RESTRICTIONS TO CONTINUOUS AND POINTWISE DISCONTINUOUS FUNCTIONS

Abstract

We compare some of the restriction properties that can be found throughout the literature. For example, theorem 10 is a common generalization of three theorems: Blumberg’s theorem [2], Baldwin’s strengthening of Blumberg’s theorem [1], and a related Brown-Prikry’s result [8] on Marczewski’s (s)-measurable functions.

1 Introduction

In 1922 Blumberg [2] proved that for every function \( f : \mathbb{R} \to \mathbb{R} \) there exists a dense set \( X \subseteq \mathbb{R} \), such that \( f|_X \) is continuous. Since then many similar results involving domains and codomains other than \( \mathbb{R} \) were obtained. Also many papers can be found, where “continuous” was changed to “differentiable” or “pointwise discontinuous” (i.e., \( f : X \to \mathbb{R} \) is pointwise discontinuous (abbreviated PWD) if \( \{ x \in X : f \text{ is continuous at } x \} \) is dense in \( X \), see [10] p.105). For a recent comprehensive account of these results see [6]. In this note we would like to compare some restriction properties of real functions defined on separable metric spaces. 

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set $W \subseteq X$ such that the restricted function $f|_{W}$ is continuous or pointwise discontinuous. The following six different notions of largeness associated with an ideal $\mathcal{J}$ can be found in restriction theorems stated in [6], [5], [1], [8], [14], and other papers. $W$ is a subset of $X$.

\[ W \text{ is non-} \mathcal{J} \text{ dense in } X \ (\text{D}) \]

\[ W \text{ is non-} \mathcal{J} \text{ dense in } W \ (\text{DI}) \quad \text{cl}_X(W) \text{ is non-} \mathcal{J} \text{ dense in } X \ (\text{WD}) \]

\[ \Downarrow \]

\[ W \notin \mathcal{J} \ (\text{N}) \quad \text{cl}_X(W) \text{ is non-} \mathcal{J} \text{ dense in } \text{cl}_X(W) \ (\text{WDI}) \]

\[ \Downarrow \]

\[ \text{cl}_X(W) \notin \mathcal{J} \ (\text{WN}) \]

$W$ is non-$\mathcal{J}$ dense in $X$ if $W \cap U \notin \mathcal{J}$ for every nonempty open subset $U \subseteq X$. $\text{cl}_X(W)$ stands for the closure of $W$ in $X$. We shall refer to these properties using the bold abbreviations in parenthesis. Here is the key: $\text{D}=$non-$\mathcal{J}$-Dense, $\text{DI}=$non-$\mathcal{J}$-Dense in $\mathcal{I}$ itself, $\text{N}=$Not in $\mathcal{J}$, $\text{WN}=$Weakly Not in $\mathcal{J}$ (i.e., not in $\mathcal{J}$ after taking the closure of $W$), etc. In general all six are different classes of sets and the above diagram indicates all inclusions.

If $\mathcal{L}$ is one of those properties (i.e. $\text{D}$, $\text{DI}$, ..., $\text{WN}$), we define a Continuous Restriction Property (C-RP) or a PointWise Discontinuous Restriction Property (PWD-RP) related to $\mathcal{L}$. Namely, a function $f : X \to \mathbb{R}$ has a $\mathcal{L}$-C-RP [resp. $\mathcal{L}$-PWD-RP] if there exists a set $W \in \mathcal{L}$ such that $f|_{W}$ is continuous [resp. PWD]. We shall say that a pair $(\mathcal{A}, \mathcal{J})$ has a $\mathcal{L}$-C-RP [resp. $\mathcal{L}$-PWD-RP] if every $\mathcal{A}$-measurable function $f : X \to \mathbb{R}$ has the same property. $(\mathcal{A}, \mathcal{J})$ has $\mathcal{A}$-$\mathcal{L}$-C-RP [resp. $\mathcal{A}$-$\mathcal{L}$-PWD-RP] if the witness set $W$ can be found in $\mathcal{A}$.

Let $\mathcal{B}(X)$ be the family of all Borel subsets of $X$ and let $\mathcal{BR}(X)$ be the family of all sets with Baire property while $\mathcal{M}(X)$ is the ideal of all subsets of $X$ meager in $X$. So for subsets $X_1 \subseteq X$, $\mathcal{M}(X_1)$ is the family of all relatively meager subsets of $X_1$. We have $\mathcal{M}(X_1) \subseteq \mathcal{M}(X)|_{X_1}$. For $X \subseteq \mathbb{R}$ let $\mathcal{L}(X)$ and $\mathcal{N}(X)$ be the Lebesgue measurable and null subsets of $X$. Classic theorems imply that $(\mathcal{BR}(\mathbb{R}), \mathcal{M}(\mathbb{R}))$ has $\mathcal{BR}(\mathbb{R})$-$\text{D}$-C-RP, while the $(\mathcal{L}(\mathbb{R}), \mathcal{N}(\mathbb{R}))$ only has $\mathcal{L}(\mathbb{R})$-$\text{DI}$-C-RP. (See [8] for more details.)
2 Continuous Restrictions

For an arbitrary pair \((A, J)\) on a separable metric space \(X\) we have the following implications.

\[
\begin{array}{ccc}
\text{D-C-RP} & \downarrow \\
\text{DI-C-RP} & \downarrow & \text{WD-C-RP} \\
\uparrow \ast & \downarrow & \downarrow \\
\text{N-C-RP} & \downarrow & \text{WDI-C-RP} \\
\downarrow & \downarrow & \downarrow \\
\text{WN-C-RP}
\end{array}
\]

Examples of pairs \((A, J)\) indicating that, except for \(\ast\), none of these implications may be reversed, can be easily found.

