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AN ALMOST CONTINUOUS NONEXTENDABLE FUNCTION

Abstract

An example is constructed under the Continuum Hypothesis showing that almost continuity and the Strong Cantor Intermediate Value Property do not imply extendability. This answers a question in [8]. Results about stationary sets are given for the class of extendable functions from I into I, where $I = [0,1]$.

In 1955, John Nash introduced the idea of a "connectivity" function $f : X \to Y$ by requiring the graph of its restriction $f_{|C}$ to be a connected subset of $X \times Y$ for each connected subset C of X [5]. A discontinuous connectivity function can arise from a differential equation. For example, for $x \neq 0$, the curve

$$y = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is a solution of $x^4 y'' + 2x^3 y' + y = 0$ (to which $t = \frac{1}{x}$ transforms $\ddot{y} + y = 0$).

A connectivity function $f : I \to I$ is *extendable* if there exists a connectivity function $F : I^2 \to I$ such that $F(x,0) = f(x)$ for all $x \in I$. A function $f : I \to I$ is called *almost continuous* if each open neighborhood of the graph of f in I^2 contains the graph of a continuous function on I. In [8], it was shown that an extendable function $f : I \to I$ has the Strong Cantor Intermediate Value Property (SCIVP), which means that whenever $f(x) \neq f(y)$ and E is a Cantor set between $f(x)$ and $f(y)$, there exists a Cantor set B lying between x and y such that $f(B) \subset E$ and $f_{|B}$ is continuous. It was also shown that there exists an almost continuous function $f : I \to I$ which has a perfect road at every point and does not have the SCIVP. The following result answers the question asked there.

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Theorem 1. *Under* CH*, there exists an almost continuous function* $f : I \to I$ *which has the* SCIVP *but is not extendable.*

PROOF. Let $\mathcal{D} = \{A_\alpha : \alpha \in A\}$ denote the collection of all dense G_δ-subsets of I and $\mathcal{X} = \{K_\alpha : \alpha \in A\}$ denote the collection of all minimal blocking sets of I^2. A "minimal blocking set" of I^2 is a smallest closed subset of I^2 that misses some function from I into I but meets every continuous function from I into I [4]. We may suppose A is well ordered so that each α in A is preceded by less than c-many elements of A. Under CH, then each α in A is preceded by countably many elements of A. Let $\alpha \in A$. According to the Baire Category Theorem, $\cap\{A_\beta : \beta \le \alpha\} \in \mathcal{D}$. The x-projection $\Pi_1(K_\alpha)$ of K_α being a nondegenerate interval $[a,b]$ [4], there is by transfinite induction a Cantor set C_α disjoint from $\cup\{C_\beta : \beta < \alpha\}$ such that

$$C_\alpha \subset [a,b] \cap \big(\cap\{A_\beta : \beta \le \alpha\}\big)\,.$$

For each $x \in C_\alpha$, define

$$g(x) = \max \Pi_2\big(K_\alpha \cap (\{x\} \times I)\big)\,,$$

where Π_2 denotes the y-projection. Then $g_{|C_\alpha}$ is upper semicontinuous, therefore in Baire class 1, and consequently Marczewski measurable, which means that a perfect set like C_α has a perfect subset P_α such that $g_{|P_\alpha}$ is continuous. For argument's sake, we may suppose $g(P_\alpha)$ is nowhere dense. For, if $g(P_\alpha)$ contains an interval $[c,d]$ and if D is a Cantor set in $[c,d]$, then the closed set $g^{-1}(D) \cap P_\alpha$ has c-many points and therefore contains a Cantor set P. So $g(P)$ is nowhere dense as desired. Define $f = g$ on P_α for each $\alpha \in A$, and define

$$f\big(I \setminus \cup\{P_\alpha : \alpha \in A\}\big) = 0\,.$$

By construction, $f : I \to I$ is dense in I^2 and almost continuous because f meets every blocking set of I^2.

