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ON BOREL MEASURABLE FUNCTIONS THAT ARE VBG AND (N)

Abstract

The Banach-Zarecki Theorem states that $VB \cap (N) = AC$ for continuous functions on a closed set. Hence it is a linear space. In this article we show that $VB \cap (N)$ is a linear space on any real Borel set. Hence $VBG \cap (N)$ will also be a real linear space for Borel measurable functions defined on an interval. As a consequence of this result, we show that the AK_N integral of Gordon ([3]) is well defined. We also give answers to Gordon's questions in [3].

1 Preliminaries

We denote by $|X|$ the outer measure of the set X. Let $\mathcal{C}$ denote the class of continuous functions. We denote by $\mathcal{C}_{ap}$ the class of all approximately continuous functions on an interval, by $\mathcal{B}_1$ the Baire one functions, and by $\mathcal{DB}_1$ the Darboux Baire one functions. A function $F : E \to \mathbb{R}$ is said to satisfy Lusin's condition (N), if $|F(Z)| = 0$ whenever $Z \subset E$ with $|Z| = 0$. For the definitions of VB and AC see [7].

Definition 1. Let $E \subseteq [a, b]$. A function $F : E \to \mathbb{R}$ is said to be ACG (respectively VBG, $\mathcal{C}G$) on E if there exists a sequence of sets $\{E_n\}$ with $E = \cup_n E_n$, such that F is AC (respectively VB, $\mathcal{C}$) on each E_n. If in addition the sets E_n are supposed to be closed, we obtain the classes $[ACG]$, $[VBG]$, $[\mathcal{C}G]$. Note that ACG used here differs from that of [7] (because in our definition continuity is not assumed).

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Definition 2. ([7], p. 108). Let E be a real set, $x_o \in E$ an accumulation point of E, and let $F : E \to \mathbb{R}$. Let

$$\overline{F}_E(x_o) = \limsup_{x \to x_o, x \in E} \frac{F(x) - F(x_o)}{x - x_o} \text{ and } \underline{F}_E(x_o) = \liminf_{x \to x_o, x \in E} \frac{F(x) - F(x_o)}{x - x_o}.$$

They are called respectively the upper and lower derivatives of F at x_o relative to the set E. When they are equal (finite or infinite) their value is termed the derivative of F at x_o relative to the set E, and is denoted by $F'_E(x_o)$.

Definition 3. Let E be a real set, $x_o \in E$ a right accumulation point of E, and let $F : E \to \mathbb{R}$. Let

$$\overline{F}^+_E(x_o) = \limsup_{x \searrow x_o, x \in E} \frac{F(x) - F(x_o)}{x - x_o} \text{ and } \underline{F}^+_E(x_o) = \liminf_{x \searrow x_o, x \in E} \frac{F(x) - F(x_o)}{x - x_o}.$$

They are called respectively the upper and lower right Dini derivatives of F at x_o relative to the set E. When they are equal (finite or infinite) their value is termed the right Dini derivative of F at x_o relative to the set E, and is denoted by $F^+_E(x_o)$. Similarly we define $\overline{F}^-_E(x_o)$, $\underline{F}^-_E(x_o)$ and $F^-_E(x_o)$ whenever x_o is a left accumulation point of E. Clearly, if x_o is a bilateral accumulation point of E, then $F'_E(x_o)$ exists (finite or infinite) if and only if the four Dini derivatives of F relative to the set E agree at x_o.

Lemma 1. ([7], p. 223.) *Let E be a real set and let $F : E \to \mathbb{R}$. If the function F is VB, then $F'_E(x)$ exists and is finite for almost all $x \in E$. Moreover, $|F(Z)| = 0$, where $Z = \{x \in E : F'_E(x)$ (finite or infinite) does not exist$\}$.*

Definition 4. The point x_o is called a point of condensation of a real set E if every open interval (a, b) containing x_o contains an uncountable set of points of E ([6], p. 52). We can define right and left versions of this as follows: x_o is called a point of right (respectively left) condensation of E if for every $\delta > 0$ the set $(x_o, x_o + \delta) \cap E$ (respectively $(x_o - \delta, x_o) \cap E$) is uncountable. x_o is called a bilateral condensation point of E if it is simultaneously a right and a left condensation point of E.