2.1 \(A = \mathcal{P}(X)\)

In 1923 W. Sierpinski and A. Zygmund \cite{17} proved that whenever \(|X| = \aleph\), then there exists a function \(z : X \to \mathbb{R}\) such that \(z|_Y\) is not continuous for any \(Y \in [X]^\aleph\). This implies that under CH \((\mathcal{P}(X), J)\) can not have N-C-RP for any \(\sigma\)-ideal \(J\) containing all singletons. Without CH however \((\mathcal{P}(\mathbb{R}), [\mathbb{R}]^{\leq \omega})\) as well as \((\mathcal{P}(\mathbb{R}), \mathcal{M}(X))\) may have D-C-RP. (See \cite{1}, \cite{15}, and Theorem 2 below.) In ZFC Bradford and Goffman \cite{3} proved that whenever an ideal \(J\) does not contain open sets, then \((\mathcal{P}(X), J)\) has WD-C-RP iff \(X\) has no meager open subsets. In general we have the following theorem.

**Theorem 1.** \((\mathcal{P}(X), J)\) has WDI-C-RP.

**Proof.** Let \(f : X \to \mathbb{R}\) and suppose that \((\mathcal{P}(X), J)\) does not have the WDI-C-RP. By Brown’s theorem 2, \cite{5} p.132, we may assume that there exists a subset \(X_1 \subseteq X, X_1 \notin J\) such that \(\mathcal{M}(X_1) \subseteq J|_{X_1}\). Take \(X'_1 = X_1 \setminus \bigcup\{V \subseteq X_1 : V \text{ is open in } X_1 \text{ and } V \in J\}\). We have \(\mathcal{M}(X'_1) \subseteq \mathcal{M}(X_1) \subseteq J\) and the last does not contain open subsets of \(X'_1\). Hence we may apply the above mentioned Bradford-Goffman theorem, \cite{3} p. 667, to \(X'_1\) and obtain a dense subset \(W \subseteq X'_1\), such that \(f|_W\) is continuous. Clearly \(\text{cl}_X(W) \supseteq X'_1\) and whenever \(U\) is open in \(X, U \cap \text{cl}_X(W) \neq \emptyset\), then \(U \cap X'_1 \notin J\) by the definition of \(X'_1\).

It is known (see \cite{5}, p. 128) that for uncountable separable metric spaces \(X\) and any \(f : X \to \mathbb{R}\) there exists a set \(W \subseteq X\) such that \(f|_W\) is continuous and
\[ |W \cap U| \geq \omega \] for every nonempty open set \( U \). Observe that by taking \( J = [X]^{\leq \omega} \) in Theorem 1 above we obtain proposition (C) of [5] and additional property that \( \text{cl}_X(W) \) is uncountably dense in itself.

If \( J_1 \) and \( J_2 \) are ideals on a set \( X \) and \( Y \subseteq X \), then we say that \( J_1 \) is orthogonal to \( J_2 \) on \( Y \) if \( Y = Y_1 \cup Y_2 \) where \( Y_i \in J_i \), \( i = 1, 2 \). We write “\( J_1 \perp J_2 \) on \( Y \)”. Let us consider the following property of a space \( X \) and an ideal \( J \):

\[
X = X_1 \cup X_2 \text{ where } X_1 \in \mathcal{M}(X) \text{ and } \mathcal{M}(X_2) \subseteq J. \tag{1}
\]

It follows from Theorem 1 of [5] that if open subsets of \( X \) do not have property (1), then \( (\mathcal{P}(X), J) \) has \( \text{D-PWD-RP} \). In this context the following theorem is somewhat surprising.

**Theorem 2.** Suppose that \( X \) and \( J \) satisfy (1) and that \( J \perp \mathcal{M}(X) \) on any open set. Let \( f : X \to \mathbb{R} \) be such that for every Borel set \( B \in \mathcal{B}(X) \setminus J \) the restricted function \( f|_B \) has \( \text{N-C-RP} \) with respect to \( J|_B \). Then \( f \) has \( \text{D-C-RP} \) with respect to \( J \).

**Proof.** Let \( X = X_1 \cup X_2 \) be a partition as in (1). By enlarging \( X_1 \) to a Borel meager set we may assume that \( X_1, X_2 \in \mathcal{B}(X) \). Let \( \mathcal{U} = (U_n)_{n<\omega} \) be an open basis for \( X_2 \). Non-orthogonality of \( J \) and \( \mathcal{M}(X) \) on open sets implies that \( U_n \notin J \). Since \( U_n \) is Borel in \( X \), by the \( \text{N-C-RP} \) of \( f|_{U_n} \) we obtain sets \( A_n \subseteq U_n \), \( A_n \notin J \) such that \( f|_{A_n} \) is continuous. Let \( T_n = \{ x \in U_n : \exists E_{\text{open}} \subset U_n (x \in E \text{ and } A_n \cap E \in J) \} \). \( X \) is separable so \( T_n \cap A_n \in J \) and \( \text{cl}_{U_n}(T_n) \setminus T_n \in \mathcal{M}(X_2) \subseteq J \). Furthermore, since \( X_1 \) is meager, \( J \) and \( \mathcal{M}(X) \) are non-orthogonal on \( U_n \). Take \( V_n = U_n \setminus \text{cl}_{U_n}(T_n) \) and observe that \( C_n = A_n \cap V_n \) is nonempty and non-\( J \) dense in \( V_n \) for all \( n < \omega \).

