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HARMONIC SINGULARITY AT INFINITY IN $\mathbb{R}^n$

Abstract

Some properties of harmonic functions defined outside a compact set in $\mathbb{R}^n$ are given. From them is deduced a generalized form of Liouville theorem in $\mathbb{R}^n$ which is known to be equivalent to an improved version of the classical Bôcher theorem on harmonic point singularities.

1 Introduction

A generalized form of the classical Bôcher theorem on the harmonic point singularity in $\mathbb{R}^n$, $n \geq 2$, is given in Ishikawa, Nakai and Tada [8]. This is equivalent to (Kelvin transformation) a generalized Liouville theorem: *If u is a harmonic function in $\mathbb{R}^n$, $n \geq 2$, such that* $\liminf_{|x|\to\infty} \frac{u(x)}{|x|} \geq 0$, *then u is a constant* (P. Bourdon [4]).

In this note, we obtain these two theorems as consequences of some equivalent properties of harmonic functions defined outside a compact set in $\mathbb{R}^n$. These developments are based on our earlier papers [7] and [2].

In particular, we give a proof of the above mentioned Liouville theorem that uses the arguments given by M. Brelot [6]; this proof is different from the one given in [7] where a reference to the Divergence theorem is made. In the special case of the complex plane $\mathbb{C}$ this theorem has been proved in [2] using the Carathéodory inequality; here we add a simple proof, valid in $\mathbb{R}^n$, $n \geq 2$, that appeals to the Poisson representation.

2 Harmonic Functions Outside a Compact Set

Given a locally integrable function $\varphi(x)$ defined outside a compact set in $\mathbb{R}^n$, let $M(r, \varphi)$ stand for the mean-value of $\varphi(x)$ on $|x| = r$ for large r.

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Lemma 2.1. *Let $f(x)$ be a function defined outside a compact set in $\mathbb{R}^n$, $n \geq 2$, such that*

$$\liminf_{|x|\to\infty} \frac{f(x)}{|x|} \geq 0\,.$$

Then there exists a locally integrable function $\varphi(x)$ such that $f(x) \geq \varphi(x)$ outside a compact set and $M(r, |\varphi|) = o(r)$ when $r \to \infty$.

Proof. Let $\liminf_{|x|\to\infty} \frac{f(x)}{|x|} = \lambda \geq 0$. If $\lambda > 0$, we can take $\varphi \equiv 0$. Let us suppose $\lambda = 0$. For an integer m, there exists a compact K_m such that

$$f(x) > -\frac{1}{m}|x| \quad \text{in} \quad K_m^c\,.$$

Choose r_m so that $K_m \subset \{x : |x| < r_m\}$. Then choose r_{m+1} so that $r_{m+1} > r_m$ and $K_{m+1} \subset \{x : |x| < r_{m+1}\}$. Now, define $\varphi(x)$ for $|x|$ large as

$$\varphi(x) = -\frac{1}{m}|x| \quad \text{if} \quad r_m < |x| \leq r_{m+1}\,.$$

Then outside a compact set, $\varphi(x)$ is a locally integrable function such that

$$\lim_{|x|\to\infty} \frac{|\varphi(x)|}{|x|} = 0 \quad \text{and} \quad M(r, |\varphi|) = o(r) \quad \text{when} \quad r \to \infty\,.$$

Also $f(x) \geq \varphi(x)$ for $|x|$ large. □

Lemma 2.2. *Let $u(x)$ be a bounded harmonic function outside a compact set in $\mathbb{R}^n$, $n \geq 2$. Then $\lim_{|x|\to\infty} u(x)$ is finite.*

Proof. This is a classical result. See, for example, p. 195 and p. 201 in M. Brelot [6]. □

Theorem 2.3. *Let $u(x)$ be a harmonic function defined outside a compact set in $\mathbb{R}^n$, $n \geq 2$. Then the following are equivalent:*

1) *$u(x) = o(|x|)$ when $|x| \to \infty$.*

2) *$\liminf_{|x|\to\infty} \frac{u(x)}{|x|} \geq 0$.*

3) *There exists a locally integrable function $\varphi(x)$ such that $u(x) \geq \varphi(x)$ outside a compact set, and $M(r, |\varphi|) = o(r)$ when $r \to \infty$.*

4) *$\lim_{|x|\to\infty} u(x)$ is finite if $n \geq 3$ and $\lim_{|x|\to\infty}\big(u(x) - \alpha \log |x|\big)$ is finite for some α if $n = 2$.*

PROOF. 1) $\Rightarrow$ 2): Evident.

