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CHARACTERIZATIONS OF VBG $\cap$ (N)

Abstract

We show that $VBG \cap (N)$ is equivalent with Sarkhel and Kar's class $(PAC)G$ on an arbitrary real set. Hence $VBG \cap (N)$ is an algebra on that set. In Theorem 4, we give three characterizations for $VBG \cap (N)$ on an arbitrary real set. It follows that Gordon's AK_N-integral [3] is a special case of the PD-integral [7] of Sarkhel and De (Remark 3). In Theorem 3 we obtain the following surprising result: *a Lebesgue measurable function f is VBG on E if and only if f is VBG on any null subset of E*. We also find seven characterizations of $VBG \cap (N)$ for Lebesgue measurable functions (see Theorem 5). For continuous functions on a closed set, we obtain several characterizations of the class ACG. Using a different technique, we obtain other characterizations of $VBG \cap (N)$ for a Lebesgue measurable function (see Theorem 8).

1 Introduction

The purpose of this paper is to give some characterizations of $VBG \cap (N)$ on an arbitrary real set.

In [8], Sarkhel and Kar introduced the class (PAC), showing that it is contained in $[VBG] \cap (N)$ and it is an algebra on any real set. Moreover, (PAC) is equivalent to the class $[VBG] \cap (N)$ on a closed set. It is clear now that $(PAC)G$ (generalized (PAC)) is contained in $VBG \cap (N)$. Surprisingly, the converse is also true. We show that $VBG \cap (N)$ is equivalent to $(PAC)G$ on an arbitrary real set. Hence $VBG \cap (N)$ is an algebra on that set. In fact in Theorem 4, we give three characterizations for $VBG \cap (N)$ on an arbitrary real set. It follows that Gordon's AK_N-integral [3] is a special case of the PD-integral [7] of Sarkhel and De (Remark 3).

In Theorem 3 we obtain the following surprising result: *a Lebesgue measurable function f is VBG on a set E if and only if f is VBG on any null subset of E*.

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As a consequence of Theorems 3 and 4, we find seven characterizations of $VBG \cap (N)$ for Lebesgue measurable functions (Theorem 5). One of them asserts that: *a Lebesgue measurable function f is $VBG \cap (N)$ on a set E if and only if f is $VBG \cap (N)$ on any null subset of E.*

For continuous functions on a compact set, we obtain several characterizations of the class ACG, such as: *a continuous function f is ACG on a compact set E if and only if f is $(PAC)G$ on any null subset of E* (Corollary 4).

In the last two sections, we give five enhancements of $V(f;E)$ (the ordinary variation of a function f on a set E): $\nu_f^1(E)$, $\nu_f^2(E)$, $\nu_f^3(E)$, $\nu_f^4(E)$ and $\nu_f^5(E)$. For each of these set-functions we obtain another characterization of $VBG \cap (N)$ for a Lebesgue measurable function (see Theorem 8): *a Lebesgue measurable function $f : E \to \mathbb{R}$ is $VBG \cap (N)$ if and only if for every null subset Z of E, there is a sequence $\{Z_n\}_n$ whose union is Z, such that $\nu_f^i(Z_n) = 0$ for each n.*

2 Preliminaries

We denote by $m^*(X)$ the outer measure of the set X and by $m(A)$ the Lebesgue measure of A, whenever $A \subset \mathbb{R}$ is Lebesgue measurable. For the definitions of VB, VBG, AC and Lusin's condition (N), see [5].

Definition 1. Let E be a real compact set, $c = \inf(E)$, $d = \sup(E)$ and $f : E \to \mathbb{R}$. Let $\{(c_k, d_k)\}_k$ be the intervals contiguous to E and let

$$f_E : [c,d] \to \mathbb{R}, \quad f_E(x) = \begin{cases} f(x) & \text{if } x \in E \\ \text{linear} & \text{on each } [c_k, d_k]\,. \end{cases}$$

Definition 2. ([6]). A sequence $\{E_n\}$ of sets whose union is E is called an E-form with parts E_n. If, moreover, each part E_n is closed in E (i.e., $E_n = P_n \cap E$, where P_n is a closed set; so $P_n = \overline{E}_n$), then the E-form is said to be closed. An expanding E-form is called an E-chain.

Definition 3. Let E be a real set and $f : E \to \mathbb{R}$.

- f is said to be $[VBG]$ (respectively $[ACG]$) on E if there is a closed E-form $\{E_n\}$ such that f is VB (respectively AC) on each E_n.
- f is said to be ACG on E if there is an E-form $\{E_n\}$ such that f is AC on each E_n. Note that ACG here differs from the definition given in [5], because f is not supposed to be continuous.

Lemma 1. ([8]). *For every closed E-form $\{E_n\}$, there is a closed E-chain $\{Q_n\}$ such that $Q_n = \cup_{k \le n} Q_{kn}$, where $Q_{kn} \subseteq Q_{km} \subseteq E_k$ for all k and for*

$m \geq n \geq k$, and $d(Q_{in}, Q_{jn}) \geq 1/n$ for $i \neq j$. (Here d denotes the usual metric distance).

3 The Conditions PAC, (PAC), [PAC]

Definition 4. Let $Q \subset \mathbb{R}$, $f : Q \to \mathbb{R}$, $E \subseteq Q$ and $r > 0$. Put

- (**Sarkhel, Kar**, [8]) $V(f; E; r) = \sup\{\sum_{i=1}^{n} |f(b_i) - f(a_i)| : \{[a_i, b_i]\}_{i=1}^{m}$ is a finite set of nonoverlapping closed intervals with the endpoints in E and $\sum_{i=1}^{m}(b_i - a_i) < r\}$;
- (**Sarkhel, Kar**, [8]) $V(f; E; 0) = \inf_{r>0} V(f; E; r)$;
- (**Sarkhel, Kar**, [8]) $PV(f; E) = \inf\{\sup_n V(f; E_n; 0) : \{E_n\}$ is an E-chain$\}$;
- $[PV](f; E) = \inf\{\sum_n V(f; E_n; 0) : \{E_n\}$ is a closed E-form$\}$;
- $\mu_f(E) = \inf\{\sum_n V(f; E_n; 0) : \{E_n\}$ is an E-form$\}$.

Remark 1. Let E be a real set and $f : E \to \mathbb{R}$. Then f is AC on E if and only if $V(f; E; 0) = 0$ ([8], p. 337).

Definition 5. ([8]). Let E be a real set and $f : E \to \mathbb{R}$.

- (**Sarkhel, Kar**, [8]) f is said to be (PAC) on E if $PV(f; E) = 0$.
- f is said to be $[PAC]$ on E if $[PV](f; E) = 0$.
- f is said to be PAC on E if $\mu_f(E) = 0$ (our definition is different from that of [7]; see Remark 3).
- (**Sarkhel, De**, [7]) f is said to be $(PAC)G$ on E if there is an E-form $\{E_n\}$ such that f is (PAC) on each E_n.
- f is said to be $[PAC]G$ on E if there is an E-form $\{E_n\}$ such that f is $[PAC]$ on each E_n.

Theorem 1 (Sarkhel)**.** ([8]). *Let P be a real set, $f, g : P \to \mathbb{R}$, $E \subseteq P$, $a, b \in \mathbb{R}$. We have each of the following assertions.*

(i) $PV(af + bg; E) \leq |a| \cdot PV(f; E) + |b| \cdot PV(g; E)$.

(ii) *If $PV(g; E) = 0$, then $PV(f + g; E) = PV(f; E)$.*

(iii) *If $m^*(E) = 0$, then $m^*(f(E)) \leq PV(f; E)$.*

(iv) *If $PV(f; E) < +\infty$, then $f \in [VBG]$ on E.*

(v) $PV(f;E) \le \sum_n PV(f;E_n)$ *whenever* $\{E_n\}$ *is a closed E-form.*

(vi) If $f,g \in (PAC)$ *on* E*, then* $f \cdot g \in (PAC)$ *on* E.