Now let \( W_n = C_n \setminus \bigcup_{k<n} \text{cl}_X(V_k) \) and \( W = \bigcup_{n<\omega} W_n \). Notice that \( W_n = (V_n \cap W) \setminus \bigcup_{k<n} \text{cl}_X(V_k) \). Hence \( W_n \) are open in \( W \). \( f|_{W_n} \) is continuous for all \( n < \omega \) which implies that \( f|_W \) is also continuous.

To see that \( W \) is not-\( J \) dense in \( X \) take an arbitrary nonempty open set \( T \subseteq X \). Since \( X_2 \) is residual in \( X \), \( T_2 = X_2 \cap T \) contains some \( U_k \). Let \( k_0 = \min \{ k : V_k \cap T_2 \neq \emptyset \} \). We clearly have \( C_{k_0} \cap T_2 \notin J \) but also \( W_{k_0} \cap T_2 \notin J \) as all sets of the form \( \text{cl}_X(V_k) \setminus V_k \) are nowhere dense in \( X_2 \) and are in \( J \) by (1). Naturally \( W \cap T \notin J \).

For separable spaces Shelah’s theorem 1.4, [15], p. 8, gives the following:

**Theorem 3.** (Shelah [15]) It is relatively consistent with ZFC that for every function \( f : 2^\omega \to 2^\omega \) there exists a non-meager subset of \( A \subseteq 2^\omega \) such that \( f|_A \) is continuous.

Suppose that \( X \) is a complete space. Shelah’s theorem 3 implies that whenever \( B \notin \mathcal{M}(X) \) is a Borel subset of \( X \), then there exists a set \( A \in \mathcal{M}(X) \) such that \( f|_{X \setminus B} \) is continuous.
$\mathcal{P}(B) \setminus \mathcal{M}(B)$ such that $f|_A$ is continuous. Theorems 3 and 2 yield the following fact.

**Corollary 4.** It is consistent that for any complete space (or a Borel subset of a complete space without open meager sets) $X$ the pair $(\mathcal{P}(X), \mathcal{M}(X))$ has D-C-RP.

**Remark 1.** It is worth noting that ideals which are ccc in Borel sets have property (1). Suppose that $\mathcal{J}$ is any ccc in Borel sets ideal on $X$ (i.e. $\mathcal{B}(X) \setminus \mathcal{J}$ does not contain uncountable pairwise disjoint subfamilies) and suppose that $\mathcal{M}(X) \not\subseteq \mathcal{J}$. Let $X^0_\alpha \in \mathcal{M}(X) \setminus \mathcal{J}$ be Borel. For an ordinal $\alpha$ try to find a set $X^\alpha \subseteq \mathcal{B}(X \setminus \bigcup_{\beta<\alpha} X^\beta) \cap (\mathcal{M}(X) \setminus \mathcal{J})$. By the ccc property this attempt must fail after $\alpha_0 < \omega_1$ steps. Sets $X_1 = \bigcup_{\alpha<\alpha_0} X^\alpha$ and $X_2 = X \setminus X_1$ have the desired properties.

Corollary 4 shows that CH cannot be eliminated from Theorem 1 of [5]. Namely in Shelah’s model $(\mathcal{P}(\mathbb{R}), \mathcal{M}(\mathbb{R}))$ has D-C-RP (in particular it has D-PWD-RP) and $\mathbb{R}$ does not satisfy condition (B') of [5] with property $\mathbb{P} = \mathcal{M}(\mathbb{R})$.

### 2.2 $\mathcal{A}$-Measurable Functions

Now we would like to prove a theorem similar to 2 without assuming (1). To compensate for that we are going to work with $\mathcal{A}$-measurable functions and assume $\mathcal{A}$-N-C-RP of $f|_A$ for all $A \in \mathcal{A} \setminus \mathcal{J}$ i.e., assume that there exists a set $B \in \mathcal{A} \setminus \mathcal{J}$ such that $f|_B$ is continuous. Following Bradford and Goffman [3] (see also [13]) we introduce the following definitions: Let $E \subseteq X$, and let $x \in X$. Then $x$ is non-$\mathcal{J}$ relative to $E$ if for every open $V \ni x$ we have $E \cap V \notin \mathcal{J}$. $x$ is $\mathcal{J}$-heavy relative to $E$ if there exists an open set $U \ni x$ such that all $y \in U$ are non-$\mathcal{J}$ relative to $E$. The first two lemmas are straightforward generalizations of Lemmas 2 and 3 of [3].

**Lemma 5.** Any subset $E \subseteq X$ can be written as a disjoint union of sets $E = A \cup B_1 \cup B_2$ such that all members of $A$ are $\mathcal{J}$-heavy relative to $E$, $B_1 \in \mathcal{J}$, and $B_2$ is nowhere dense in $X$.

**Proof.** Let us define $B_1 = \{ x \in E : \exists U_{open} \subseteq X \ x \in U \ \& \ (U \cap E) \in \mathcal{J} \}$. $X$ is separable. Hence $B_1 \in \mathcal{J}$. Now let $B_2 = \{ x \in E : x$ is non-$\mathcal{J}$ but not $\mathcal{J}$-heavy relative to $E \}$. Take an arbitrary open set $T \subseteq X$ and let $x \in B_2 \cap T$. Since $x$ is not $\mathcal{J}$-heavy, there exists $y \in T$ which is not non-$\mathcal{J}$ relative to $E$. So there exists an open neighborhood $V$ of $y$ such that $E \cap V \notin \mathcal{J}$. We must have $V \cap B_2 = \emptyset$, which shows that $B_2$ is nowhere dense. Clearly points of $E$ that are not in $B_1$ nor $B_2$ are $\mathcal{J}$-heavy. \[\Box\]
For $f : X \to \mathbb{R}$ we define

$$H_f(X, \mathcal{J}) = \{x \in X : \forall K_{\text{open}} \ni f(x) (x \text{ is } \mathcal{J}-\text{heavy relative to } f^{-1}(K))\} \quad (2)$$

Properties of $H_f(X, \mathcal{J})$ were studied by Piotrowski [13] in a more general context.

