Real Analysis Exchange
Vol. 23(2), 1997–1998, pp. 395–420

Miroslav Krbec,* Institute of Mathematics, Czech Acad. Sci., Žitná 25, 115 67 Prague 1, Czech Republic e-mail: `krbecm@matsrv.math.cas.cz`
Thomas Schott,† Fakultät für Mathematik und Informatik, Friedrich-Schiller-Universität, UHH, 17. OG., 07740 Jena, Germany e-mail: `schott@minet.uni-jena.de`

EMBEDDINGS OF WEIGHTED SOBOLEV SPACES IN THE BORDERLINE CASE

Abstract

We prove a version of Schur's lemma for operators with positive kernels on weighted L_p spaces and apply the result to Riesz potentials of first order to get weighted generalizations of Trudinger's limiting embedding.

1 Introduction

In this paper we shall pursue a study of the limiting behavior of linear integral operators with a non-negative kernel $K(x,y)$,

$$Tf(x) = \int_{\Omega} K(x,y) f(y)\, dy,$$

with the goal to establish limiting embedding of weighted Sobolev spaces into weighted Orlicz spaces. The study of limiting embedding draws a lot of attraction now and new aspects and applications to various branches of analysis appear; see e.g. [FLS], [EK], [ET], $\dots$. To our knowledge, the weighted case has not been considered in the literature up to now.

Let $1 < p \le q < \infty$ and let Ω be a measurable subset of n-dimensional Euclidean space $\mathbb{R}^n$. Unless otherwise stated we shall assume that $n \ge 3$. Let us agree for notational convenience that all functions in the sequel will be

Key Words: positive kernel, Riesz potential, weighted Sobolev space, weighted Orlicz space, limiting embedding
Mathematical Reviews subject classification: 46E35
Received by the editors July 27, 1995
*The first author was supported in part by grant No. 201/94/1066 of GA ČR.
†A great deal of this work has been carried out during visits of authors in Jena and Prague. We wish to thank both places for the hospitality.

non-negative. If K is a non-negative kernel defined on $\mathbb{R}^n \times \mathbb{R}^n$ and $\alpha > 0$, then we let

$$T_\alpha f(x) = \int_\Omega K(x,y)^\alpha f(y)\,dy.$$

To keep the notation unified we put $T_1 = T$. Let $T_\alpha^* f(y) = \int_\Omega K(x,y)^\alpha f(x)\,dx$. Then T_α^* is the dual operator to T_α; that is,

$$\int_\Omega T_\alpha f(x) g(x)\,dx = \int_\Omega f(x) T_\alpha^* g(x)\,dx.$$

Let v and w be weights in Ω, that is, locally integrable and a.e. positive functions in Ω. By a duality argument, the inequality

$$\left(\int_\Omega |Tf(x)|^q w(x)\,dx\right)^{1/q} \le c\left(\int_\Omega |f(x)|^p v(x)\,dx\right)^{1/p}, \tag{1.1}$$

holds for each non-negative f iff the same is true for

$$\begin{aligned} &\left(\int_\Omega |T^* f(x)|^{p'} v(x)^{-p'/p}\,dx\right)^{1/p'} \\ &\le c\left(\int_\Omega |f(x)|^{q'} w(x)^{-q'/q}\,dx\right)^{1/q'}. \end{aligned} \tag{1.2}$$

If c_1^* and c_2^* denote the best possible constants in (1.1) and (1.2), respectively, then $c_1^* = c_2^*$.

The limiting behavior of the operators we are going to study can suitably be expressed in terms of weighted Orlicz spaces. For our purposes it will be sufficient to use the concept of the Orlicz space in its very classical form (See, e.g. [KR] for the non-weighted case, which, nevertheless, in the context of this paper does not essentially differ from the unweighted case.), namely, a Young function will be any convex, even, non-negative function on $\mathbb{R}^1$, increasing on $[0,\infty)$ and such that $\lim_{t\to 0_+} \Phi(t)/t = \lim_{t\to\infty} t/\Phi(t) = 0$. Given a Young function Φ and weights w_1, w_2 on Ω, we define the weighted modular by

$$m_\Phi(f, w_1, w_2) = \int_\Omega \Phi(w_1(x) f(x)) w_2(x)\,dx$$

and the weighted Orlicz space $L_\Phi(\Omega, w_1, w_2)$ as the space of all locally integrable f on Ω for which the Luxemburg norm

$$\|f\|_{\Phi,\Omega,w_1,w_2} = \inf\{\lambda > 0;\ m_\Phi(f/\lambda, w_1, w_2) \le 1\}$$

is finite. For $w_1 \equiv w_2 \equiv 1$ we shall simply write $L_\Phi(\Omega)$.