Corollary 1. *Let* E *be a real set and* $\mathcal{A} = \{f : E \to \mathbb{R} : f \in (PAC) \text{ on } E\}$*. Then* $\mathcal{A}$ *is an algebra.*

PROOF. See Theorem 1, (i), (vi). □

Proposition 1. *Let* Q *be a real set,* $f : Q \to \mathbb{R}$ *and* $E \subseteq Q$*. We have:*

(i) $\mu_f(E) \le [PV](f;E)$;

(ii) $PV(f;E) \le [PV](f;E)$;

(iii) $[PV](f;E) \le \sum_n [PV](f;E_n)$ *whenever* $\{E_n\}$ *is a closed E-form;*

(iv) $\mu_f : \mathcal{P}(Q) \to [0,+\infty]$ *is a metric outer measure.*

PROOF. (i) This is obvious.

(ii) Suppose that $[PV](f;E) = M < +\infty$ (if $M = +\infty$, there is nothing to prove). Then for $\epsilon > 0$, it follows that there exist a closed E-form $\{E_n\}$ and a sequence of positive numbers $\{r_n\}$ such that $\sum_n V(f;E_n;r_n) < M + \epsilon$. By Lemma 1, there exists a closed E-chain $\{Q_n\}$ such that $Q_n = \cup_{k=1}^n Q_{kn}$, $Q_{kn} \subseteq Q_{km} \subseteq E_k$ for all k and $m \ge n \ge k$ and

$$d(Q_{in}, Q_{jn}) \ge \frac{1}{n} \quad \text{for } i \ne j\,. \tag{1}$$

Let $\rho_n = \min\left\{r_1, r_2, \ldots, r_n, \frac{1}{2n}\right\}$. Let $\{[a_p,b_p]\}_{p=1}^q$ be a finite set of nonoverlapping closed intervals with the endpoints in Q_n and $\sum_{p=1}^q (b_p - a_p) < \rho_n$. By (1), both endpoints of an interval $[a_p,b_p]$ belong to some Q_{in}. It follows that

$$\sum_{p=1}^q |f(b_p) - f(a_p)| \le \sum_{i=1}^n V(f;Q_{in};\rho_n) \le \sum_{i=1}^n V(f;E_i;r_i) < M + \epsilon \quad \text{for all } n\,.$$

Therefore $PV(f;E) \le M$.

(iii) We may suppose that $\sum_n [PV](f;E_n;0) < +\infty$ (otherwise there is nothing to prove). Let $\epsilon > 0$. Then for every positive integer k, there exists a closed E_k-form $\{E_{kn}\}$ and a sequence of positive numbers $\{r_{kn}\}$ such that

$$\sum_n V(f;E_{kn};r_{kn}) < [PV](f;E_k) + \frac{\epsilon}{2^k}\,.$$

But $\{E_{kn}\}$ is a closed E-form, and

$$\sum_k \sum_n V(f; E_{kn}; r_{kn}) < \epsilon + \sum_k [PV](f; E_k)\,.$$

It follows that $[PV](f;E) \leq \epsilon + \sum_k [PV](f;E_k)$. Since ϵ is arbitrary, we obtain that $[PV](f;E) \leq \sum_k [PV](f;E_k)$.

(iv) Clearly $\mu_f(\emptyset) = 0$ and μ_f is an increasing set-function; i.e., $\mu_f(A) \leq \mu_f(B)$ whenever $A \subseteq B$. As in (iii) we obtain that

$$\mu_f(\cup_n E_n) \leq \sum_n \mu_f(E_n)\,. \tag{2}$$

Let E_1, E_2 be such that $d(E_1;E_2) = r > 0$. Suppose that $\mu_f(E_1 \cup E_2) < +\infty$ (if $\mu_f(E_1 \cup E_2) = +\infty$, by (2), it follows that $\mu_f(E_1 \cup E_2) = \mu_f(E_1) + \mu_f(E_2)$). For $\epsilon > 0$ there exist a $E_1 \cup E_2$-form $\{P_n\}$ and a sequence of positive numbers $\{r_n\}$ such that

$$\sum_n V(f; P_n; r_n) < \mu_f(E_1 \cup E_2) + \epsilon\,.$$

Let $P_{1n} = E_1 \cap P_n$, $P_{2n} = E_2 \cap P_n$ and $\rho_n = \min\{r_n, r\}$. Then

$$\mu_f(E_1) + \mu_f(E_2) \leq \sum_n V\big(f; P_{1n}; \frac{\rho_n}{2}\big) + \sum_n V\big(f; P_{2n}; \frac{\rho_n}{2}\big) \leq$$

$$\leq \sum_n V(f; P_n; \rho_n) \leq \sum_n V(f; P_n; r_n) \leq \mu_f(E_1 \cup E_2) + \epsilon\,.$$

Since ϵ is arbitrary and μ_f is an outer measure, we obtain that $\mu_f(E_1 \cup E_2) = \mu_f(E_1) + \mu_f(E_2)$. □

Lemma 2. *Let E be a real set and $f : E \to \mathbb{R}$. If $f \in [ACG]$ on E, then $f \in [PAC]$ on E.*

PROOF. Let $\epsilon > 0$. Since $f \in [ACG]$ on E, there exists a closed E form $\{E_n\}$ such that $f \in AC$ on each E_n. For $\epsilon/2^n$, let $r_n > 0$ be given by the fact that $f \in AC$ on E_n. Then $\sum_n V(f; E_n; r_n) < \sum_n \epsilon/2^n = \epsilon$. Hence $[PV](f;E) = 0$. Therefore $f \in [PAC]$ on E. □

4 Characterizations of $[\mathbf{VBG}] \cap (\mathbf{N})$ on a Closed Set

Lemma 3. *Let E be a real compact set, $f : E \to \mathbb{R}$, $x_0 \in E$ and $\epsilon > 0$. If $f \in VB$ on E, then there exists $\delta > 0$ such that*

$$V(f; E \cap (x_0, x_0 + \delta)) < \epsilon \quad \textit{and} \quad V(f; E \cap (x_0 - \delta, x_0)) < \epsilon\,.$$

Moreover, if $\{I_n\}_n$ is a sequence of abutting closed intervals such that $\cup_n I_n = (x_0 - \delta, x_0)$ or $\cup_n I_n = (x_0, x_0 + \delta)$, then $\sum_n V(f; E \cap I_n) < \epsilon$.

PROOF. Let $a = \inf E$, $b = \sup E$ and $F : [a, b] \to \mathbb{R}$, $F(x) = f_E(x)$ (see Definition 1). Then $F \in VB$ (see for example Corollary 2.7.2, (ii) of [1]). Let $V_F : [a, b] \to \mathbb{R}$,

$$V_F(x) = \begin{cases} 0 & \text{if } x = a \\ V(F; [a, x]) & \text{if } x \in (a, b] \end{cases}$$

Since V_F is an increasing function on $[a, b]$, $V_F(x_0-) = \ell^-$ and $V_F(x_0+) = \ell^+$ exist and are both finite. It follows that there exists a $\delta > 0$ such that

$$V_F((x_0 - \delta, x_0)) \subset (\ell^- - \epsilon, \ell^-) \quad \text{and} \quad V_F((x_0, x_0 + \delta)) \subset (\ell^+, \ell^+ + \epsilon))\,.$$

Let $\alpha, \beta \in E$, $\alpha < \beta$. Then

$$|f(\beta) - f(\alpha)| = |F(\beta) - F(\alpha)| \le V(F; [\alpha, \beta]) = V_F(\beta) - V_F(\alpha)\,.$$

Therefore $V(f; E \cap (x_0, x_0 + \delta)) \le \epsilon$. Clearly

$$\sum_n V(f; E \cap I_n) \le \sum_n V(F; I_n) = \sum_n V_F(\beta_n) - V_F(\alpha_n) < \epsilon\,,$$

where $\{I_n\}_n = \{[\alpha_n, \beta_n]\}_n$ are as in the hypotheses. □

Lemma 4. *Let E be a real compact set and $f : E \to \mathbb{R}$. If $f \in VB \cap (N)$, then $[PV](f; E) = 0$; i.e., $f \in [PAC]$ on E.*