**Lemma 6.** Let $f : X \to \mathbb{R}$. There exist sets $B_1 \in \mathcal{J}$ and $B_2 \in \mathcal{M}(X)$ such that $H_f(X, \mathcal{J}) = X \setminus (B_1 \cup B_2)$.

**Proof.** Let $(G_n)_{n<\omega}$ be an open basis in $\mathbb{R}$ and let $S_n = f^{-1}(G_n) = A^n \cup B^n_1 \cup B^n_2$ where the last union is like in lemma 5. Take $B_1 = \bigcup_{n<\omega} B^n_1$ and $B_2 = \bigcup_{n<\omega} B^n_2$. Now select an arbitrary $x \in X \setminus (B_1 \cup B_2)$ and an open set $K \ni f(x)$. Find $n < \omega$ such that $G_n \subseteq K$ and $f(x) \in G_n$. $x$ is $\mathcal{J}$-heavy relative to $S_n$ so in particular it is $\mathcal{J}$-heavy relative to the bigger set $f^{-1}(K)$. For the other inclusion take $K = G_n$, $n < \omega$ and it follows immediately. \hfill \square

For any ideal $\mathcal{J}$ on a metric space $X$ we define $\mathcal{J}^*$ to be the $\sigma$-ideal generated by $\mathcal{J}$ and $\mathcal{M}(X)$. The next lemma is easy to verify.

**Lemma 7.** Let $Z$ be a separable metric space and let $\mathcal{J}$ be an ideal on $Z$ with $\mathcal{J} \not\subseteq \mathcal{M}(Z)$ on any open set. If $U \subseteq Z$ is open and $V \subseteq U$ is non-$\mathcal{J}^*$ dense in $U$, then $\mathcal{J} \not\subseteq \mathcal{M}(V)$ on any open subset of $V$.

**Lemma 8.** Let $Z$ be a zero-dimensional separable metric space. Assume that $\mathcal{J}$ is a $\sigma$-ideal and $A \supseteq \mathcal{J} \cup \mathcal{B}(Z)$ is a $\sigma$-algebra on $Z$. Suppose that an $A$-measurable function $f : Z \to \mathbb{R}$ and $Y \in A$ are such that

1) $\mathcal{J} \not\subseteq \mathcal{M}(Z)$ on any open subset of $Z$

2) $Y$ is non-$\mathcal{J}$ dense in itself

3) $Y \subseteq H_f(Z, \mathcal{J}^*)$

4) $f|_Y$ is continuous

5) $\forall A \in A \setminus \mathcal{J} \exists B \in A|_A \setminus \mathcal{J} f|_B$ is continuous.

If $\varepsilon > 0$, then there exist pairwise disjoint open subsets $U = (U_n)_{n<\omega}$ of $Z$ and subsets $Y_n \subseteq V_n \subseteq U_n$, $Y_n \in A$ such that

6) $\text{diam}(U_n) < \varepsilon$

7) $\bigcup U$ is dense in $Z$

8) $V_n$ are non-$\mathcal{J}^*$ dense in $U_n$
9) $Y_n$ are non-$\mathcal{J}$ dense in itself

10) $f|_{Y_n}$ is continuous

11) $Y \subseteq \bigcup_{n<\omega} Y_n$

12) $Y_n \subseteq H_f(V_n, \mathcal{J}^*)$

13) $|f(x) - f(x')| < \varepsilon$ whenever $x, x' \in V_n$ for some $n < \omega$.

**Proof.** We shall first define the even numbered sets $U_n, V_n, and Y_n$ to satisfy condition 11) and then define the odd numbered ones to satisfy 7). Select $y \in Y$. By 3) there exists an clopen neighborhood $U'_y$ of $y$ with diam($U'_y$) < $\varepsilon$ such that

$$f^{-1}\left((f(y) - \frac{\varepsilon}{2}, f(y) + \frac{\varepsilon}{2})\right) \text{ is non-$\mathcal{J}^*$ dense in } U'_y. \hspace{1cm} (3)$$

4) implies the existence of a clopen set $U''_y \ni y$ such that $|f(x) - f(x')| < \frac{\varepsilon}{2}$ whenever $x, x' \in Y \cap U''_y$. Let $U_y = U'_y \cap U''_y$ and observe that

$$Y \cap U_y \subseteq f^{-1}((f(y) - \frac{\varepsilon}{2}, f(y) + \frac{\varepsilon}{2}) \hspace{1cm} (4)$$

Then $(U_y)_{y \in Y}$ is a clopen cover of $Y$. There is a countable set $\{y_n : n < \omega\} \subseteq Y$ such that $(U_{y_n})_{n<\omega}$ is a subcover of $Y$. Set $G_n = U_{y_n} \setminus \bigcup_{k<n} U_k$. Then $(G_n)_{n<\omega}$ is a disjoint open cover of $Y$ and by possibly deleting some sets we may assure that $G_n \cap Y \neq \emptyset$ for all $n < \omega$. For each $n < \omega$ we put $U_{2n} = G_n$, $V_{2n} = U_{2n} \cap f^{-1}((f(y_{2n}) - \frac{\varepsilon}{2}, f(y_{2n}) + \frac{\varepsilon}{2}))$, and $Y_{2n} = U_{2n} \cap Y$.

