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A COUNTEREXAMPLE OF AN EXTREMELY CHAOTIC FUNCTION

Abstract

We exhibit an example of continuous function f on the interval, which is extremely chaotic but not transitive, disproving a conjecture by Bruckner and Hu.

Bruckner and Hu stated the following result: *Under the Continuum Hypothesis, a continuous function f defined on $I_0 = [0,1]$ is chaotic if and only if f^2 is nomadic* [1].

Here "nomadic" means transitive and "chaos" means extremal chaos in the following sense. There is an uncountable set S such that, for every two different points $x, y \in S$,

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| = 1, \tag{1}$$

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0. \tag{2}$$

In this note, we show that the assertion above is not true. Recall, that a function f on I_0 is called d-chaotic, if there exists an uncountable set S such that

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| = d \tag{3}$$

and (2) are satisfied for any two different points x, y in S. A basic set for f is any maximal infinite ω-limit set $\tilde\omega$ of f containing a periodic point. Such a set is indecomposible if, for every $n \in \mathbb{N}, \tilde\omega$ is a basic set of f^n.

Lemma 1. *A basic set $\tilde\omega$ is indecomposible if and only if for any interval J such that $J \cap \tilde\omega$ is infinite there exists $W = lim_{n\to\infty} f^n(J)$ with $\overline{W} \supset \tilde\omega$.*

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PROOF. Cf. Theorem 3.7 (vi) in [2]. □

Lemma 2. *Let $\tilde{\omega}$ be a basic set of f. Then there exist an $n \in \mathbb{N}$, and a system of nonoverlapping periodic portions $\omega_0, \dots, \omega_n$ of $\tilde{\omega}$, which form a single orbit such that $\tilde{\omega} = \omega_0 \cup \ldots \cup \omega_n$ and, for every i, the set ω_i is an indecomposible basic set of $g = f^n$.*

PROOF. Cf. Theorem 3.7 (vi) in [2]. □

The following theorem is our main result.

Theorem 1. *Let $f : I_0 \to I_0$ be a continuous function, $\tilde{\omega}$ its basic set, possessing an indecomposible portion of diameter d. Then f is d-chaotic.*

PROOF. Let ω be an indecomposible periodic portion of $\tilde{\omega}$ with period p such that $\operatorname{diam} \omega = d$. Then $g = f^p : \omega \to \omega$. It suffices to prove that g is d-chaotic. Take two systems of compacted intervals A_i, B_i such that they are contained in the interior of the convex hull of ω, for every $i \in \mathbb{N}$, and such that

$$\lim_{n\to\infty} \operatorname{diam} A_n = \lim_{n\to\infty} \operatorname{diam} B_n = 0, \tag{4}$$

$$\lim_{n\to\infty} \operatorname{diam} (A_n \cup B_n) = d. \tag{5}$$

By Lemma 1, for every $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that $g^{kn}(A_n) \cap g^{kn}(B_n) \supset A_{n+1} \cup B_{n+1}$. Denote $C = \bigcap_{n\in\mathbb{N}} g^{-kn}(A_n \cup B_n)$. To any $x \in C$ assign its code $\alpha(x) = x_0x_1x_2 \ldots \in \{0,1\}^{\mathbb{N}}$ such that $x_n = 0$ if $g^{kn}(x) \in A_n$, and $x_n = 1$ if $g^{kn}(x) \in B_n$.

Let $\mathcal{F}$ be the family of all subsets $D \subset C$ such that if x, y are different points in D with codes $\alpha(x) = \{x_i\}$ and $\alpha(y) = \{y_i\}$, then both sets $\{i; x_i \neq y_i\}$ and $\{i; x_i = y_i\}$ are infinite. Now if $S \in \mathcal{F}$ is uncountable then g, and consequently f, is d-chaotic. Indeed, since, for infinitely many n, both $g^{kn}(x)$ and $g^{kn}(y)$ belong to exactly one of the sets A_n, B_n, (4) implies (2). Similarly, (5) implies (3).

It remains to show that $\mathcal{F}$ contains an uncountable set S. Since any one-point set belongs to $\mathcal{F}$ we have $\mathcal{F} \neq \emptyset$. If $\{P_t\}_{t\in T}$ is a chain of sets in $\mathcal{F}$, ordered by inclusions, i.e., $P_t \subset P_s$ if $t < s$, then it has an upper bound $P = \bigcup_{t\in T} P_t \in \mathcal{F}$. So by the Zorn's lemma, $\mathcal{F}$ has a maximal element M. If M is countable, denote by A the set of $y \in C$ with $\alpha(y) = \{y_i\}$ such that there is a point $x \in M$ with $\alpha(x) = \{x_i\}$ for which $\{i; y_i = x_i\}$ is finite, and by B the set of $y \in C$ with $\alpha(y) = \{y_i\}$ such that there is a point $x \in M$ with $\alpha(x) = \{x_i\}$ for which $\{i; y_i \neq x_i\}$ is finite. Both these sets are evidently countable. Take $z \in C \setminus (A \cup B \cup M)$. Then $M \cup \{z\} \in \mathcal{F}$, contrary to the maximality of M. □

Corollary 1. *Function defined on* $[0,1]$ *by*

$$g(x) = \begin{cases} 3x & x \in [0, \frac{1}{3}] \\ 1 & x \in (\frac{1}{3}, \frac{2}{3}) \\ 3(1-x) & x \in [\frac{2}{3}, 1] \end{cases}$$

is extremely chaotic.

To see this note that the Cantor middle-third set is a basic set for g. On the other hand, g is not transitive. This shows that the above mentioned result (Corollary 4.9 in [1]) is not true.

References

[1] A.M. Bruckner and T. Hu, *On scrambled sets for chaotic functions*, Trans. Amer. Math. Soc., **301** (1987) 289–297.

[2] B. Schweizer and J. Smítal, *Measures of chaos and a spectral decomposition of dynamical systems on the interval*, Trans. Amer. Math Soc., **344** (1994) 737–754.