2) $\Rightarrow$ 3): Lemma 2.1.

3) $\Rightarrow$ 4): Given a harmonic function u outside a compact set in $\mathbb{R}^n$, there exists a harmonic function v in $\mathbb{R}^n$ such that

(a) $u(x) - v(x) - \alpha \log |x|$ is bounded outside a compact set for some α if $n = 2$, and

(b) $u(x) - v(x)$ is bounded outside a compact set if $n \geq 3$.

To prove (a) and (b) we can use the series expansions for $u(x)$ as given in M. Brelot [6]. (A general result of this form applicable even to Riemann surfaces and to some other harmonic spaces is given in Rodin and Sario [9]; see also [1]). Consequently, the assumption on $u(x)$ implies that the harmonic function $v(x)$ in $\mathbb{R}^n$ satisfies the condition that outside a compact set

$$v(x) \geq \varphi(x) - \alpha \log |x| - \beta \quad \text{in } \mathbb{R}^2$$

and

$$v(x) \geq \varphi(x) - \beta_o \quad \text{in} \quad \mathbb{R}^n\,, \quad n \geq 3$$

(where β and β_o are constants). In either case, $v(x) \geq \psi(x)$ outside a compact set K where $\psi(x)$ is a locally integrable function such that

$$M\big(r, |\psi|\big) = o(r) \quad \text{when} \quad r \to \infty\,.$$

Since $v(x) \geq -|\psi(x)|$ in $\mathbb{R}^n \setminus K$, $v^- \leq |\psi|$; also $|v| = v + 2v^-$ and $M(r, v) = v(0)$. Hence $M(r, |v|) = o(r)$ when $r \to \infty$. This implies that v is a constant. For this, we almost reproduce a proof given in M. Brelot [6], p. 194. (Later we give another proof using the Poisson representation).

Write $v(x) = \sum_0^\infty a_p(\theta) r^p$, where $|x| = r$ and $a_p(\theta)$'s are Laplace functions of order p of the point θ on the unit sphere. Since $M(r, |v|) = o(r)$, $M(r, va_p) = o(r)$ when $r \to \infty$. But $M(r, va_p) = r^p M(1, a_p^2)$. Hence

$$r^{p-1} M(1, a_p^2) \to 0 \quad \text{when} \quad r \to \infty\,.$$

This implies that $a_p = 0$ if $p \geq 1$. Hence $v(x) \equiv a_0$.

Going back to (a) and (b) above, we deduce that $u(x) - \alpha \log |x|$ is bounded outside a compact set if $n = 2$ and $u(x)$ itself is bounded outside a compact set if $n \geq 3$.

Finally, an appeal to Lemma 2.2 proves that 3) $\Rightarrow$ 4).

4) $\Rightarrow$ 1): Evident.

This completes the proof of the theorem. $\square$

3 Some Consequences of the Theorem

In this section, we obtain some corollaries of Theorem 2.3 which include Liouville and Bôcher theorems in $\mathbb{R}^n$.

Corollary 3.1. *Let $u(x)$ be a harmonic function that is bounded on one side in $|x-a|>\rho$ in $\mathbb{R}^n$. Then in $|x-a|\geq r>\rho$,*

a) if $n=2$, $u(x)=\alpha\log|x|+$ a bounded harmonic function*, and*

b) if $n\geq 3$, $u(x)$ is bounded.

Corollary 3.2. *Let $u(x)$ be a harmonic function defined outside a compact set in $\mathbb{R}^n$. If $M(r,u^+)=o(r)$ when $r\to\infty$, then*

$$|u|=\begin{cases} O(\log r) & \textit{if } n=2\\ O(1) & \textit{if } n\geq 3\,. \end{cases}$$

Proof. As mentioned in the proof of 3) $\Rightarrow$ 4) of Theorem 2.3, there exists a harmonic function v in $\mathbb{R}^n$ such that outside a compact set

a) in $\mathbb{R}^2$, $u(x)=v(x)+\alpha\log|x|+b(x)$ and

b) in $\mathbb{R}^n$, $n\geq 3$, $u(x)=v(x)+b(x)$

where $b(x)$ stands for a bounded harmonic function.

