of Functions of a Real Variable and the Theory of Fourier Series*, Vol. 1, Dover, New York, 1957.

[7] A. Khintchine, *Recherches sur la structure des fonctions mesurables*, Fund. Math., **9** (1927), 217–279.

[8] R. J. O'Malley, *The set where an approximate derivative is a derivative*, Proc. Amer. Math. Soc., **54** (1976), 122–124.

[9] S. Marcus, *On a theorem of Denjoy and on approximate derivative*, Monatsh. Math., **66** (1962), 435–440.

[10] S. Saks, *Theory of the Integral* , Dover, New York, 1964.

[11] D. N. Sarkhel and A. B. Kar, *(PVB) function and integration*, J. Austral. Math. Soc., Ser. A, **36** (1984), 335–353.

[12] D. N. Sarkhel and P. K. Seth, *On some generalized approximative relative derivatives*, Rendiconti Circolo Mat. Palermo, **35**(2) (1986), 5–21.

[13] H. K. Sen, *Darboux's property and its applications*, Proc. Benares Math. Soc., (N.S.) **2** (1940), 17–23.

[14] B. S. Thomson, *On full covering properties*, Real Analysis Exch., **6** (1980-1981), 77–93.

[15] B. S. Thomson, *Derivation bases on the real line, II*, Real Analysis Exch., **8** (1982-1983), 278–442.

[16] G. P. Tolstoff, *Sur la dérivée approximative exacte*, Rec. Math. (Mat. Sb.), (N.S.) **4** (1938), 499–504.

[17] C. E. Weil, *On approximate and Peano derivatives*, Proc. Amer. Math. Soc., **20** (1969), 487–490.

[18] C. E. Weil, *A property for certain derivatives*, Indiana Univ. Math. J., **23** (1973-74), 527–536.

[19] Z. Zahorski, *Sur la prémière dérivée*, Trans. Amer. Math. Soc., **69** (1950), 1–54.