Lemma 2. *Let C be a real set and $D \subset C$, D at most countable. A point $x \in \mathbb{R}$ is a right (resp. left; bilateral) condensation point of C, if and only if it is a right (resp. left; bilateral) condensation point of $C \setminus D$.*

PROOF. Let $x \in \mathbb{R}$ and let $\delta > 0$. We have $C \cap (x, x + \delta) = ((C \setminus D) \cap (x, x + \delta)) \cup (D \cap (x, x + \delta))$. Now the assertion follows immediately. $\square$

Lemma 3. *Every uncountable set $E \subseteq \mathbb{R}$ can be represented as $E = Q \cup D$, where D is at most countable and each point of Q is a bilateral condensation point of Q.*

PROOF. Let $P = \{x \in \mathbb{R} : x \text{ is a condensation point of } E\}$ and $P_1 = \{x \in E \cap P : x \text{ is isolated on the right or on the left for } E \cap P\}$. Then P_1 is at most countable (see [7] p. 260). The set $D_1 = E \setminus P$ is also at most countable (see Corollary 2 of [6], p. 53). Let $Q = (E \cap P) \setminus P_1$ and $D = P_1 \cup D_1$. Then $E = Q \cup D$ and the set D is at most countable. Let $x_o \in Q$ and let $\delta > 0$. Since x_o is not right isolated for $E \cap P$, it follows that $(x_o, x_o + \delta) \cap (E \cap P) \neq \emptyset$. Let $y_o \in (x_o, x_o + \delta) \cap (E \cap P)$. Then $(x_o, x_o + \delta) \cap E$ is uncountable. But $E = Q \cup D$; so $(x_o, x_o + \delta) \cap E = ((x_o, x_o + \delta) \cap Q)) \cup ((x_o, x_o + \delta) \cap D)$. Therefore $(x_o, x_o + \delta) \cap Q$ is uncountable and thus x_o is a right condensation point for Q. Similarly it follows that x_o is a left condensation point for Q. □

Lemma 4. *Let E be a real uncountable set, and let $F : E \to \mathbb{R}$, $F \in VB$. Then there exists $Q \subseteq E$ such that each point of Q is a bilateral condensation point of Q, $F_{|Q}$ is continuous on Q and $E \setminus Q$ is at most countable. Consequently, $F \in (N)$ on E if and only if $F \in (N)$ on Q.*

PROOF. By Lemma 4.1 of [7] (p. 221) there exists $G : \mathbb{R} \to \mathbb{R}$, $G \in VB$, such that $G(x) = F(x)$ for each $x \in E$. It follows that the set of discontinuity points of G is countable in each compact interval $[a, b]$. Since $\mathbb{R} = \cup_{n=1}^{\infty} [-n, n]$, it follows that $D_1 = \{x \in \mathbb{R} : G \text{ is discontinuous at } x\}$ is countable. Then $F_{|(E \setminus D_1)}$ is continuous on $E \setminus D_1$. By Lemma 3, $E \setminus D_1$ can be represented as $E \setminus D_1 = Q \cup D_2$, where D_2 is at most countable and each point of Q is a bilateral condensation point of Q. Let $D = D_1 \cup D_2$. Then $E = Q \cup D$, $F_{|Q}$ is continuous on Q and D is countable. □

Corollary 1. *Let E be a real uncountable set, and let $F_1, F_2 : E \to \mathbb{R}$, $F_1, F_2 \in VB$. Then there exists $Q \subseteq E$ such that each point of Q is a bilateral condensation point of Q, $(F_1)_{|Q}, (F_2)_{|Q}$ are continuous on Q and $E \setminus Q$ is at most countable.*

PROOF. For F_i there exists Q_i such that each point of Q_i is a bilateral condensation point of Q_i, $(F_i)_{|Q_i}$ is continuous on Q_i and $E \setminus Q_i$ is at most countable, $i = 1, 2$ (see Lemma 4). Let $Q = Q_1 \cap Q_2$. Then $(F_1)_{|Q}$ and $(F_2)_{|Q}$ are continuous on Q and $Q_1 \setminus Q \subseteq E \setminus Q \subseteq (E \setminus Q_1) \cup (E \setminus Q_2)$. Hence $E \setminus Q$ and $Q_1 \setminus Q$ are both at most countable. By Lemma 2 (taking $C = Q_1$ and $D = Q_1 \setminus Q$), it follows that each point of Q is a bilateral condensation point of Q. □

Theorem 1 (Souslin)**.** ([4], p. 396). *Let X and Y be two complete metric separable spaces, and let A be a Borel subset of X. If $F : A \to Y$ is a* *continuous one-to-one function, then $F(A)$ is a Borel set. (In fact F may be supposed to be only a Borel measurable function, see [4], p. 397).*