When $n=2$, $m(r,u)=v(0)+\alpha\log r+M(r,b)=o(r)$ when $r\to\infty$. Hence if $M(r,u^+)=o(r)$, then $M(r,|u|)=o(r)$. By Theorem 2.3, it follows that $\lim_{|x|\to\infty}\big(u(x)-\alpha\log|x|\big)$ is finite. Hence $|u|=O(\log r)$ when $r\to\infty$.

When $n\geq 3$, $M(r,u)=v(0)+M(r,b)=O(1)$ when $r\to\infty$. Hence if $M(r,u^+)=o(r)$, we show as before that $\lim_{|x|\to\infty}u(x)$ is finite. Hence $|u|=O(1)$ when $r\to\infty$. ☐

Corollary 3.3. (Liouville's Theorem [4], [7]) *Let h be a harmonic function in $\mathbb{R}^n$, $n\geq 2$. Then the following are equivalent:*

1) *$h(x)=o(|x|)$ when $|x|\to\infty$.*

2) $\liminf_{|x|\to\infty}\frac{h(x)}{|x|}\geq 0$.

3) *There exists a locally integrable function $\varphi(x)$ such that $h(x)\geq\varphi(x)$ outside a compact set and $M(r,|\varphi|)=o(r)$ when $r\to\infty$.*

4) *h is a constant.*

PROOF. In view of Theorem 2.3, it is enough to remark that when $n = 2$, $h(x) - \alpha \log |x|$ tends to a finite limit when $|x| \to \infty$. Hence

$$M\big(r, h(x) - \alpha \log |x|\big) = h(0) - \alpha \log r$$

tends to a finite limit when $r \to \infty$; consequently, $\alpha = 0$.

Thus for all $n \geq 2$, $h(x)$ tends to a finite limit at the point at infinity. Hence by the maximum principle, h is a constant. □

Remark. The above generalized form of the Liouville theorem in the complex plane $\mathbb{C}$ was proved in [2] using the Carathéodory inequality. Equally simple is the following proof using Poisson kernel, which we give in $\mathbb{R}^n$, $n \geq 2$.

PROOF. Assume $h(x)$ is harmonic in $\mathbb{R}^n$, $n \geq 2$ and $M(r, |h|) = o(r)$ when $r \to \infty$. Let $x, y \in \mathbb{R}^n$, $|y| = r$. Let α_n be the surface area of the unit sphere in $\mathbb{R}^n$ and $d\sigma_n(y)$ the surface area on $S_n(r) = \{y : |y| = r\}$ in $\mathbb{R}^n$. Then,

$$\big|h(x) - h(0)\big| = \left| \frac{1}{\alpha_n r^{n-1}} \int_{S_n(r)} \Big(\frac{|y|^{n-2}(|y|^2 - |x|^2)}{|y - x|^n} - 1 \Big) h(y)\, d\sigma_n(y) \right|.$$

Now

$$P(x, y) = \frac{|y|^{n-2}(|y|^2 - |x|^2)}{|y - x|^n} - 1 = O\Big(\frac{1}{|y|}\Big)$$

when $|y| \to \infty$, for x in a compact set. (See M. Brelot [5] p. 134 for the expansion of $|y - x|^{-n}$ as a uniformly convergent series.) That is, $|P(x, y)| \leq A/|y|$ when $|y|$ is large, for some constant A. Hence

$$\big|h(x) - h(0)\big| \leq \frac{A}{r} M(r, |h|) \to 0$$

when $|y| = r \to \infty$, by hypothesis. Thus $h(x) = h(0)$ for all x. □

Corollary 3.4. (Bôcher's Theorem [8], [2], [7]) *Let $u(x)$ be a harmonic function in $0 < |x| < 1$ in $\mathbb{R}^n$, $n \geq 2$. Then the following are equivalent:*

1) $\lim_{|x|\to 0} |x|^{n-1} u(x) = 0$.

2) $\liminf_{|x|\to 0} |x|^{n-1} u(x) \geq 0$.

3) *There exists a locally integrable function $\varphi(x)$ such that $u(x) \geq \varphi(x)$ and $M(r, |\varphi|) = o(r^{1-n})$ when $r \to 0$.*

4) *$u(x) = v(x) + \alpha E_n(x)$ in $0 < |x| < 1$, where $v(x)$ is harmonic in $|x| < 1$ and $E_n(x)$ is the fundamental solution of the Laplacian Δ in $\mathbb{R}^n$.*

PROOF. In view of Theorem 2.3, an application of the Kelvin transformation proves the corollary. □

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