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ON A CONJECTURE OF AGRONSKY AND CEDER CONCERNING ORBIT-ENCLOSING ω-LIMIT SETS

Abstract

In [1] the following conjecture was stated:

> A continuum $K \subset E^k$ is an orbit-enclosing ω-limit set if and only if it is arcwise connected.

The main aim of this paper is to disprove this conjecture by giving an example of an orbit-enclosing ω-limit set S in E^2 (cf. Theorem 3 below) which is not arcwise connected. Moreover, we show that S can be chosen with non-empty interior, and the mapping F, with respect to which S is an orbit-enclosing ω-limit set can be chosen as a triangular map.

1 Terminology and Notation

Suppose A is a topological space, $x \in A$, $f : A \to A$ a continuous map, and $\mathbf{N}$ the set of natural numbers. We will use $f^n(x)$ to denote the *n-th iteration of x under f*. By the *trajectory of x under f* we mean the set $\gamma(x,f) = \{f^n(x);\ n \in \mathbf{N} \cup \{0\}\}$. By $\omega(x,f)$, called the *ω-limit set with respect to f and x*, we mean the set of limit points of the sequence $\{f^n(x)\}_{n=0}^{\infty}$. We say that $\omega(x,f)$ is *orbit-enclosing* if $\gamma(x,f) \subseteq \omega(x,f)$. We say that a subset B of A is an *orbit-enclosing ω-limit set (with respect to f and x)* if there exists a continuous map $f : A \to A$ and a point $x \in A$ such that $B = \omega(x,f)$ and $\omega(x,f)$ is orbit-enclosing. If for any nonvoid subsets U and V of A, both relatively open in A, there exists $n \in \mathbf{N}$ such that $f^n(U) \cap V \neq \emptyset$, then we say that f is *topologically transitive* on A (or briefly, *transitive*). By a *continuum*

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we mean any compact connected set which contains more than one point. A set $M \subset A$ is *arcwise connected* if each two points in M belong to some homeomorph of $[0,1]$ which lies in M. A map F from $A_1 \times A_2$ into itself is called *triangular* if it is of the form $F(x,y) = (f(x), g(x,y))$.

2 Main Results

Let $[0,1]^2$ denotes the unit square, W the curve $y = \frac{1}{2} + \frac{1}{2}\sin\frac{\pi}{x-2}$ for $x \in [1,2)$, and $T = \{2\} \times [0,1]$ the vertical line. Put $S = [0,1]^2 \cup W \cup T$. Clearly, the continuum S is not arcwise connected, because no homeomorph of $[0,1]$ lying in S intersects both T and $[0,1]^2 \cup W$.

To prove that S is an orbit-enclosing ω-limit set, we will use the following key result.

Theorem 1. *Let $D = [0,1]^2 \cup \big([1,\infty) \times \{\frac{1}{2}\}\big)$. For $x \in [0,\infty)$ put $\overline{x} = (x, \frac{1}{2})$. Then there exists a continuous triangular surjective map $\varphi : D \to D$ which is transitive and for which* $\lim_{x\to\infty} |\varphi(\overline{x}) - \overline{x}| = 0$.

The proof of the theorem, including the construction of φ, is postponed to Section 3. Now, let D, W and φ be as above. Define map $h : D \to [0,1]^2 \cup W$ by

$$h(t) = \begin{cases} t & \text{if } t \in [0,1]^2, \\ \left(2 - \frac{1}{t}, \frac{1}{2} + \frac{1}{2}\sin(-t\pi)\right) & \text{otherwise.} \end{cases}$$

Obviously, h is bijective. Let $F : S \to S$ be given by

$$F = \begin{cases} h \circ \varphi \circ h^{-1} & \text{on } [0,1]^2 \cup W, \\ id & \text{on } T. \end{cases}$$

Remark 1. *The construction of φ in the Section* 3 *ensures that φ is a triangular map so that F is a triangular map, too.*

We will take advantage of the following theorem to show that the set S is an orbit-enclosing ω-limit set with regard to F.

Theorem 2. ([2, p. 105]) *Let $A \subset E^n$ be a nonvoid compact set. Then there exists a continuous map $f : A \to A$ such that f is topologically transitive on A if and only if A is an orbit-enclosing ω-limit set.*

Theorem 3. *The set S is an orbit-enclosing ω-limit with regard to F.*

PROOF. The map F defined above is continuous, since $\lim_{x\to\infty} |\varphi(\overline{x}) - \overline{x}| = 0$. Transitivity of φ on the set D implies transitivity of F on the set $[0,1]^2 \cup W$, and hence, on the closure S of $[0,1]^2 \cup W$. □

3 Proof of Theorem 1

The construction of the map φ from Theorem 1 will be divided in three steps. *Step* 1. *A transitive map* $\tau_1 : [0,1]^2 \to [0,1]^2$. Let $C \subset [0,1]$ be the Cantor set. It is known that each point $x \in C$ can be written in the triadic system uniquely in the form $x = x_1x_2x_3\cdots = \frac{x_1}{3^1} + \frac{x_2}{3^2} + \frac{x_3}{3^3} + \dots$, where $x_i \in \{0,2\}$, $i = 1,2,3,\dots$ (see [5]).