Assumption 2) implies that $Y_{2n}$ is non-$\mathcal{J}$ dense in itself. 4) gives continuity of $f|_{Y_{2n}}$. Inclusion (4) shows that

$$Y_{2n} \subseteq V_{2n} \hspace{1cm} (5)$$

and condition (3) implies that $V_{2n}$ is non-$\mathcal{J}^*$ dense in $U_{2n}$. Since $(G_n)_{n<\omega}$ was a cover of $Y$, the union $\bigcup_{n<\omega} Y_{2n} = Y$. 13) follows from the definition of $V_{2n}$. Hence we have verified all conditions except 7) and 12). While 7) will be taken care of by the odd $U_{n-s}$, 12) for even indices follows from the following claim.

**Claim:** $Y_{2n} \subseteq H_f(V_{2n}, \mathcal{J}^*)$ for all $n < \omega$.

Let $x \in Y_{2n}$ and let $K \ni f(x)$ be an open subset of $\mathbb{R}$. Take $K_1 = K \cap (f(y_{2n}) - \frac{\varepsilon}{2}, f(y_{2n}) + \frac{\varepsilon}{2})$. By (5) $f(x) \in (f(y_{2n}) - \frac{\varepsilon}{2}, f(y_{2n}) + \frac{\varepsilon}{2})$; so $f(x) \in K_1$. By the assumption 3) there exists an open subset $U \subseteq Z$ such that $f^{-1}(K_1) \cap U$ is non-$\mathcal{J}^*$ dense in $U$. It follows that $f^{-1}(K_1) \cap U \cap G_n$ is non-$\mathcal{J}^*$ dense in
\( \hat{U} = U \cap G_n \). But since \( f^{-1}(K_1) \cap G_n \) is a subset of \( V_{2n} \), we obtain that \( f^{-1}(K) \cap V_{2n} \cap \hat{U} \) is non-\( J^* \) dense in \( \hat{U} \). Thus \( x \in H_f(V_{2n}, J^*) \).

To define the odd \( U_n \supseteq V_n \supseteq Y_n \) we proceed as in lemma 4 of [3]. Let \( R = \{ z_\alpha : \alpha < \kappa \} \) for some \( \kappa \leq \epsilon \) be a well-ordering of \( H_f(Z \setminus \cl_z(\bigcup_{\alpha < \omega} U_{2n}), J^*) \).

Orthogonality of \( J \) and \( M(Z) \) on every open set and lemma 6 imply that \( R \) is dense in \( Z \setminus \cl_z(\bigcup_{\alpha < \omega} U_{2n}) \). Suppose that we have defined sets \( U'_\alpha \supseteq V'_\alpha \supseteq Y'_\alpha \) for all \( \alpha < \beta < \omega_1 \). Let \( z_\beta_0 \) be the first element of \( R \cap Z \setminus \cl_z(\bigcup_{\alpha < \omega} U_{2n} \cup \bigcup_{\alpha < \beta} U'_\alpha) \).

Let \( U'_\beta \) be a neighborhood of \( z_\beta_0 \) disjoint from \( \cl_z(\bigcup_{\alpha < \omega} U_{2n} \cup \bigcup_{\alpha < \beta} U'_\alpha) \), of diameter less than \( \epsilon \), and such that \( V'_\beta = f^{-1}(f(z_\beta_0) - \frac{\epsilon}{2}, f(z_\beta_0) + \frac{\epsilon}{2}) \cap U'_\beta \) is non-\( J^* \) dense in \( U'_\beta \). It follows from Lemma 6 that \( H_f(V'_\beta, J^*) \) contains a subset \( A \in A \setminus J \). By assumption 5), there exists a non-\( J \) dense in itself subset \( V'_\beta \in A \setminus J \) such that \( f|_{V'_\beta} \) is continuous.

After countably many steps, say \( \gamma < \omega_1 \), the choice of \( z_{\gamma} \) will no longer be possible and this is when \( U_{\alpha} \supseteq U_{\alpha} \) will become dense in \( Z \setminus \cl_z(\bigcup_{\alpha < \omega} U_{2n}) \).

It suffices to renumerate sets \( U'_\alpha, V'_\alpha \), and \( Y'_\alpha, \alpha < \gamma \) as \( U_{2n+1}, V_{2n+1}, \) and \( Y_{2n+1}, n < \omega \). \( \square \)

**Lemma 9.** If \( A \) contains all Borel subsets of \( X \) and an \( A \)-measurable function \( f : X \to \mathbb{R} \) has \( N \)-C-RP with respect to \( J \), then it also has \( A \)-N-C-RP with respect to the same ideal.

**Proof.** Suppose that \( W \in \mathcal{P}(X) \setminus J \) is such that \( f|_W \) is continuous. There exists a \( G \) subset \( G \subseteq X \) and a continuous function \( g : G \to \mathbb{R} \) such that \( f|_W \subseteq g \), see [9]. Since the difference \( f|_G - g \) is also \( A \)-measurable, the set \( W_1 = (f|_G - g)^{-1}({0}) \) is in \( A \setminus J \). Clearly \( f|_{W_1} \) is continuous. \( \square \)

**Theorem 10.** Let \( X \) be a separable metric space and let \( J \) a \( \sigma \)-ideal on \( X \), \( J \subseteq M(X) \) on every open set. Suppose that \( A \supseteq J \cup B(X) \) is a \( \sigma \)-algebra on \( X \). If \( f : X \to \mathbb{R} \) is \( A \)-measurable and \( f|_A \) has \( N \)-C-RP whenever \( A \in A \setminus J \), then \( f \) has \( A \)-D-C-RP.