PROOF. The proof is similar to that of the second part of Theorem 3.6 of [8]. Since f is VB on E, it follows that f is continuous nearly everywhere on E. Let $d_1, d_2, \dots$ be the discontinuity points of f. By Lemma 3, for $\epsilon > 0$ and for each d_n we can find some intervals $I_n = (p_n, d_n)$ and $J_n = (d_n, q_n)$ such that $\sum_k (V(f; E \cap I_{nk}) + V(f; E \cap J_{nk})) < \epsilon/2^{n+1}$, whenever $\{I_{nk}\}_k$ and $\{J_{nk}\}_k$ are two sequence of closed intervals abutting end to end, with $\cup_k I_{nk} = I_n$ and $\cup_k J_{nk} = J_n$. It follows that $Q = E \setminus \cup_n (I_n \cup J_n)$ is a compact set and $f_{|Q}$ is $\mathcal{C} \cap VB \cap (N) = AC$. (See the Banach–Zarecki Theorem; here $\mathcal{C}$ denotes the class of continuous functions.) Therefore $f \in AC$ on Q. For $\epsilon/2$ let $r_0 > 0$ be given by the fact that $f \in AC$ on Q. Then

$$V(f; Q; r_0) + \sum_n \sum_k \big(V(f; E \cap I_{nk}) + V(f; E \cap J_{nk})\big) < \frac{\epsilon}{2} + \sum_n \frac{\epsilon}{2^{n+1}} = \epsilon\,.$$

Therefore $[PV](f; E) = 0$. □

Theorem 2. *Let E be a real compact set and $f : E \to \mathbb{R}$. The following assertions are equivalent.*

(i) *$f \in [PAC]$ on E.*

(ii) $f \in (PAC)$ *on* E.

(iii) $f \in [VBG] \cap (N)$ *on* E.

PROOF. (i) $\Rightarrow$ (ii) See Proposition 1, (ii).

(ii) $\Rightarrow$ (iii) See Theorem 1, (iii), (iv).

(iii) $\Rightarrow$ (i) Since $f \in [VBG] \cap (N)$, there is a closed E-form $\{E_n\}$ (Clearly each E_n is a closed set, because E is closed.) such that $f \in VB \cap (N)$ on each E_n. By Lemma 4, $f \in [PAC]$ on each E_n and by Proposition 1, (iii), $f \in [PAC]$ on E. □

Remark 2. Theorem 2, (ii), (iii) is due to Sarkhel and Kar (see Theorem 3.6 of [8]).

5 Characterizations of VBG and ACG for Lebesgue Measurable Functions

Lemma 5. *Let* E *be a real compact set and* $f : E \to \mathbb{R}$ *a continuous function. The following assertions are equivalent.*

(i) $f \in VBG$ *(respectively* ACG*) on* E.

(ii) $f \in VBG$ *(respectively* ACG*) on* Z*, whenever* Z *is a null subset of* E.

PROOF. This follows by Proposition 1.9.1, (iii) of [1], if we put $P_1 =$ the class of continuous functions, and $P = VB$ (respectively AC). □

Theorem 3. *Let* E *be a bounded Lebesgue measurable set, and* $f : E \to \mathbb{R}$ *a Lebesgue measurable function. The following assertions are equivalent.*

(i) f *is* VBG *(respectively* ACG*) on* E.

(ii) f *is* VBG *(respectively* ACG*) on* Z*, whenever* Z *is a null subset of* E.

PROOF. (i) $\Rightarrow$ (ii) This is obvious.

(ii) $\Rightarrow$ (i) By Lusin's theorem (see [4], p. 106 or [5], p. 72), there exists an increasing sequence $\{E_n\}_n$ of closed subsets of E such that $m(\cup E_n) = m(E)$ and $f_{|E_n}$ is continuous. Let $Z = E \setminus (\cup_n E_n)$. Then Z is a null subset of E. Hence f is VBG (resp. ACG) on Z. Fix some n. By Lemma 5, f is VBG (resp. ACG) on E_n. Therefore f is VBG (resp. ACG) on E. □

6 Characterizations of VBG $\cap$ (N) on a Real Set

Lemma 6. *Let $f : [a,b] \to \mathbb{R}$, $f \in VB$ on $[a,b]$. Consider the curve*

$$C : X(t) = t; \quad Y(t) = f(t), \quad t \in [a,b]$$

and let $Z = \{x \in [a,b] : f^{'}(x)$ does not exist (finite or infinite)$\}$. Let $S : [a,b] \to \mathbb{R}$, where $S(x)$ is the length of the curve C on the interval $[a,x]$. Then $m^*(S(Z)) = 0$.

PROOF. Let $C_f = \{x \in [a,b] : f$ is continuous at $x\}$. Then $[a,b] \setminus C_f$ is countable (see [4], p. 219). Let $N = Z \cap C_f$. Then $m^*(S(N)) = 0$ (see [5], pp. 125–126). It follows that $m^*(S(Z)) = 0$. $\square$

Lemma 7. *Let $f : [a,b] \to \mathbb{R}$, $f \in VB$ on $[a,b]$. Let $Z = \{x \in [a,b] : f^{'}(x)$ does not exist (finite or infinite)$\}$. Then $\mu_f(Z) = 0$, i.e. $f \in PAC$ on Z.*

PROOF. By Lemma 6, $m^*(S(Z)) = 0$. For $\epsilon > 0$, there exists an open set G such that $S(Z) \subset G$ and $m(G) < \epsilon$. Let $\{(\alpha_i, \beta_i)\}_i$ be the components of G (a component is a maximal open interval contained in G). We may suppose without loss of generality that $S(Z) \cap (\alpha_i, \beta_i) \neq \emptyset$. Let $Z_i = \{x \in Z : S(x) \in (\alpha_i, \beta_i)\}$. Since we always have that $|f(\beta) - f(\alpha)| \leq S(\beta) - S(\alpha)$ and S is strictly increasing, it follows that $V(f; Z_i) < \beta_i - \alpha_i$ and

$$\mu_f(Z) < \sum_i V(f; Z_i) < \sum_i (\beta_i - \alpha_i) < \epsilon\,.$$

Since ϵ is arbitrary, we obtain that $\mu_f(Z) = 0$. $\square$

Lemma 8. *Let $E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$. If $f \in ACG$ on E, then $\mu_f(E) = 0$.*

PROOF. Let $\epsilon > 0$. Since $f \in ACG$ on E, there exists an E-form $\{E_n\}$ such that f is AC on each E_n. For $\epsilon/2^n$, let $r_n > 0$ be given by the fact that f is AC on E_n. Then $\sum_n V(f; E_n; r_n) < \epsilon$. Hence $\mu_f(E) = 0$. $\square$

Lemma 9. *Let $E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$. Suppose that $m^*(f(E)) = 0$ and that there exists an E-form $\{E_n\}$ such that f is monotone on each E_n. Then $\mu_f(E) = 0$.*

PROOF. Since $m^*(f(E)) = 0$ it follows that $m^*(f(E_n)) = 0$ for each n. Let $\epsilon > 0$ and let $G_n = \cup_i (\alpha_{ni}, \beta_{ni})$ be an open set such that $f(E_n) \subset G_n$ and $m^*(G_n) < \epsilon/2^n$, where $\{(\alpha_{ni}, \beta_{ni})\}_i$ is a sequence of nonoverlapping open intervals. Let $E_{ni} = \{x \in E_n : f(x) \in (\alpha_{ni}, \beta_{ni})\}$. Then

$$\sum_n \sum_i V(f; E_{ni}) < \sum_n \sum_i (\beta_{ni} - \alpha_{ni}) < \epsilon\,.$$