Suppose $x \in C$, $x = x_1x_2x_3\dots$. We define the map $\psi : C \to [0,1]^2$ by the relation $\psi(x) = (x'_1x'_3x'_5\dots, x'_2x'_4x'_6\dots) \equiv (x^*, y^*)$, where $x'_i = \frac{x_i}{2}$ and x^*, y^* are written in the dyadic system.

The map ψ is obviously surjective and continuous.

Now we will introduce the following notions:

Contiguous interval $U \subset [0,1]$ *of order* n, $n \in \mathbf{N}$, is an arbitrary closed interval of length $1/3^n$, that contains just two points of C (these points are obviously the end points of the interval). *Non-contiguous interval* $J \subset [0,1]$ *of order* n, $n \in \mathbf{N}$, is the closure of one of the intervals, complementary to the union of all contiguous intervals of order $\leq n$. For any intervals J_0, J_1, $J_0 < J_1$ denotes that J_0 lies on the left of J_1.

The next lemma is easy and follows immediately from the properties of the Cantor set.

Lemma 1. *Let $k \geq 1$. Let J be a non-contiguous interval of order $2k$, let $J_0 < J_1$ be the non-contiguous intervals of order $2k+1$ contained in J, and $J_{00} < J_{01}$ the non-contiguous intervals of order $2k+2$ contained in J_0. Then*

(i) *$\psi(J)$ is a square K of size $1/2^k \times 1/2^k$.*

(ii) *$\psi(J_0)$ and $\psi(J_1)$ are rectangles K_0 and K_1, each of size $1/2^{k+1} \times 1/2^k$, forming the left and right half of K, respectively.*

(iii) *$\psi(J_{00})$ and $\psi(J_{01})$ are squares, each of size $1/2^{k+1} \times 1/2^{k+1}$, forming the lower and upper half of K_0, respectively.*

(iv) *The end-points of the contiguous interval $J \setminus (J_0 \cup J_1)$ are mapped onto the end-points of $\psi(J_0) \cap \psi(J_1)$.*

(v) *The end-points of the contiguous interval $J_0 \setminus (J_{00} \cup J_{01})$ are mapped onto the end-points of $\psi(J_{00}) \cap \psi(J_{01})$.*

Let $\sigma : [0,1] \to [0,1]^2$ be the piecewise linear map given by $\sigma(0) = [\frac{1}{2}, 1]$, $\sigma(\frac{1}{4}) = [1, \frac{3}{4}]$, $\sigma(\frac{3}{4}) = [0, \frac{1}{4}]$, $\sigma(1) = [\frac{1}{2}, 0]$. Extend the map ψ, which is defined on C, to $[0,1]$ as follows. Let J and K be as in Lemma 1. If $U \subset J$ is the contiguous interval of order $2k+1$, and $U_0 \subset J_0$ the contiguous interval of order $2k+2$, then let the graph of $\psi : U \to K$ be the affine copy of σ, and the graph of $\psi : U_0 \to [0,1]^2$ the set $\psi(J_{00}) \cap \psi(J_{01})$, i. e. a horizontal line of length $1/2^{k+1}$.

Define a map $\tau_1 : [0,1]^2 \to [0,1]^2$ by $\tau_1 = \psi \circ \pi$, where π is the projection to the x-axis.

In the sequel, we say that a map $\varphi : X \to X$ is expansive with a coefficient $s > 1$, if there is an $\varepsilon > 0$ such that for any set $A \subset X$ with $diam(A) < \varepsilon$, $diam(\varphi(A)) > s \cdot diam(A)$.

Lemma 2. *Let U be a contiguous interval of order n, and L a subinterval of U. Then $\pi \circ \psi|_L$ is expansive and the coefficient of expansion $s_n = |(\pi \circ \psi)(L)|/|L|$ being such that*

$$6 \cdot (9/2)^k \geq s_{2k+1} \geq 3 \cdot (9/2)^k, \quad k = 0, 1, 2, \dots, \tag{1}$$

$$s_{2k} = (9/2)^k, \quad k = 1, 2, \dots . \tag{2}$$

PROOF. If $n = 2k+1$, then U is mapped by $\pi \circ \psi$ to an interval of length $1/2^k$ (cf. (i) of Lemma 1). Moreover, $\pi \circ \psi$ on U is two-to-one and piecewise linear with constant slope. This implies (1). Formula (2) follows similarly by the fact that for $n = 2k$, $\pi \circ \psi$ maps U linearly onto an interval of length $1/2^k$ (cf. (ii) of Lemma 1). □

Lemma 3. *The map $\tau_1 : [0,1]^2 \to [0,1]^2$ is a triangular map which is surjective, continuous and transitive.*

PROOF. The map $\tau_1 = \psi \circ \pi$ is surjective, since ψ is surjective and clearly, τ_1 is triangular.