2 $VBG \cap (N)$ is a Real Linear Space for Borel Measurable Functions

Lemma 5. *Let E be a real set and let $F : E \to \mathbb{R}$. Let $A = \{x \in E : F'_E(x)$ exists and is finite$\}$. Then $F \in (N)$ on A.*

Proof. For $n = 1, 2, \dots$ let $A_n = \{x \in A : |F(t) - F(x)| \le n|t - x|$ whenever $t \in [x - 1/n, x + 1/n] \cap E\}$. Let $A_{n,i} = [i/n, (i+1)/n] \cap A_n$ for each integer i. Then $A = \cup_n A_n = \cup_n \cup_i A_{n,i}$. Let n and i be such that $A_{n,i}$ contains at least two points $x_1 < x_2$. Then $|F(t) - F(x_1)| \le n(t - x_1)$, for every $t \in [x_1, x_2] \cap E$. For $t = x_2$ we obtain that $|F(x_2) - F(x_1)| \le n(x_2 - x_1)$. It follows that F is a Lipschitz function on $A_{n,i}$. Hence $F \in (N)$ on A. □

Lemma 6. *Let E be a real set and let $F : E \to \mathbb{R}$, $F \in VB$. Let*

$$E^{+\infty} = \{x \in E : F'_E(x) = +\infty\} \text{ and } E^{-\infty} = \{x \in E : F'_E(x) = -\infty\}.$$

Then $F \in (N)$ on E if and only if $|F(E^{-\infty} \cup E^{+\infty})| = 0$.

Proof. Let $A = \{x \in E : F'_E(x)$ exists and is finite$\}$ and $Z = \{x \in E : F'_E(x)$ (finite or infinite) does not exist$\}$. Then $E = A \cup Z \cup E^{-\infty} \cup E^{+\infty}$. By Lemma 1, it follows that $F \in (N)$ on Z and $|E^{+\infty}| = |E^{-\infty}| = 0$. By Lemma 5, $F \in (N)$ on A. Therefore $F \in (N)$ on E if and only if $|F(E^{-\infty} \cup E^{+\infty})| = 0$. □

Lemma 7. *Let E be a Borel set such that each of its points is a bilateral accumulation point of E, and let $F : E \to \mathbb{R}$ be a Borel measurable function. Then $\overline{F}^+_E(x)$, $\underline{F}^+_E(x)$, $\overline{F}^-_E(x)$ and $\underline{F}^-_E(x)$ are Borel measurable functions. Therefore $E^{+\infty}$ and $E^{-\infty}$ are Borel measurable sets.*

Proof. For each pair (m, n) of positive integers with $n > m$, let

$$D_{n,m}(F : x) = \sup\left\{\frac{F(t) - F(x)}{t - x} : t \in \left(x + \frac{1}{n}, x + \frac{1}{m}\right) \cap E\right\}.$$

(Here we consider $\sup \emptyset = 0$.) Now the proof is as that of Theorem 4.3 of [7], p. 113. (Also see the proof of Theorem 2.1 of [1], pp. 54-55.) □

Theorem 2. *Let E be a real Borel set and $F_1, F_2 : E \to \mathbb{R}$. If $F_1, F_2 \in VB \cap (N)$, then $F_1 + F_2 \in VB \cap (N)$. Therefore $VB \cap (N)$ is a real linear space on E.*

PROOF. Clearly $F = F_1 + F_2 \in VB$. It remains to show that $F \in (N)$. If E is a countable set, then there is nothing to prove. Suppose that E is uncountable. By Lemma 4 and Corollary 1 it follows that we may suppose without loss of generality that each point of E is a bilateral condensation point of E (Therefore each point of E is a bilateral accumulation point of E.) and F_1, F_2 are continuous on E. Hence F is continuous on E. Suppose on the contrary that $F \notin (N)$ on E. By Lemma 6 it follows for example that $|F(E^{+\infty})| > 0$. For $n = 1, 2, \ldots$ let

$$E_n = \left\{ x \in E^{+\infty} : \frac{F(y) - F(x)}{y - x} \geq 1 \text{ whenever } y \in \left(x, x + \frac{1}{n}\right] \cap E \right\}$$

and for each integer i let $E_{n,i} = [i/n, (i+1)/n] \cap E_n$. Then $E^{+\infty} = \cup_n E_n = \cup_{n,i} E_{n,i}$. Consider n and i such that $|F(E_{n,i})| > 0$.