Therefore $\mu_f(E) = 0$. $\square$

Lemma 10. *Let E be a real bounded set and $f : E \to \mathbb{R}$. If $f \in VB \cap (N)$ on E, then $\mu_f(E) = 0$.*

PROOF. Let $a, b \in \mathbb{R}$ such that $E \subseteq [a, b]$. Since $f \in VB$ on E, there exists a function $G : [a, b] \to \mathbb{R}$ such that $G_{|E} = f$ and $G \in VB$ on $[a, b]$ (see Lemma 4.1 of [5], p. 221). Let $A = \{x \in [a, b] : G^{'}(x)$ does not exist (finite or infinite)$\}$. By Lemma 7, we obtain that $\mu_G(A) = 0$. Hence $\mu_f(A \cap E) = 0$. Let $B = \{x \in E : G^{'}(x) = \pm\infty\}$. Then $m^*(B) = 0$. Since $f \in (N)$ on E, we have that $m^*(f(B)) = 0$. Also there exists a B-form $\{B_n\}$ such that G is monotone on each B_n (see for example the proof of Theorem 10.1 of [5], p. 235). By Lemma 9, it follows that $\mu_f(B) = 0$. Let $C = \{x \in [a, b] : G^{'}(x)$ exists and is finite$\}$. Then $G \in AC^*G \subset ACG$ on C. Hence by Lemma 8, $\mu_G(C) = 0$; so $\mu_f(E \cap C) = 0$. Thus $\mu_f(E) = 0$ (see Proposition 1, (iv)). □

Lemma 11. *Let Q be a real compact set, $E \subseteq Q$, $f : Q \to \mathbb{R}$ and $r > 0$. If f is continuous on Q, then $V(f; E; r) = V(f; \overline{E}; r)$.*

PROOF. We always have $V(f; E; r) \le V(f; \overline{E}; r)$. We show the converse. Let $V(f; E; r) = M < +\infty$ (if $M = +\infty$ there is nothing to prove). Let $\{[a_i, b_i]\}_{i=1}^m$ be a finite set of nonoverlapping closed intervals with the endpoints in $\overline{E}$ and $\sum_{i=1}^m (b_i - a_i) < r$. Since f is continuous on Q, for $\epsilon > 0$, there exists $a_i^*, b_i^* \in E$ such that $\{[a_i^*, b_i^*]\}_{i=1}^m$ are nonoverlapping closed intervals, with

$$\sum_{i=1}^m (b_i^* - a_i^*) < r, \quad |f(a_i) - f(a_i^*)| < \epsilon/2m \;\text{ and }\; |f(b_i) - f(b_i^*)| < \epsilon/2m\,.$$

It follows that

$$\sum_{i=1}^m |f(b_i) - f(a_i)| \le \sum_{i=1}^m \big(|f(b_i) - f(b_i^*)| + |f(b_i^*) - f(a_i^*)| + |f(a_i) - f(a_i^*)|\big) < M + \epsilon\,.$$

Since ϵ is arbitrary, it follows that $V(f; \overline{E}; r) \le M$. □

Lemma 12. *Let E be a real bounded set and $f : E \to \mathbb{R}$. If $f \in VB$ and $\mu_f(E) = 0$, then $[PV](f; E) = 0$. Hence $PV(f; E) = 0$.*

PROOF. Let $\epsilon > 0$. Since $\mu_f(E) = 0$, there exist an E-form $\{E_n\}$ and a sequence of positive numbers $\{r_n\}$ such that

$$\sum_n V(f; E_n; r_n) < \frac{\epsilon}{2}\,. \tag{3}$$

Let $a, b \in \mathbb{R}$ such that $E \subseteq [a, b]$. Since $f \in VB$ on E, there exists a function $F : [a, b] \to \mathbb{R}$ such that $F_{|E} = f$ and $F \in VB$ on $[a, b]$ (see Lemma 4.1 of [5],

p. 221). Let $D = \{d_n\}$ be the set of all discontinuity points of F. For each d_n there exist $I_n = (p_n, d_n)$ and $J_n = (d_n, q_n)$ (see Lemma 3) such that

$$\sum_k \big(V(F; E \cap I_{nk}) + V(F; E \cap J_{nk})\big) < \frac{\epsilon}{2^{n+1}}\,, \tag{4}$$

whenever $\{I_{nk}\}_k$ and $\{J_{nk}\}_k$ are two sequences of closed intervals abutting end to end with $\cup_k I_{nk} = I_n$ and $\cup_k J_{nk} = J_n$. Let $Q = [a,b] \setminus (\cup_n (I_n \cup J_n))$. Then Q is a compact set and

$$F_{|Q} \text{ is continuous.} \tag{5}$$

Let $Q_n = \overline{Q \cap E_n}$. Clearly $\{E \cap Q_n\}_n \cup \{E \cap I_{nk}\}_{n,k} \cup \{E \cap J_{nk}\}_{n,k}$ is a closed E-form. By (3), (4), (5) and Lemma 11, it follows that

$$\sum_n V(F; Q_n; r_n) + \sum_n \sum_k V(F; E \cap I_{nk}) + \sum_n \sum_k V(F; E \cap J_{nk}) < \epsilon\,.$$

Thus $[PV](f; E) = 0$. That $PV(f; E) = 0$ follows by Proposition 1, (ii). □

Corollary 2. *Let E be a real bounded set and $f : E \to \mathbb{R}$. If $f \in VB \cap (N)$, then $[PV](f; E) = 0$.*

PROOF. By Lemma 10, $f \in VB$ and $\mu_f(E) = 0$. Now by Lemma 12, it follows that $[PV](f; E) = 0$. □

Lemma 13. *Let $E \subset \mathbb{R}$ and $f : E \to \mathbb{R}$. If $\mu_f(E) < +\infty$, then $f \in VBG$ on E.*

PROOF. Since $\mu_f(E) < +\infty$, there exist an E-form $\{E_n\}$ and a sequence $\{r_n\}$ of positive numbers such that $\sum_n V(f; E_n; r_n) < \mu_f(E) + 1$. Hence $V(f; E_n; r_n) < \mu_f(E) + 1$. Then $f \in VB$ on each E_{nk}, $k = 0, \pm1, \pm2, \pm3, \dots$, where

$$E_{nk} = E_n \cap \left[k\frac{r_n}{2}, (k+1)\frac{r_n}{2}\right].$$

It follows that $f \in VBG$ on E. □

Theorem 4. *Let E be a real bounded set and $f : E \to \mathbb{R}$. The following assertions are equivalent.*

(i) $f \in VBG \cap (N)$ *on* E.

(ii) f *is* $[PAC]G$ *on* E.

(iii) f *is* PAC *on* E.

(iv) f *is* $(PAC)G$ *on* E.

PROOF. (i) $\Rightarrow$ (ii) Since $f \in VBG \cap (N)$ on E, there exists an E-form $\{E_n\}$ such that $f \in VB \cap (N)$ on each E_n. By Corollary 2, f is $[PAC]$ on each E_n. Therefore f is $[PAC]G$ on E.

(ii) $\Rightarrow$ (iii) See Proposition 1, (i), (iv).

(iii) $\Rightarrow$ (ii) By Lemma 13, $f \in VBG$ on E. Then there is an E-form $\{E_n\}$ such that $f \in VB$ on each E_n. Clearly $\mu_f(E_n) = 0$. By Lemma 12, we obtain that $[PV](f; E_n) = 0$. Hence $f \in [PAC]G$ on E.

(ii) $\Rightarrow$ (iv) See Proposition 1, (ii).

(iv) $\Rightarrow$ (i) Since f is $(PAC)G$ on E, there exists an E-form $\{E_n\}$ such that $PV(f; E_n) = 0$ for each n. By Theorem 1, (iii), (iv), it follows that $f \in [VBG] \cap (N)$ on each E_n. Hence $f \in VBG \cap (N)$ on E. $\square$

Remark 3. Sarkhel and De introduced the following condition (Definition 5.1 of [7] or the remark on p. 337 of [8]).