If $1 \le p < \infty$, and v is a weight on a domain Ω, then $W_0^{1,p}(\Omega, v)$ is the standard weighted Sobolev space defined as a completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|f\|_{1,p;\Omega,v} = \left(\int_\Omega |\nabla f(x)|^p v(x)\, dx\right)^{1/p}.$$

Write $W_0^{1,p}(\Omega) = W_0^{1,p}(\Omega, 1)$.

Different positive constants will sometimes be denoted by the same symbol c provided a misunderstanding cannot occur.

2 Bounds for Positive Integral Operators in Weighted Spaces

We start with a sufficient condition for the validity of (1.1).

Lemma 2.1. *Let $1 < p \le q < \infty$ and let $r = r(p,q)$ be defined by*

$$\frac{1}{r} = \frac{1}{p'} + \frac{1}{q}. \tag{2.1}$$

Assume that there exist a positive function φ and $c > 0$ such that

$$T_r^*[w(T_r\varphi)^{q/p'}](x) \le cv(x)^{q/p}\varphi(x)^{q/p'} \qquad a.e.\ in\ \Omega. \tag{2.2}$$

Then (1.1) holds with $c_1^* \le c^{1/q}$.

PROOF. By Hölder's inequality we have

$$\begin{aligned} Tf(x) \le &\left(\int_\Omega K(x,y)^r \varphi(y)\, dy\right)^{1/p'} \\ &\times \left(\int_\Omega K(x,y)^{(1-r/p')p}\varphi(y)^{-p/p'} f(y)^p\, dy\right)^{1/p}, \end{aligned}$$

whence

$$\begin{aligned} \left(\int_\Omega |Tf(x)|^q w(x)\, dx\right)^{p/q} \le &\left\{\int_\Omega w(x)(T_r\varphi)(x)^{q/p'} \right.\\ &\left.\times \left[\int_\Omega K(x,y)^{pr/q}\varphi(y)^{-p/p'} f(y)^p\, dy\right]^{q/p} dx\right\}^{p/q}. \end{aligned} \tag{2.3}$$

By Minkowski's inequality, the right hand side of (2.3) is not greater than

$$\int_\Omega \varphi(y)^{-p/p'} f(y)^p \left[\int_\Omega K(x,y)^r w(x) (T_r\varphi)(x)^{q/p'}\, dx \right]^{p/q} dy$$

and by our assumption (2.2) the last integral can be estimated from above by

$$c^{p/q} \int_\Omega v(y) f(y)^p\, dy.$$

It is also clear that $c_1^* \le c^{1/q}$. The proof is complete. $\square$

Remark 2.2. For $n = 1$, $\Omega = (0,\infty)$ and $Tf(x) = \int_0^x f(t)\, dt$, we get the result in Gurka [G]. The case $p = q$ in Lemma 2.1 can be found in Kerman and Sawyer [KS]. Observe that in both cases the existence of a positive function φ satisfying (2.2) is also necessary for (1.1).

Corollary 2.3. *Let r be defined by* (2.1) *and suppose that there exist a positive function φ and a constant $c > 0$ such that*

$$T_r \left[v^{-p'/p} (T_r^* \varphi)^{p'/q} \right](x) \le c w(x)^{-p'/q} \varphi(x)^{p'/q} \quad a.e.\ in\ \Omega. \tag{2.4}$$

Then (1.1) *holds with $c_1^* \le c^{1/p'}$.*

PROOF. This is easy to see now as (2.4) implies (1.2), and this in turn gives (1.1). $\square$

Applying Corollary 2.3 to the Riesz potential operator of the first order,

$$If(x) = \int_B |x-y|^{1-n} f(y)\, dy$$

where B is the unit ball in $\mathbb{R}^n$, $n \ge 3$, we get the following.