We show that F is strictly increasing on $E \cap \overline{E}_{n,i}$. If $x_1, x_2 \in E_{n,i}$ and $x_1 < x_2$, then

$$\frac{F(x_2) - F(x_1)}{x_2 - x_1} \geq 1. \tag{1}$$

Let $x_o, y_o \in E \cap \overline{E}_{n,i}$, $x_o < y_o$. Then there exists a sequence $\{x_k\}$ of points in $E_{n,i}$ converging to x_o, and a sequence $\{y_k\}$ of points in $E_{n,i}$ converging to y_o, such that $x_k < y_k$ for each k. By (1), $F(y_k) - F(x_k) \geq y_k - x_k$. Since F is continuous on E, it follows that $F(y_o) - F(x_o) \geq y_o - x_o$; so F is strictly increasing on $E \cap \overline{E}_{n,i}$.

Let $P = E^{+\infty} \cap \overline{E}_{n,i}$. Therefore F is strictly increasing on P. By Lemma 7, P is a Borel set and since F is continuous on P, it follows that $F(P)$ is also a Borel set (see Theorem 1); so $F(P)$ is a Lebesgue measurable set with positive measure. Then $F(P)$ contains a compact set Q of positive measure. Let $P_1 = P \cap F^{-1}(Q)$. Since F is strictly increasing on P_1, it follows that $F_{|P_1}$ has an inverse on P_1; namely $(F_{|P_1})^{-1} : Q \to P_1$, that is strictly increasing. But the set $Q_1 = \{x \in Q : (F_{|P_1})^{-1} \text{ is discontinuous at } x\}$ is countable. Let $G \supset Q_1$ be an open set such that $|G| < |Q|/2$. Then $Q_2 = Q \setminus G$ is a compact set of positive measure and $(F_{|P_1})^{-1}$ is continuous on Q_2. Hence $P_2 = (F_{|P_1})^{-1}(Q_2) \subseteq P_1 \subseteq P$ is a compact set (because any continuous function maps a compact set into a compact set). But $F_1, F_2 \in VB \cap (N) \cap \mathcal{C}$ on P_2; so by the Banach-Zarecki Theorem (see [7], p. 227), $F_1, F_2 \in AC$ on P_2. It follows that $F \in AC \subsetneqq (N)$ on P_2. But $F(P_2) = Q_2$, $|Q_2| > 0$ and $|P_2| = 0$, a contradiction. □

Corollary 2. *Let P be a Borel measurable subset of $\mathbb{R}$. Then the set $\mathcal{A} = \{F : P \to \mathbb{R} : F \in VBG \cap (N)$ and F is a Borel measurable function$\}$ is a real linear space.*

Proof. For $F_1, F_2 \in \mathcal{A}$, there exists a sequence $\{P_k\}_k$ of sets, such that $\cup_k P_k = P$ and $F_1, F_2 \in VB$ on each P_k. Let $G_{k,i} : \mathbb{R} \to \mathbb{R}$, $G_{k,i} = F_i$ on P_k and $G_{k,i} \in VB$ on $\mathbb{R}$, $i = 1, 2$. (This is possible - see for example [7], Lemma 4.1, p. 221.) Let $E_{k,i} = \{x \in P : F_i(x) = G_{k,i}(x)\}$. Since a VG function on $\mathbb{R}$ is Borel measurable and since F_1 and F_2 are Borel measurable functions too, it follows that each $E_{k,i}$ is a Borel set and contains P_k. Then $E_k = E_{k,1} \cap E_{k,2}$ is a Borel set containing P_k. By Theorem 2, $F = F_1 + F_2 \in VB \cap (N)$ on each E_k. Therefore on each P_k. It follows that $F \in VBG \cap (N)$ on P. □

Corollary 3 (Sarkhel and Kar, [11]). *$[VBG] \cap (N)$ is a real linear space on a real compact set.*

Proof. Let Q be a real compact set and $F_1, F_2 : Q \to \mathbb{R}$, $F_1, F_2 \in [VBG]$ on Q. Then there exists a sequence $\{Q_n\}$ of closed sets such that $Q = \cup_n Q_n$ and $F_1, F_2 \in VB \cap (N)$ on each Q_n. By Theorem 2, $F_1 + F_2 \in VB \cap (N)$ on each Q_n. Hence $F_1 + F_2 \in [VBG] \cap (N)$ on Q. □

This result was first obtained by Sarkhel and Kar (see Corollary 3.1.1 and Theorem 3.6 of [11]).