- A function $f : E \to \mathbb{R}$, $E \subset \mathbb{R}$, is said to be **PAC** on E, if there is a countable subset E_1 of E such that f is (PAC) on $E \setminus E_1$.

Clearly **PAC** differs from PAC defined in the present paper. Also in [7] the following is proved.

- f is said to be **PACG** on E if there exists an E-form $\{E_n\}$ such that f is **PAC** on each E_n.

Clearly $(PAC)G \subset$ **PACG** (see Definition 5). In [7] (see Theorem 5.2 and the proof of Theorem 5.3), Sarkhel and De showed that **PACG** $\subseteq VBG \cap (N)$. By Theorem 4, (i), (iv), it follows that **PACG** $= [PAC]G = (PAC)G = PAC = VBG \cap (N)$ on a real bounded set E.

In [3], Gordon introduced the AK_N integral.

- A function $f : [a, b] \to \mathbb{R}$ is said to be AK_N integrable, if there is a function $F : [a, b] \to \mathbb{R}$ such that $F \in VBG \cap (N) \cap$(approximately continuous) and $F'_{ap} = f$ $a.e.$ on $[a, b]$.

However, in his proof of the uniqueness of this integral he neglected to show that the difference of two functions belonging to $F \in VBG \cap (N) \cap$ (approximately continuous) is still (N). In [2] we show that $VBG \cap (N)$ is a linear space for Borel measurable functions and give a complete proof that the AK_N integral is well defined.

In [7] (Definition 7.1), Sarkhel and De introduced the PD-integral:

- A function $f : [a, b] \to \mathbb{R}$ is said to be PD integrable, if there is a function $F : [a, b] \to \mathbb{R}$ such that $F \in$ **PACG** $\cap$ (proximally continuous) and $F'_{ap} = f$ $a.e.$ on $[a, b]$.

Since the class of approximately continuous functions (see [7]) contains strictly the class of approximately continuous functions and **PACG** $= VBG \cap (N)$, it follows that the AK_N-integral is a special case of the PD-integral.

Corollary 3. *Let E be a real bounded set and $\mathcal{A} = \{f : E \to \mathbb{R} : f \in VBG \cap (N)$ on $E\}$. Then $\mathcal{A}$ is an algebra.*

PROOF. Let $f, g \in \mathcal{A}$, $\alpha, \beta \in \mathbb{R}$. By Theorem 4, (i), (iv), we obtain that $f, g \in (PAC)G$ on E. Then there exists an E-form $\{E_n\}$, such that $f, g \in (PAC)$ on each E_n. Hence $PV(f; E_n) = PV(g; E_n) = 0$ for each n. By Theorem 1, (i), $PV(\alpha f + \beta g; E_n) = 0$. Hence $\alpha f + \beta g \in (PAC)G = VBG \cap (N)$ on E (see Theorem 4, (i), (iv)). It follows that $\mathcal{A}$ is a real linear space. But $f \cdot g \in (PAC)$ on each E_n (see Theorem 1, (vi)); so $f \cdot g \in (PAC)G = VBG \cap (N)$ on E (see Theorem 4, (i), (iv)). Therefore $\mathcal{A}$ is an algebra. □

Theorem 5. *Let E be a bounded Lebesgue measurable set, and let $f : E \to \mathbb{R}$. If f is a Lebesgue measurable function the following assertions are equivalent.*

(i) $f \in VBG \cap (N)$ *on* E

(ii) $f \in [PAC]G$ *on* E.

(iii) $f \in PAC$ *on* E.

(iv) $f \in (PAC)G$ *on* E.

(v) $f \in VBG \cap (N)$ *on* Z, *whenever* Z *is a null subset of* E.

(vi) $f \in [PAC]G$ *on* Z, *whenever* Z *is a null subset of* E.

(vii) $f \in PAC$ *on* Z, *whenever* Z *is a null subset of* E.

(viii) $f \in (PAC)G$ *on* Z, *whenever* Z *is a null subset of* E.

PROOF. (i) $\Leftrightarrow$ (v) follows by Theorem 3; (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) and (v) $\Leftrightarrow$ (vi) $\Leftrightarrow$ (vii) $\Leftrightarrow$ (viii) follow by Theorem 4. □

Corollary 4. *Let E be a real compact set and $f : E \to \mathbb{R}$. If f is continuous on E, then the following assertions are equivalent.*

(i) $f \in ACG$ *on* E.

(ii) $f \in VBG \cap (N)$ *on* E.

(iii) $f \in [PAC]$ *on* E.

(iv) $f \in (PAC)$ *on* E.

(v) *$f \in PAC$ on E;*

(vi) *$f \in ACG$ on Z, whenever Z is a null subset of E.*

(vii) *$f \in [PAC]$ on Z, whenever Z is a null subset of E.*

(viii) *$f \in (PAC)$ on Z, whenever Z is a null subset of E.*

(ix) *$f \in PAC$ on Z, whenever Z is a null subset of E.*

Proof. (i) $\Leftrightarrow$ (ii) See Theorem 6.8 of [5], p. 228.

(ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) Since f is continuous, it follows that $VBG = [VBG]$. Now the assertions follow by Theorem 2.

(ii) $\Leftrightarrow$ (v) See Theorem 5, (i), (iii).

(i) $\Leftrightarrow$ (vi) See Lemma 5.

(vi) $\Rightarrow$ (vii) Let Z be a null subset of E such that $f \in ACG$ on Z. Since f is continuous on E, it follows that $f \in [ACG]$ on Z. By Lemma 2, $f \in [PAC]$ on Z.

(vii) $\Rightarrow$ (viii) See Proposition 1, (ii).

(viii) $\Rightarrow$ (ii) See Theorem 5, (viii), (i).

(ix) $\Leftrightarrow$ (ii) See Theorem 5, (vii), (i). □

7 Enhancements of the Ordinary Variation

Definition 6. Let E be a real bounded set, $f : E \to \mathbb{R}$ and $\delta : E \to (0, +\infty)$. Put

- $V^1_\delta(f; E) = \sup\{\sum_{i=1}^n |f(b_i) - f(a_i)| : \{[a_i, b_i]\}_{i=1}^n$ is a finite set of nonoverlapping closed intervals with the endpoints in E, such that $0 < b_i - a_i < \min\{\delta(a_i), \delta(b_i)\}\}$;
- $V^2_\delta(f; E) = \sup\{\sum_{i=1}^n \mathcal{O}(f; [a_i, b_i] \cap E) : \{[a_i, b_i]\}_{i=1}^n$ is a finite set of nonoverlapping closed intervals with the endpoints in E, such that $0 < b_i - a_i < \min\{\delta(a_i), \delta(b_i)\}\}$;
- $V^3_\delta(f; E) = \sup\{\sum_{i=1}^n \mathcal{O}(f; [a_i, b_i] \cap E) : \{[a_i, b_i]\}_{i=1}^n$ is a finite set of nonoverlapping closed intervals; there exists $x_i \in [a_i, b_i] \cap E$ such that $[a_i, b_i] \subset (x_i - \delta(x_i), x_i + \delta(x_i))\}$;
- $V^4_\delta(f; E) = \sup\{\sum_{i=1}^n \mathcal{O}(f; [a_i, b_i] \cap E) : \{[a_i, b_i]\}_{i=1}^n$ is a finite set of nonoverlapping closed intervals; there exists $x_i \in \{a_i, b_i\} \cap E$ such that $[a_i, b_i] \subset (x_i - \delta(x_i), x_i + \delta(x_i))\}$;
- $V^5_\delta(f; E) = \sup\{\sum_{i=1}^n \mathcal{O}(f; [a_i, b_i] \cap E) : \{[a_i, b_i]\}_{i=1}^n$ is a finite set of nonoverlapping closed intervals with the endpoints in E, such that $0 < b_i - a_i < \delta(a_i) + \delta(b_i)$;

- $\nu^i_f(E) = \inf_\delta\{V^i_\delta(f;E)\}$, $i = 1,2,3,4,5$.