**Proof.** Without loss of generality we may assume that \( X \) is zero-dimensional. (It suffices to remove a meager set to assure that.) Let \( \Delta = 2^{<\omega} \setminus \{0\} \). We construct three trees of subsets of \( X : (U_\tau)_{\tau \in \Delta}, (V_\tau)_{\tau \in \Delta} \), and \( (Y_\tau)_{\tau \in \Delta} \).

**Claim.** There exists an \( A \)-measurable non-\( J \) dense in itself subset \( Y \subseteq H_f(X, J^*) \) such that \( f|_Y \) is continuous.

\( A \supseteq J \cup B(X) \) and Lemma 6 applied to \( J^* \) imply that \( H_f(X, J^*) \in A \setminus J^* \).

\( f|_{H_f(X, J^*)} \) has \( N \)-C-RP so there exists a set \( Y' \subseteq H_f(X, J^*) \), \( Y' \notin J^* \) such that \( f|_{Y'} \) is continuous. Since \( A \supseteq B(X) \), Lemma 9 may be used to extend \( Y' \) to a subset of \( H_f(X, J^*) \), \( Y'' \in A \) such that \( f|_{Y''} \) is still continuous. Define \( Y = Y'' \setminus \{ x \in Y'' : \exists E \subseteq \text{open} \ X(x \in E \text{ and } E \cap Y'' \in J) \} \). \( Y \in A \) is nonempty because \( X \) is separable and non-\( J \) dense in itself.
To obtain the first level of the three trees apply Lemma 8 with \( Z = X, Y = Y \), and \( \varepsilon = 1 \). Now let \( k > 0 \). To obtain sets \( (U_{\tau^{-n}})_{\tau \in \omega^k, n \in \omega} \), \( (V_{\tau^{-n}})_{\tau \in \omega^k, n \in \omega} \), and \( (Y_{\tau^{-n}})_{\tau \in \omega^k, n \in \omega} \), from level \( k + 1 \) we apply Lemma 8 for each \( \tau \in \omega^k \) with \( Z = V_\tau, Y = Y_\tau \) and \( \varepsilon = \frac{1}{k+1} \). Then simply put \( U_{\tau^{-n}} = U_n, V_{\tau^{-n}} = V_n, \) and \( Y_{\tau^{-n}} = Y_n \), where \( U_n, V_n, \) and \( Y_n \) are from the Lemma and satisfy conditions 6)-13). Observe that the assumption 1) is preserved from one step to another due to Lemma 7.

Now let \( W = \bigcup_{k \in \omega} \bigcup_{\tau \in \omega^k} Y_\tau \). It is easy to see that for every \( k \in \omega \) the union \( \bigcup_{\tau \in \omega^k} U_\tau \) is dense in \( X \). To show that \( W \) is non-\( \mathcal{J} \) dense in \( X \) let \( T \) be a nonempty open subset on \( X \). Due to decreasing diameters of \( U_\tau \) there exists \( k \in \omega \) and a \( \tau \in \omega^k \) such that \( U_\tau \subseteq T \). This implies that \( T \cap W \supseteq Y_\tau \notin \mathcal{J} \).

It suffices to verify that \( f|_W \) is continuous. Let \( x \in W \). For almost all \( k \in \omega \) there exist sequences \( \tau \in \omega^k \) such that \( x \in Y_\tau \). \( Y_\tau \subseteq V_\tau \cap W \) and \( V_\tau \cap W \) is open in \( W \) with \( \text{diam}(f(V_\tau)) < \frac{1}{k} \).

The following applications illustrate the strength of theorem 10.

**Corollary 11.** (H. Blumberg [2]) If \( X \) is a Baire space, then \( (\mathcal{P}(X), \{\emptyset\}) \) has \( D\text{-C-RP} \).

**Proof.** Apply Theorem 10 with \( \mathcal{A} = \mathcal{P}(X) \) and \( \mathcal{J} = \{\emptyset\} \).

Let \( \omega \leq \kappa < \epsilon \). It is well known (see [16]) that if \( X \in [\mathbb{R}]^\kappa \) and \( f : X \to \mathbb{R} \), then, under Martin’s Axiom, there exists a set \( Y \in [X]^\kappa \) such that \( f|_Y \) is continuous. Theorem 10 gives the following.

**Corollary 12.** (S. Baldwin [1]) Assume Martin’s Axiom. Let \( \omega < \kappa < \epsilon \), cf(\( \kappa \)) > \( \omega \). Suppose that \( X \subseteq \mathbb{R} \) contains no meager open subsets and \( f : X \to \mathbb{R} \). Then \( (\mathcal{P}(X), [X]^{<\kappa}) \) has \( D\text{-C-RP} \).

**Proof.** Clearly, under Martin’s Axiom \( [X]^{<\kappa} \notin \mathcal{M}(X) \) on every open set. Apply theorem 10 with \( \mathcal{A} = \mathcal{P}(X) \) and \( \mathcal{J} = [X]^{<\kappa} \).