3 Gordon's AK_N Integral is Well Defined

Definition 5 (Gordon, [3]). A function $f : [a, b] \to \overline{\mathbb{R}}$ is said to be AK_N integrable on $[a, b]$ if there exists a function $F : [a, b] \to \mathbb{R}$ such that

(1) $F \in \mathcal{C}_{ap}$ on $[a, b]$,

(2) $F \in VBG \cap (N)$ on $[a, b]$,

(3) $F'_{ap}(x) = f(x)$ *a.e.* on $[a, b]$.

The number $F(b) - F(a)$ is called the definite AK_N integral of f, and F is called an indefinite AK_N integral of f on $[a, b]$.

Lemma 8. *The AK_N integral is well defined.*

Proof. Let $f : [a, b] \to \overline{\mathbb{R}}$ be AK_N integrable, and let F_1 and F_2 be two AK_N primitives of f. By Corollary 2, $VBG \cap (N) \cap \mathcal{C}_{ap}$ is a real linear space. Since $\mathcal{C}_{ap} \subset \mathcal{DB}_1$ (see [1], p. 21), it follows that $F_1 - F_2 \in \mathcal{DB}_1 \cap (N)$ and $(F_1 - F_2)'_{ap}(x) = 0$ *a.e.* on $[a, b]$. We have the following result of C. M. Lee (see [5] or [2], p. 146).

> If a function $F : [a, b] \to \mathbb{R}$ is $\mathcal{DB}_1 \cap (N)$ on $[a, b]$ and $F'(x) \geq 0$ *a.e.* where F is derivable, then F is increasing and AC on $[a, b]$.

By this result we obtain that $F_1 - F_2$ is a constant on $[a,b]$. Therefore the AK_N integral is well defined. $\square$

Remark 1. Gordon's proof about the AK_N integral being well defined is not complete. He seems to assume (see [3]) that $VBG \cap (N) \cap \mathcal{C}_{ap}$ is a real linear space, but the proof of this fact is not easy (see our Corollary 2).

4 Answers to Gordon's Questions of [3]

In [3] Gordon posed the following questions.

1. Is every $VBG \cap (N) \cap \mathcal{C}_{ap}$ function a $[\mathcal{C}G]$ function ?

2. Is every indefinite AP integral a $[\mathcal{C}G]$ function ?

The answer to question 1 is negative. This follows because

$$VBG \cap (N) \cap [\mathcal{C}G] \cap \mathcal{C}_{ap} \subsetneqq [VBG] \cap (N) \cap \mathcal{C}_{ap} \subseteq VBG \cap (N) \cap \mathcal{C}_{ap}\,. \tag{2}$$

Indeed, by the Banach-Zarecki Theorem ([7], p. 227) we have that $VBG \cap [\mathcal{C}G] \cap (N) = [ACG]$. Hence $VBG \cap [\mathcal{C}G] \cap (N) \cap \mathcal{C}_{ap} = [ACG] \cap \mathcal{C}_{ap}$. In [11] Sarkhel and Kar constructed a function $F : [a,b] \to \mathbb{R}$ such that

$$F \in \mathcal{C}_{ap} \cap (N) \cap [VBG],\ \text{but } F \notin ACG \text{ on } [a,b]\,. \tag{3}$$

It follows that $VBG \cap [\mathcal{C}G] \cap (N) \cap \mathcal{C}_{ap} \subsetneqq [VBG] \cap (N) \cap \mathcal{C}_{ap}$ (because if $F_{|P}$ is VB, and $F_{|\overline{P}}$ is continuous, then F is VB on $\overline{P}$, see for example [2], p. 42).

The answer to question 2 is affirmative. Let $F : [a,b] \to R$ be an AP-primitive. Then F is also a primitive for the β-Ridder integral (see Remark 5.17.3 of [2]). By the definition of the β-Ridder integral (see for example Remark 5.17.1 (ii) of [2]) it follows that $F \in \mathcal{C}_{ap} \cap [ACG]$. Therefore $F \in [\mathcal{C}G]$.

5 Questions

Starting from relations (2) and (3), the following questions arise.

1) $[VBG] \cap (N) \cap \mathcal{C}_{ap} \subsetneqq VBG \cap (N) \cap \mathcal{C}_{ap}$ on $[a,b]$?

2) Is there a function $F : [a,b] \to \mathbb{R}$ such that $F \in ACG \cap \mathcal{C}_{ap}$ and $F \notin [VBG] \cap (N) \cap \mathcal{C}_{ap}$?

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