Theorem 6. *Let E be a real bounded set, $f, g : [a,b] \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$.*

(i) $\nu^i_{\alpha f+\beta g}(E) \le |\alpha| \cdot \nu^i_f(E) + |\beta| \cdot \nu^i_g(E)$, $i = 1,2,3,4,5$.

(ii) If $\nu^i_g(E) = 0$, *then* $\nu^i_{f+g}(E) = \nu^i_f(E)$, $i = 1,2,3,4,5$.

(iii) If $\max_{x\in E}\{|f(x)|, |g(x)|\} = M < +\infty$, *then*

$$\nu^i_{f\cdot g}(E) \le M \cdot \big(\nu^i_f(E) + \nu^i_g(E)\big).$$

(iv) $PV(f;E) \le \nu^1_f(E) \le \nu^2_f(E) \le \nu^3_f(E) = \nu^4_f(E) \le \nu^5_f(E)$.

PROOF. Let $\delta_1, \delta_2 : E \to (0,+\infty)$ and let $\delta(x) = \min\{\delta_1(x), \delta_2(x)\}$.

(i) We have

$$\nu^i_{\alpha f+\beta g}(E) \le V^i_\delta(\alpha f+\beta g;E) \le |\alpha| \cdot V^i_\delta(f;E) + |\beta| \cdot V^i_\delta(g;E) \le$$
$$\le |\alpha| \cdot V^i_{\delta_1}(f;E) + |\beta| V^i_{\delta_2}(g;E).$$

Hence $\nu^i_{\alpha f+\beta g}(E) \le |\alpha| \cdot \nu^i_f(E) + |\beta| \cdot \nu^i_g(E)$.

(ii) Clearly $\nu^i_g(E) = 0$ implies that $\nu^i_{-g}(E) = 0$. By (i) we have

$$\begin{aligned}\nu^i_f(E) &= \nu^i_{f+g-g}(E) \le \nu^i_{f+g}(E) + \nu^i_{-g}(E)\\ &= \nu^i_{f+g}(E) \le \nu^i_f(E) + \nu^i_g(E) = \nu^i_f(E)\,.\end{aligned}$$

Therefore $\nu^i_{f+g}(E) = \nu^i_f(E)$.

(iii) Let $x, y \in E$, $x < y$. Then

$$\begin{aligned}\big|f(y)\cdot g(y) - f(x)\cdot g(x)\big| &= \big|g(y)\cdot(f(y)-f(x)) + f(x)\cdot(g(y)-g(x))\big|\\ &\le M\cdot\big(\big|f(y)-f(x)\big| + |g(y)-g(x)|\big)\,.\end{aligned}$$

It follows that

$$\begin{aligned}\nu^i_{f\cdot g}(E) &\le V^i_\delta(f\cdot g;E) \le M\cdot\big(V^i_\delta(f;E) + V^i_\delta(g;E)\big)\\ &\le M\cdot\big(V^i_{\delta_1}(f;E) + V^i_{\delta_2}(g;E)\big)\,.\end{aligned}$$

Therefore $\nu^i_{f\cdot g}(E) \le M\cdot(\nu^i_f(E) + \nu^i_g(E))$.

(iv) We may suppose that $\nu^1_f(E) = M < +\infty$. (If $M = +\infty$, there is nothing to prove.) For $\epsilon > 0$ there is a $\delta : E \to (0,+\infty)$ such that $V^1_\delta(f;E) < M + \epsilon$. Let $E_k = \{x \in E : \delta(x) > 1/k\}$, $k = 1,2,\ldots$. Then $\{E_k\}$ is an E-chain. Fix some k and let $\{[a_i,b_i]\}_{i=1}^m$ be a finite set of nonoverlapping closed intervals with the endpoints in E_k, $\sum_{i=1}^m(b_i - a_i) < 1/k$. Clearly

$$y - x < \min\{\delta(x), \delta(y)\}, \quad \text{whenever } x, y \in E_k \text{ and } 0 < y - x < \frac{1}{k}\,.$$

It follows that $\sum_{i=1}^{m} \mathcal{O}(f;[a_i,b_i]\cap E_k) < M+\epsilon$. Then $V(f;E_k;\frac{1}{k}) \le M+\epsilon$. Hence $V(f;E_k;0) \le M+\epsilon$, for each k. Since ϵ is arbitrary, we obtain that $PV(f;E) \le M$.

For $\delta : E \to (0,+\infty)$ we have $V^1_\delta(f;E) \le V^2_\delta(f;E) \le V^3_\delta(f;E)$. It follows that $\nu^1_f(E) \le \nu^2_f(E) \le \nu^3_f(E)$.
We show that $\nu^3_f(E) = \nu^4_f(E)$. Clearly $\nu^3_f(E) \ge \nu^4_f(E)$. Let $\{[a_i,b_i]\}_{i=1}^n$ be a finite set of nonoverlapping closed intervals such that there exists $x_i \in [a_i,b_i]\cap E$, with $[a_i,b_i] \subset (x_i-\delta(x_i), x_i+\delta(x_i))$. We may suppose without loss of generality that each $x_i \in (a_i,b_i)$. Then

$$\sum_{i=1}^{n} \mathcal{O}(f;[a_i,b_i]\cap E) \le \sum_{i=1}^{n} \mathcal{O}(f;[a_i,x_i]\cap E) + \sum_{i=1}^{n} \mathcal{O}(f;[x_i,b_i]\cap E) \le V^4_\delta(f;E).$$

Hence $V^3_\delta(f;E) \le V^4_\delta(f;E)$. Thus $\nu^3_f(E) \le \nu^4_f(E)$; so $\nu^3_f(E) = \nu^4_f(E)$.
We show that $\nu^4_f(E) \le \nu^5_f(E)$. Let $\{[a_i,b_i]\}_{i=1}^n$ be a finite set of nonoverlapping closed intervals such that there exists $x_i \in \{a_i,b_i\}\cap E$, with $[a_i,b_i] \subset (x_i - \delta(x_i), x_i+\delta(x_i))$. We may suppose without loss of generality that each $x_i = a_i$. For $\epsilon > 0$ there exists $y_i \in [a_i,b_i]\cap E$ such that

$$\mathcal{O}(f;[a_i,b_i]\cap E) \le \mathcal{O}(f;[a_i,y_i]\cap E) + \frac{\epsilon}{2^i}\,.$$

Then $y_i - a_i < \delta(a_i) + \delta(y_i)$. Hence

$$\sum_{i=1}^{n} \mathcal{O}(f;[a_i,b_i]\cap E) < \epsilon + \sum_{i=1}^{n} \mathcal{O}(f;[a_i,y_i]\cap E) < \epsilon + V^5_\delta(f;E)\,.$$

It follows that $V^4_\delta(f;E) \le V^5_\delta(f;E)$. Therefore $\nu^4_f(E) \le \nu^5_f(E)$. □

Lemma 14. *Let E be real bounded set and $f : E \to \mathbb{R}$. Then $m^*(f(E)) \le \nu^i_f(E)$, $i = 1,2,3,4,5$.*

PROOF. By Theorem 6, (iv), it is sufficient to prove the assertion for $i = 1$. Let $\nu^1_f(E) = M < +\infty$. (If $M = +\infty$, there is nothing to prove.) For $\epsilon > 0$, there exists a $\delta : E \to (0,+\infty)$ such that $V^1_\delta(f;E) < M + \epsilon/2$. Let $E_n = \{x \in E : \delta(x) > (b-a)/n\}$. Then $\{E_n\}$ is an E-chain. Fix some n and let

$$E_{nm} = E_n \cap \left[\frac{m-1}{n}(b-a), \frac{m}{n}(b-a)\right], \quad m = 1,2,\ldots,n\,.$$