A set \( S \subseteq X \) is called \( (s) \)-measurable if for every perfect set \( P \subseteq X \) there exists a perfect subset \( P' \subseteq P \) such that either \( P' \cap S = \emptyset \) or \( P' \subseteq S \). \( (s_0) \) is the ideal of hereditarily \( (s) \)-measurable sets. It is well known (see Marczewski [11]) that whenever \( X \) is complete, then \( f : X \to \mathbb{R} \) is \( (s) \)-measurable iff for every perfect set \( P \subseteq X \) there exists a perfect subset \( Q \subseteq P \) such that \( f|_Q \) is continuous. It follows that \( f \) is \( (s) \)-measurable iff \( f|_A \) has \( (s) \)-N-C-RP whenever \( A \in (s) \setminus (s_0) \). The following corollary follows from the more general theorem 3 of [8]:

**Corollary 13.** (Brown and Prikry [8]) If \( X \) is a complete space without isolated points, then \( ((s), (s_0)) \) has \( (s)\text{-C-RP} \).
Proof. It is well known that \((s_0) \triangleq M(X)\) on any open subset of \(X\) and that \((s)\) contains all Borel subsets. Theorem 10 completes the proof. \qed

3 PWD Restrictions

Now we would like to look at the diagram of pointwise discontinuous restriction properties.

\[
\begin{array}{ccc}
\text{D-PWD-RP} & \checkmark & (1) \\
\text{DI-PWD-RP} & \checkmark & (2) \\
\text{WD-PWD-RP} & \checkmark & (3) \\
\text{N-PWD-RP} & \checkmark & (4) \\
\text{WDI-PWD-RP} & \checkmark & (5) \\
\text{WN-PWD-RP} & \checkmark & (6) \\
\end{array}
\]

Clearly for any class \(\mathcal{L}\) the \(\mathcal{L}\)-C-RP implies the corresponding \(\mathcal{L}\)-PWD-RP. In addition to that, the following properties are equivalent.

- \(\text{WD-C-RP} \iff \text{WD-PWD-RP}\)
- \(\text{WDI-C-RP} \iff \text{WDI-PWD-RP}\)
- \(\text{WN-C-RP} \iff \text{WN-PWD-RP}\)

Hence, we are going to focus on the left side of the diagram. The original Blumberg’s theorem [2] implies that \((P(\mathbb{R}), \mathcal{J})\) has WD-C-RP for any ideal \(\mathcal{J}\) without open sets. The following theorem shows that WD-C-RP \(\not\Rightarrow\) N-PWD-RP ((4) can not be turned by \(-90^\circ\)).

**Theorem 14.** Let \(\mathcal{J} = \{M \cup C : M \in \mathcal{M}(\mathbb{R}) \text{ and } C \in \mathcal{P}(\mathbb{R})^{<c}\}\). Then \((P(\mathbb{R}), \mathcal{J})\) does not have the N-PWD-RP.

Proof. Let \(z : \mathbb{R} \to \mathbb{R}\) be the Zygmund-Sierpinski function [17]. Let \(A \subseteq \mathbb{R}\) and suppose that \(G = \{x \in A : z|_A\text{ is continuous at } x\}\) is dense in \(A\). Since \(G\) is a relative \(G_\delta\) subset of \(A\), \(A \setminus G \in \mathcal{M}(A) \subseteq \mathcal{J}\). \(z\) is the Zygmund-Sierpinski function; so \(G \in [A]^{<c} \subseteq \mathcal{J}\). It follows that \(A \in \mathcal{J}\).
Example 15. Implication (1) can not be reversed. Let $K \subseteq \mathbb{R}$ be a nowhere dense perfect set. Take $X = K \cup \mathbb{Q}$ and $J = [X]^{\leq \omega}$. $X = \bigcup_{n<\omega} X_n$ where $X_n$ are pairwise disjoint and nowhere dense in $X$. Assume $X_1 = K$. Let $f : X \to \mathbb{R}$, \( f(x) = n \) for $x \in X_n$. Well known arguments (see [3] p. 667) shows that $(\mathcal{P}(X), J)$ does not have D-PWD-RP. On the other hand $(\mathcal{P}(K), [K]^{\leq \omega})$ has D-PWD-RP (see [4]); so $(\mathcal{P}(X), J)$ has D-PWD-RP.

Remark 2. Assume that $J$ contains all singletons. Under CH $(\mathcal{P}(X), J)$ has DI-PWD-RP iff it has N-PWD-RP. Suppose that $(\mathcal{P}(X), J)$ does not have DI-PWD-RP. By Brown’s theorem 2 of [5] $X = \bigcup_{n<\omega} X_n$ where $M(X_n) \subseteq J$. Take $z : X \to \mathbb{R}$ to be the Zygmund-Sierpinski function on $X$. Suppose that $z|_A$ is pointwise discontinuous for some $A \in \mathcal{P}(X) \setminus J$. There exists an $n < \omega$ such that $A \cap X_n \notin J$. We can find a set $B$, $A \supseteq B \supseteq A \cap X_n$ such that $z|_B$ is PWD and $|B \setminus (A \cap X_n)| \leq \omega$. The set $G = \{x \in B : z|_B$ is continuous at $x\}$ is a dense $G_\delta$ in $B$. Hence $B \setminus G \in M(B) \subseteq J$. Since $B \notin J$, $|G| > \omega$ which contradicts the Zygmund-Sierpinski property under CH.