Let $\{U_n\}_{n=0}^{\infty}$ be the sequence of contiguous intervals. Then ψ is continuous since each of the maps $\psi|_C$, $\psi|_{U_n}$ is continuous, and $\lim_{n\to\infty} diam(\psi(U_n)) = 0$. Finally, $\tau_1 = \psi \circ \pi$ is continuous as a composition of two continuous maps.

It remains to show that the map $\tau_1 = \psi \circ \pi$ is transitive, or equivalently that $\pi \circ \psi$ is transitive. So, let $L' \subset [0,1]$ be an interval. Obviously, there always exists an interval $L \subset L'$ such that $L \subset U$, U is a contiguous interval. Suppose $|L| = 1/3^n$, $n \geq 3$, and put $M = (\pi \circ \psi)(L)$. According to (1) and (2), $|M| \geq 9|L|$. Let U_0 be the contiguous interval with which M has the longest intersection.

For any interval K of length $1/3^i$ the longest interval of set $K \setminus C$ has length at least $1/3^{i+1}$, then the set $M \setminus C$ contains an interval N of length at least $1/3^{n-1}$. By the induction we instantly get that there exists $k \in \mathbf{N}$ such that $(\pi \circ \psi)^k(L) \setminus C$ contains an interval of length at least $1/9$. The rest of the proof is obvious. □

Step 2. *A transitive map $\tau_2 : [1,\infty) \times \{\frac{1}{2}\} \to [1,\infty) \times \{\frac{1}{2}\}$.* For short, for any $x \in [0,\infty)$, put $\overline{x} = \{x\} \times \{\frac{1}{2}\}$, and similarly define $\overline{J}$ for any interval J, etc. Furthermore, let $a_0 = 1$, $a_n = 1 + \frac{1}{4} + \frac{1}{5} + \dots + \frac{1}{n+3} = 1 + \sum_{k=1}^{n} \frac{1}{k+3}$.

Let τ_2 be the piecewise linear map given by

$$\tau_2(\overline{a}_0) = \overline{a}_2, \qquad \tau_2(\overline{a}_1) = \overline{a}_0,$$
$$\tau_2(\overline{a}_{2k}) = \overline{a}_{2k+2}, \quad \tau_2(\overline{a}_{2k+1}) = \overline{a}_{2k-1}, \ k = 1, 2, 3, \dots .$$

The following lemma is obvious.

Lemma 4. *The map* $\tau_2 : \overline{[1, \infty)} \to \overline{[1, \infty)}$ *is continuous and transitive.*

Step 3. *A map* $\varphi : D \to D$.

Define φ by $\quad \varphi(x) = \tau_1(x) \quad$ if $\pi(x) \in [0, 1] \setminus [\frac{7}{9}, \frac{8}{9}]$,
$\qquad\qquad\qquad \varphi(x) = \tau_2(x) \quad$ if $x \in \overline{[a_2, \infty)}$,

and let φ be piecewise linear on $\big([\frac{7}{9}, \frac{8}{9}] \times [0, 1]\big) \cup \overline{[1, a_2]}$, given by $\varphi(\frac{7}{9} \times [0, 1]) = \overline{1}$, $\varphi(\frac{37}{45} \times [0, 1]) = \overline{2}$, $\varphi(\frac{8}{9} \times [0, 1]) = \overline{(\frac{1}{2})}$, and $\varphi(\overline{a}_0) = (1, 1)$, $\varphi(\overline{a}_1) = \overline{(\frac{1}{2})}$, $\varphi(\overline{a}_2) = \overline{a}_4$.

PROOF OF THEOREM 1. By Lemmas 3 and 4, φ is a continuous triangular map, which is surjective and transitive, and

$$\begin{aligned}\lim_{x \to \infty} |\varphi(\overline{x}) - \overline{x}| &= \lim_{k \to \infty} |\varphi(\overline{a}_{2k+1}) - \overline{a}_{2k+1}| = \lim_{k \to \infty} |\overline{a}_{2k-1} - \overline{a}_{2k+1}| \\ &= \lim_{k \to \infty} \Big| \frac{1}{2k+3} + \frac{1}{2k+4} \Big| = 0 \,. \end{aligned}$$ □

Remark 2. *If we do not require the set* S *has non-empty interior, then it can be simply the union of the curve* W *and the vertical line* T *defined above, and it is necessary to slightly modify the map* F *from the Theorem* 1 *on the surroundings of the point* $[1, \frac{1}{2}]$.

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