Corollary 2.4. *Let $1/r = 1/q + 1/n'$ and suppose that there is a positive function φ and $C > 0$ such that*

$$\begin{aligned} C = \sup_{x\in B} \left[\frac{w(x)}{\varphi(x)} \right]^{n'/q} \int_B |x-y|^{r(1-n)} v(y)^{-n'/n} \\ \times \left(\int_B |y-z|^{r(1-n)} \varphi(z)\, dz \right)^{n'/q} dy < \infty. \end{aligned} \tag{2.5}$$

Then

$$\left(\int_B |If(x)|^q w(x)\,dx\right)^{1/q} \le C^{1/n'}\left(\int_B |f(x)|^n v(x)\,dx\right)^{1/n}. \tag{2.6}$$

3 Weighted Estimates for Riesz Potentials

The condition of Lemma 2.1, general as it may be, contains the assumption about a suitable function φ; a verification might be rather difficult in particular cases. Here we are going to deal with Riesz potentials, we establish weighted estimates and get information about the asymptotic behavior of the constants estimating the norms of the Riesz potentials when q tends to infinity.

We start with a technical lemma which is of crucial importance in the sequel.

Lemma 3.1. *Let B be the unit ball in $\mathbb{R}^n$,*

$$F(x) = \int_B |x-y|^\alpha |y|^\beta \left(\log \frac{e}{|y|}\right)^\gamma dy, \qquad x \in B. \tag{3.1}$$

Then the following statements hold.

(i) *If $\gamma = 0$, then there are $c > 0$ and $c_1 > 0$ such that*

$$F(x) \le c_1 c^{\alpha+\beta+n}\left(\frac{1}{\alpha+n} + \frac{1}{\beta+n} + \frac{1}{\alpha+\beta+n}\right), \quad x \in B,$$

provided $-n < \alpha < 1-n$, $\beta > -n$, $\alpha+\beta+n > 0$.

(ii) *If $\gamma = 0$, then there is $c > 0$ such that*

$$F(x) \le c\left(\frac{1}{\alpha+n} + \frac{1}{\beta+n} + \frac{1}{|\alpha+\beta+n|}\right)|x|^{\alpha+\beta+n}, \quad x \in B,$$

provided $-n < \alpha < 1-n$, $\beta > -n$, $\alpha+\beta+n < 0$.

(iii) *Let $-1 < \gamma_0 < \gamma_1$. Then there is $c > 0$ such that*

$$F(x) \le \frac{c}{\alpha+n}\left(\log \frac{e}{|x|}\right)^{\gamma+1}, \quad x \in B,$$

provided $-n < \alpha < 1-n$, $\beta > -n$, $\alpha+\beta+n = 0$, $\gamma_0 < \gamma < \gamma_1$.

(iv) *Let* $-n < \gamma < -1$ *and* $\beta = -n$. *Then there is* $c > 0$ *such that*

$$F(x) \le \frac{c}{\alpha+n}|x|^\alpha \left(\log \frac{e}{|x|}\right)^{\gamma+1}, \quad x \in B,$$

provided $-n < \alpha < 1-n$.

(v) *Let* $\gamma_0 < -1$. *Then there is* $c > 0$ *such that*

$$F(x) \le c\left(\frac{1}{\alpha+n} + \frac{1}{|\gamma+1|}\right), \quad x \in B,$$

provided $-n < \alpha < 1-n$, $\alpha+\beta+n = 0$, $\gamma_0 < \gamma < -1$.

The constants c *and* c_1 *may depend on* n, γ_0 *and* γ_1 *but they are independent of* α, β *and* γ.

PROOF. The key is a suitable decomposition of B and establishing the respective bounds. Given $x \in B$, let

$$\begin{aligned} B_x^1 &= \{y \in B;\ |y| \le |x|/2\}, \\ B_x^2 &= \{y \in B;\ |x|/2 < |y| \le 2|x|\}, \\ B_x^3 &= \{y \in B;\ |y| > 2|x|\}, \end{aligned}$$

and

$$F_j(x) = \int_{B_x^j} |x-y|^\alpha |y|^\beta \left(\log \frac{e}{|y|}\right)^\gamma dy, \qquad x \in B, \quad j = 1,2,3.$$

Let $-n < \alpha < 1-n$, $\beta \ge -n$ and $\gamma \in \mathbb{R}^1$. If $y \in B_x^1$, then $|x-y| \ge |x|/2$ and

$$F_1(x) \le c_n 2^{-\alpha}|x|^\alpha \int_0^{|x|/2} r^{\beta+n-1}\left(\log \frac{e}{r}\right)^\gamma dr. \tag{3.2}$$