Clearly $m^*(f(E_n)) \le \sum_{m=1}^{n} m^*(f(E_{nm}))$. For each m let $x_m, y_m \in E_{nm}$, $x_m < y_m$, such that

$$\mathcal{O}(f;E_{nm}) \le |f(y_m) - f(x_m)| + \frac{\epsilon}{2^{m+1}}\,.$$

Clearly $y_m - x_m < \min\{\delta(x_m), \delta(y_m)\}$. Then

$$\sum_{m=1}^{n} m^*(f(E_{nm})) \le \sum_{m=1}^{n} \mathcal{O}(f;E_{nm}) \le \sum_{m=1}^{n} |f(y_m)-f(x_m)| + \sum_{m=1}^{n} \frac{\epsilon}{2^{m+1}} < V_\delta^1(f;E) + \frac{\epsilon}{2} \le M+\epsilon\,.$$

Therefore $m^*(f(E_n)) \le \sum_{m=1}^{n} m^*(f(E_{nm})) \le M+\epsilon$. Since $\{E_n\}$ is increasing and $\cup E_n = E$, it follows that $m^*(f(E)) \le M+\epsilon$. Since ϵ is arbitrary, we obtain that $m^*(f(E)) \le M$. $\square$

Lemma 15. *Let E be a real bounded set and $f : E \to \mathbb{R}$. Then $\nu_f^1(E) \le \sum_n \nu_f^1(E_n)$, whenever $\{E_n\}$ is a closed E-form.*

PROOF. Suppose that $\sum_n \nu_f^1(E_n) = M < +\infty$. (If $M = +\infty$ there is nothing to prove.) For $\epsilon > 0$, let $\delta_n : E_n \to (0,+\infty)$ such that $V_{\delta_n}^1(f;E_n) < \nu_f^1(E_n) + \epsilon/2^n$. By Lemma 1, there is a closed E-chain $\{Q_n\}$ such that $Q_n = \cup_{k\le n} Q_{kn}$, where $Q_{kn} \subset Q_{km} \subset E_k$ for all k and $m \ge n \ge k$, and $d(Q_{in}, Q_{jn}) \ge 1/n$ for $i \ne j$. Let $\delta : E \to (0,+\infty)$,

$$\delta(x) = \begin{cases} \delta_1(x) & \text{if } x \in Q_1 \\ \min\left\{\frac{1}{2n}, \delta_1(x), \dots, \delta_n(x), d(x, Q_{n-1})\right\} & \text{if } x \in Q_n \setminus Q_{n-1},\ n \ge 2. \end{cases}$$

(Since Q_{n-1} is closed in E we have that $d(x,Q_{n-1}) > 0$ for $x \in Q_n \setminus Q_{n-1}$.) Let $\{[a_p,b_p]\}_{p=1}^q$ be a finite set of nonoverlapping closed intervals with the endpoints in E, such that $b_p - a_p < \min\{\delta(a_p), \delta(b_p)\}$, $p = 1,2,\dots,n$. Fix some p and let s be the first positive integer such that $a_p, b_p \in Q_s$. Suppose for example that $a_p \in Q_s \setminus Q_{s-1}$. From the definition of δ it follows that $b_p \in Q_s \setminus Q_{s-1}$. Then

$$\delta(a_p) \le \min\left\{\frac{1}{2s}, \delta_1(a_p), \dots, \delta_s(a_p)\right\}$$

and

$$\delta(b_p) \le \min\left\{\frac{1}{2s}, \delta_1(b_p), \dots, \delta_s(b_p)\right\}.$$

Let $k \le s$ be such that $a_p \in Q_{ks}$. Since $b_p - a_p < \delta(a_p) < 1/s$, it follows that b_p belongs to the same $Q_{ks} \subset E_k$. Clearly $b_p - a_p < \min\{\delta_k(a_p), \delta_k(b_p)\}$. It follows that

$$\sum_p |f(b_p) - f(a_p)| \le \sum_k \left(\nu_f^1(E_k) + \frac{\epsilon}{2^k}\right) < M + \epsilon\,.$$

Hence $V_\delta^1(f;E) \le M+\epsilon$. Thus $\nu_f^1(E) \le M+\epsilon$. Since ϵ is arbitrary, we obtain that $\nu_f^1(E) \le M$. □

8 A Characterization of a Lebesgue Measurable Function f, Satisfying VBG ∩ (N), Using $\mu_{\mathbf{f}}^{\mathbf{i}}$

Lemma 16. *Let $f:[a,b]\to\mathbb{R}$ and let $Z=\{x\in[a,b]: f$ is continuous at x; $f^{'}(x)$ does not exist (finite or infinite)$\}$. If $f\in VB$ on $[a,b]$, then $\nu_f^i(Z)=0$, $i=1,2,3,4,5$.*

PROOF. By Theorem 6, (iv), it is sufficient to prove the assertion for $i=5$. By Lemma 6, $m^*(S(Z))=0$. For $\epsilon>0$, there exists an open set G such that $S(Z)\subset G=\cup_i(\alpha_i,\beta_i)$ and $m(G)<\epsilon$, where $\{(\alpha_i,\beta_i)\}_i$ are nonoverlapping open intervals and $S(Z)\cap(\alpha_i,\beta_i)\neq\emptyset$. Since f is continuous at each point of Z, using Theorem 8.4 of [5] (p. 123), it follows that S is continuous at each point of Z. Let $Z_i=\{x\in Z: S(x)\in(\alpha_i,\beta_i)\}$. Clearly $Z=\cup Z_i$. Then there exists a $\delta:Z\to(0,+\infty)$ such that $S((x-\delta(x),x+\delta(x)))\subset(\alpha_i,\beta_i)$ whenever $x\in Z_i$. Let $\{[a_j,b_j]\}_{j=1}^m$ be a finite set of nonoverlapping closed intervals with the endpoints in Z and $0<b_j-a_j<\delta(a_j)+\delta(b_j)$. Fix some $[a_j,b_j]$ and let $c_j\in(a_j,a_j+\delta(a_j))\cap(b_j-\delta(b_j),b_j)$. Suppose that $a_j\in Z_i$ and $b_j\in Z_k$. Then $S(c_j)\in(\alpha_i,\beta_i)\cap(\alpha_k,\beta_k)$; so $i=k$ and $b_j\in Z_i$ too. Consequently $S(b_j)\in(\alpha_i,\beta_i)$. Hence $\alpha_i<S(a_j)<S(b_j)<\beta_i$ (because S is strictly increasing). But $|f(y)-f(x)|\le S(y)-S(x)$ whenever $a\le x<y\le b$. It follows that $\sum_{j=1}^m \mathcal{O}(f;[a_j,b_j]\cap Z)\le\sum_{j=1}^m S(b_j)-S(a_j)\le\sum_i(\beta_i-\alpha_i)<\epsilon$. Then $V_\delta^5(f;Z)\le\epsilon$. Hence $\nu_\delta^5(Z)\le\epsilon$. Since ϵ is arbitrary, $\nu_f^5(Z)=0$. □

Lemma 17. *Let E be a bounded null set and $f:E\to\mathbb{R}$. If $f\in AC$ on E, then $\nu_f^i(E)=0$, $i=1,2,3,4,5$.*

PROOF. By Theorem 6, (iv), it is sufficient to prove the assertion for $i=5$. For $\epsilon>0$ let $\delta>0$ be given by the fact that $f\in AC$ on E. Since E is a null set, there exists an open set G such that $E\subset G$ and $m(G)<\delta$. Let $\eta:E\to(0,+\infty)$ be such that $(x-\eta(x),x+\eta(x))\subset G$. Let $\{[a_i,b_i]\}_{i=1}^m$ be a finite set of nonoverlapping closed intervals with the endpoints in E, such that $0<b_i-a_i<\eta(a_i)+\eta(b_i)$. Clearly $[a_i,b_i]\subset G$. It follows that $\sum_{i=1}^m(b_i-a_i)<m(G)$. Since $\sum_{i=1}^m\mathcal{O}(f;[a_i,b_i]\cap E)<\epsilon$, it follows that $V_\eta^5(f;E)\le\epsilon$, and consequently $\nu_f^5(E)\le\epsilon$. Since ϵ is arbitrary, we obtain that $\nu_f^5(E)=0$. □