4 Baire, Lebesgue, and Other Measurable Functions

If $\mathcal{A}$ is the Baire, Lebesgue, universally measurable, or other classic $\sigma$-algebra of sets, then restriction properties for $(\mathcal{A}, J_\mathcal{A})$, where $J_\mathcal{A}$ is the ideal of sets hereditarily in $\mathcal{A}$, are discussed in [8]. Here we look at restriction properties for $\mathcal{A}$ with arbitrary ideals $J$ other than $J_\mathcal{A}$.

If $X$ is meager on itself, then $BR(X) = \mathcal{P}(X)$ and this case has been discussed above. It follows from a well known theorem of Nikodym [12] that if $X$ does not contain nonempty meager open subsets, then $(BR(X), M(X))$ has $BR(X)$-D.C-RP. Using the same technique we can show that $(BR(X) \triangle J, J^\bot)$ has $(BR(X) \triangle J)$-D.C-RP as long as $J \perp M(X)$ on every open set. It remains to examine pairs $(BR(X), J)$ where $J \perp M(X)$. In such case there exists a nowhere dense set $F \notin J$. It is easy to find a discrete set $D \subseteq X \setminus F$, such that $cl_X(D) \supseteq F$. This last observation may be applied in a more general situation and yields the following facts.

Proposition 16. If $M(X) \notin J$, then the pair $(\mathcal{P}(X), J)$ has $BR(X)$-N-PWD-RP.

Corollary 17. $(BR(X), J)$ has N-PWD-RP for any $\sigma$-ideal $J$.

In general N-PWD-RP is the best restriction property that we can hope for $(BR(X), J)$ where $J \perp M(X)$.

Example 18. Let $X = \mathbb{R} = P \cup S$ where $P$ is some nowhere dense perfect set. Let $\mathcal{M}(P) = \{P' \cup P'' : P' \in \mathcal{M}(P) \text{ and } P'' \in [P]^{<\omega}\}$. Define $J = \{P' \cup S' : P' \in \mathcal{M}(P) \text{ and } S' \in [S]^{<\omega}\}$. There is no $\sigma$-ideal $J$ in $\mathcal{P}(X)$ such that $(\mathcal{P}(X), J)$ has $BR(X)$-N-PWD-RP.
Clearly \( M(X) \perp J \) on \( X \). \((\mathcal{BR}(X), J)\) does not have DI-PWD-RP because of the following \( \mathcal{BR}(X)\)-measurable function

\[
f(x) = \begin{cases} 
0 & \text{if } x \in S \\
z(x) & \text{if } x \in P,
\end{cases}
\]

where \( z \) is the Zygmund-Sierpinski function on \( P \). If \( W \) was a non-\( J \)-dense in itself, then \( W \subseteq P \). If \( f|_W \) was PWD, then the set \( G = \{x \in W : f|_W \text{ is continuous at } x\} \) is a dense \( G_\delta \) subset of \( W \) so \( W \setminus G \in \mathcal{M}(P) \). This implies that \( G \notin J \) and in particular \(|G| = c\), but that contradicts the Zygmund-Sierpinski property.

From Proposition 16 we easily obtain

**Corollary 19.** If \( X \) has positive outer measure, then \((\mathcal{P}(X), \mathcal{N}(X))\) has \( \mathbf{N}\)PWD-RP.

Here also no stronger restriction property is provable due to Example 18. Corollary 19 also follows from Theorem E of [7] on points of differentiability. More counterexamples for other \( \sigma \)-algebras follow from the next theorem.

**Theorem 20.** Let \( A \) be a \( \sigma \)-algebra of subsets of \( \mathbb{R} \) and assume that there exists a set \( X \notin \mathcal{MC} = \{M \cup C : M \in \mathcal{M}(\mathbb{R}) \text{ and } C \in [\mathbb{R}]^{<\omega}\} \) such that \( \mathcal{P}(X) \subseteq A \). If we define \( \mathcal{MC}_X = \{A \subseteq \mathbb{R} : A \cap X \in \mathcal{MC}\} \), then \((A, \mathcal{MC}_X)\) does not have the \( \mathbf{N}\)-PWD-RP.

**Proof.** Follow Example 18 with \( P = X \).

**Corollary 21.** There exists a \( \sigma \)-ideal \( J \) on \( \mathbb{R} \) such that \((\mathcal{L}(\mathbb{R}), J)\) does not have \( \mathbf{N}\)-PWD-RP.

**Proof.** Use Theorem 20 with \( X \) being a second category measure zero set.

**Corollary 22.** Assume CH. If \( A \) is one of the following \( \sigma \)-algebras: (s) measurable, universally measurable, or \( \mathcal{B}(\mathbb{R}) \triangle \mathcal{UN}(\mathbb{R}) = \{B \triangle N : B \in \mathcal{B}(X) \text{ and } N \text{ is universally null}\} \), then there exists a \( \sigma \)-ideal \( J \) such that \((A, J)\) does not have the \( \mathbf{N}\)-PWD-RP.

**Proof.** Use Theorem 20 with \( X \) being a Lusin set.

**Remark 3.** In the random real model \((\mathcal{B}(\mathbb{R}) \triangle \mathcal{UN}(\mathbb{R}), J)\) has \( \mathbf{N}\)-PWD-RP for all \( J \). Recall that in this model \( \mathcal{UN}(\mathbb{R}) \subseteq [\mathbb{R}]^{\leq \omega_1} \subseteq \mathcal{M}(\mathbb{R}) \). It follows that \( \mathcal{B}(\mathbb{R}) \triangle \mathcal{UN}(\mathbb{R}) \subseteq \mathcal{BR}(\mathbb{R}) \) and Corollary 17 implies \( \mathbf{N}\)-PWD-RP.

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References