We show f has the SCIVP. Suppose E is a Cantor set between different values $f(x)$ and $f(y)$ with $x < y$, and let $I_1, I_2, I_3, \ldots$ be the interval components of $I \setminus E$. A nondecreasing continuous function $h : I \to [x,y]$ can be constructed like the Cantor ternary function so that h is a different constant r_n on each I_n. Then the inverse relation

$$h^{-1} = \big\{(r,s) : (s,r) \in h\big\} \in \mathcal{X}$$

implies $h^{-1} = K_{\alpha_o}$ for some $\alpha_o \in A$, and so $f_{|P_{\alpha_o}} \subset K_{\alpha_o}$ and is continuous. $P_{\alpha_o} \setminus \{r_1, r_2, r_3, \ldots\}$ contains a Cantor set B between x and y; moreover $f(B) \subset E$ and $f_{|B}$ is continuous.

Assume f is extendable. There exists a dense G_δ-subset G of I that is f-negligible [7]; i.e., no matter how f is arbitrarily redefined on G with values in I, the resulting function is still extendable. Since $G \in \mathcal{D}$, $G = A_{\alpha_1}$, for some $\alpha_1 \in A$. Let

$$\{P_\alpha : \alpha < \alpha_1\} = \{P_1, P_2, P_3, \dots\}.$$

We may redefine $f = 0$ on A_{α_1} and the resulting function f is still extendable. Since $P_\alpha \subset A_{\alpha_1}$ for all $\alpha \geq \alpha_1$, then

$$f(I) = f\big(I \setminus \cup_{i=1}^{\infty} P_i\big) \cup \big(\cup_{i=1}^{\infty} f(P_i)\big) = \{0\} \cup \big(\cup_{i=1}^{\infty} f(P_i)\big),$$

which is a union of countably many nowhere dense subsets of the nondegenerate interval $f(I)$. Contradiction. □

Question 1. Without CH in Theorem 1, does there exist such a function in ZFC?

Question 2. With $\mathbb{R}$ replacing I in the above definitions, does there exist an almost continuous nonextendable function $f : \mathbb{R} \to \mathbb{R}$ having the SCIVP and obeying $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$?

Let $\mathcal{C}$ denote a class of functions $f : I \to I$, and let E be a subset of I. E is *stationary* for $\mathcal{C}$ if whenever $f \in \mathcal{C}$ and f is constant on E, then f is constant on all of I.

Theorem 2. *If E is stationary for the class $\mathcal{C}$ of extendable functions $f : I \to I$, then E intersects each dense G_δ-subset of I.*

Proof. Assume there is a stationary set E for $\mathcal{C}$ that misses a dense G_δ-subset A_α of I. There exists an extendable function $f : I \to I$ whose graph is dense in I^2 [1]. Let A be a dense G_δ-subset of $(0, 1)$ that is f-negligible. $A_0 = A \cap A_\alpha$ is a dense G_δ-set that is f-negligible. By [6], there is a homeomorphism $h : I \to I$ such that $I \setminus A_0 \subset h^{-1}(A_0)$ and $h^{-1}(A_0)$ is $(f \circ h)$-negligible. Since $E \subset I \setminus A_0$, E is $(f \circ h)$-negligible. Therefore $f \circ h$ can be redefined just on E to be 0, but the resulting extendable function is not 0 on all of I, contrary to E being stationary for $\mathcal{C}$. □

According to [2], the next result holds for some other classes of functions, too.

Theorem 3. *If a subset E of I intersects each nonempty perfect subset of I, then E is stationary for the class $\mathcal{C}$ of extendable functions $f : I \to I$.*

PROOF. Suppose $f \in \mathcal{C}$ and f is a constant c on E. Assume there exists an $x \in I \setminus E$ such that $f(x) \neq c$. Since f is extendable, f has a perfect road at x [3]. This means that there exists a perfect subset P of I containing x and having x as a bilateral limit point if x is not an endpoint of I such that $f_{|P}$ is continuous at x. Therefore there is a perfect subset P_0 of P containing x such that $c \notin f(P_0)$. But $E \cap P_0 \neq \emptyset$ implies $c \in f(P_0)$, a contradiction. So f equals c on I, and E is stationary for $\mathcal{C}$. □

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