Lemma 18. *Let $f:[a,b]\to\mathbb{R}$ and $A\subseteq\{x\in[a,b]: f$ is continuous at $x\}$. If f is increasing on A, then $m^*(f(A))=\nu_f^i(A)$, $i=1,2,3,4,5$.*

PROOF. We always have $m^*(f(A)) \le \nu_f^i(A)$ (see Lemma 14). We show the converse. Suppose that $m^*(f(A)) < +\infty$. (If $m^*(f(A)) = +\infty$, there is nothing to prove.) By Theorem 6, (iv), it is sufficient to prove the assertion for $i = 5$. For $\epsilon > 0$ let G be an open set such that $f(A) \subset G$ and $m^*(f(A)) + \epsilon > m(G)$. Let $G = \cup_i(\alpha_i, \beta_i)$, where $\{(\alpha_i, \beta_i)\}_i$ is a sequence of nonoverlapping open intervals. Then $A = \cup_i A_i$, where $A_i = \{x \in A : f(x) \in (\alpha_i, \beta_i)\}$. Since f is continuous at each point of A, there is a $\delta : A \to (0, +\infty)$ such that $f((x - \delta(x), x + \delta(x))) \subset (\alpha_i, \beta_i)$ if $x \in A_i$. (This is possible because f is continuous at each point of A.) Let $\{[a_j, b_j]\}_{j=1}^m$ be a finite set of nonoverlapping closed intervals with the endpoints in A, with $0 < b_j - a_j < \delta(a_j) + \delta(b_j)$. Let $c_j \in (a_j, a_j + \delta(a_j)) \cap (b_j - \delta(b_j), b_j)$. Suppose that $a_j \in A_i$ and $b_j \in A_k$. Then $f(c_j) \in (\alpha_i, \beta_i) \cap (\alpha_k, \beta_k)$; so $i = k$ and $b_j \in A_i$ too. Consequently $f(b_j) \in (\alpha_i, \beta_i)$. Since f is increasing on A, we have $\alpha_i < f(a_j) \le f(b_j) < \beta_i$ and $\sum_{j=1}^m \mathcal{O}(f; [a_j, b_j] \cap A) = \sum_{j=1}^m |f(b_j) - f(a_j)| \le \sum_i(\beta_i - \alpha_i) < m^*(f(E)) + \epsilon$. Thus $V_\delta^5(f; A) \le m^*(f(A)) + \epsilon$, and since ϵ is arbitrary, $\nu_f^5(A) \le m^*(f(A))$. $\square$

Lemma 19. *Let E be a real bounded set and $f : E \to \mathbb{R}$. If $f \in VB$ on E, then the following assertions are equivalent for $i = 1, 2, 3, 4, 5$:*

(i) $f \in (N)$ on E;

(ii) for every null set $Z \subseteq E$ there exists a Z-form $\{Z_n\}$ such that $\nu_f^i(Z_n) = 0$, for each n.

PROOF. (i) $\Rightarrow$ (ii) Since $f \in VB$ on E, there exists a function $F : [a, b] \to \mathbb{R}$ such that $F_{|E} = f$ and $F \in VB$ on $[a, b]$ (see Lemma 4.1 of [5], p. 221). Let $D = \{x \in [a, b] : F$ is not continuous at $x\}$. Then D is countable (see [4], p. 219). Let $A = \{x \in [a, b] \setminus D : F'(x)$ does not exist (finite or infinite)$\}$. By Lemma 16, $\nu_F^i(A) = 0$; so $\nu_f^i(A \cap E) = 0$. Let $B = \{x \in E \setminus D : F'(x) = \pm\infty\}$. Then $m^*(B) = 0$. Since $F \in (N)$ on E, we obtain that $m^*(F(B)) = 0$. Also, there exists a B-form $\{B_n\}$ such that F is monotone on each B_n. Then $\nu_f^i(B_n) = 0$ (see Lemma 18). Let $C = \{x \in [a, b] : F'(x)$ exists and is finite$\}$. Then $F \in AC^*G \subset ACG$ on C. It follows that there exists a C-form $\{C_n\}$ such that $F \in AC$ on each C_n. Let Z be a null subset of E and $Z_o = A \cap Z$. Then $\nu_f(Z_o) = 0$. Let $D_1 = Z \cap D = \{d_1, d_2, \ldots\}$. Clearly $\nu_f^i(\{d_k\}) = 0$ and $\nu_f^i(Z \cap B_n) = 0$. By Lemma 17, $\nu_f^i(Z \cap C_n) = 0$. Therefore we have (ii).

(ii) $\Rightarrow$ (i) This implication is always true. Let Z be a null subset of E. Then there exists a Z-form $\{Z_n\}$ such that $\nu_f^i(Z_n) = 0$. By Lemma 14, $m^*(f(Z_n)) = 0$. It follows that $m^*(f(Z)) = 0$. Hence $f \in (N)$ on E. $\square$

Theorem 7. *Let E be a real bounded set and $\mathcal{A} = \{f : E \to \mathbb{R} : f \in VB \cap (N)\}$. Then $\mathcal{A}$ is an algebra.*

PROOF. Let $f, g \in \mathcal{A}$, $\alpha, \beta \in \mathbb{R}$. Clearly $\alpha f + \beta g \in VB$. Let Z be a null subset of E. By Lemma 19, there exists a Z-form $\{Z_n\}$ such that $\nu_f^i(Z_n) = \nu_g^i(Z_n) = 0$, $i \in \{1,2,3,4,5\}$. By Theorem 6, (i) we have that $\nu^i_{\alpha f + \beta g}(Z_n) = 0$. Hence $\alpha f + \beta g \in (N)$ on E (see Lemma 19). Therefore $\mathcal{A}$ is a real linear space. Clearly $f \cdot g \in VB$ on E, and f and g are bounded on E. By Theorem 6, (iii), $\nu^i_{f \cdot g}(Z_n) = 0$, $i \in \{1,2,3,4,5\}$, and by Lemma 19, we obtain that $f \cdot g \in (N)$ on the set E. $\square$

Remark 4. As a consequence of Theorem 7, we obtain again Corollary 3.

Lemma 20. *Let E be a real bounded set and $f : E \to \mathbb{R}$. If $\nu_f^i(E) < +\infty$, $i \in \{1,2,3,4,5\}$, then $f \in [VBG]$ on E.*

PROOF. See Theorem 6, (iv) and Theorem 1, (iv). $\square$

Theorem 8. *Let E be a bounded Lebesgue measurable set and $f : E \to \mathbb{R}$ a Lebesgue measurable function. For $i \in \{1,2,3,4,5\}$. The following assertions are equivalent.*

(i) $f \in VBG \cap (N)$ on E.

(ii) for every null set $Z \subseteq E$ there is a Z-form $\{Z_n\}$ such that $\nu_f^i(Z_n) = 0$.

PROOF. (i) $\Rightarrow$ (ii) See Lemma 19,

(ii) $\Rightarrow$ (i) Let Z be a null subset of E. By hypothesis and Lemma 20, $f \in VBG$ on Z. It follows that $f \in VBG$ on E (see Theorem 3). By hypothesis and Lemma 14, we obtain that $m^*(f(Z)) = 0$. Hence $f \in (N)$ on the set E. $\square$

Remark 5. Consider the following definition.

> Let E be a real set. A function $\nu : \mathcal{P}(E) \to [0, +\infty]$ is said to be $\sigma - AC$ on E, if for every null set $Z \subseteq E$ there is a Z-form $\{Z_n\}$ such that $\nu(Z_n) = 0$.

In this terms Theorem 8 can be written as follows.

> *Let E be a bounded Lebesgue measurable set, $f : E \to \mathbb{R}$ a Lebesgue measurable function and $i \in \{1,2,3,4,5\}$. Then $f \in VBG \cap (N)$ on E if and only if ν_f^i is $\sigma - AC$ on E.*

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