Further,

$$F_2(x) \le c_\gamma 2^{|\beta|}|x|^\beta \left(\log \frac{e}{|x|}\right)^\gamma \int_{B_x^2} |x-y|^\alpha\, dy,$$

and

$$\int_{B_x^2} |x-y|^\alpha\, dy \le \int_{|z-x|\le 2|x|} |z|^\alpha\, dz \le c_n \int_0^{3|x|} r^{\alpha+n-1}\, dr,$$

whence

$$F_2(x) \le c_\gamma 2^{|\beta|} 3^{\alpha+n} \frac{1}{\alpha+n} |x|^{\alpha+\beta+n} \left(\log \frac{e}{|x|}\right)^\gamma . \tag{3.3}$$

Finally, if $y \in B_x^3$, then $|x-y| \ge |y|/2$, and using the trivial estimate $2^{-\alpha} \le 2^{2n} 2^\alpha$, we get

$$F_3(x) \le c_n 2^\alpha \int_{2|x|}^{1} r^{\alpha+\beta+n-1} \left(\log \frac{e}{r}\right)^\gamma dr \tag{3.4}$$

for every $|x| \le 1/2$.

Now the statement in (i) follows by putting together the estimates

$$F_1(x) \le c 2^{-(\alpha+\beta+n)} \frac{1}{\beta+n} |x|^{\alpha+\beta+n}, \tag{3.5}$$

$$F_2(x) \le c 3^{\alpha+|\beta|+n} \frac{1}{\alpha+n} |x|^{\alpha+\beta+n}, \tag{3.6}$$

and

$$F_3(x) \le c 2^\alpha \frac{1}{\alpha+\beta+n} .$$

As to (ii) let us first observe that (3.5) and (3.6) hold, too. Further,

$$\begin{aligned} F_3(x) &\le c 2^\alpha \frac{1}{|\alpha+\beta+n|} \left[(2|x|)^{\alpha+\beta+n} - 1\right] \\ &\le \frac{c}{|\alpha+\beta+n|} |x|^{\alpha+\beta+n}, \quad |x| \le \frac{1}{2}. \end{aligned}$$

Since $\beta \le \max(-\alpha-n, n) \le n$ and $0 < -(\alpha+\beta+n) < n$, we deduce (ii).

Prove (iii). The estimates (3.3) and (3.4) yield

$$F_2(x) \le \frac{c}{\alpha+n} \left(\log \frac{e}{|x|}\right)^\gamma \tag{3.7}$$

and

$$F_3(x) \le \frac{c}{\gamma+1} \left[\left(\log \frac{e}{|x|}\right)^{\gamma+1} - 1\right] \le c \left(\log \frac{e}{|x|}\right)^{\gamma+1} . \tag{3.8}$$

According to (3.2) we have

$$F_1(x) \le c|x|^\alpha \int_0^{|x|} r^{-\alpha-1} \left(\log \frac{e}{r}\right)^\gamma dr. \tag{3.9}$$

Put

$$G_1(t) = \int_0^t r^{-\alpha-1} \left(\log \frac{e}{r}\right)^{\gamma} dr, 0 < t < 1,$$

$$H_1(t) = t^{-\alpha} \left(\log \frac{e}{t}\right)^{\gamma+1}, \qquad 0 < t < 1.$$

Then there is $c_1 > 0$ such that

$$G_1(t) \le c_1 H_1(t), \qquad 0 < t < 1, \tag{3.10}$$

for all α and γ with $-n < \alpha < 1-n$, $\gamma_0 < \gamma < \gamma_1$. Indeed, let us consider the function $J_1(\xi, t) = \xi H_1(t) - G_1(t), \quad 0 < t < 1$. We have

$$\frac{\partial J_1(\xi,t)}{\partial t} = t^{-\alpha-1} \left(\log \frac{e}{t}\right)^{\gamma} \left[\xi|\alpha| \log \frac{e}{t} - \xi(\gamma+1) - 1\right].$$

For any fixed ξ, the function $\partial J_1/\partial t$ is positive in some neighborhood of the origin and there is at most one point $t_\xi \in (0,1)$ such that $\partial J_1/\partial t(\xi, t_\xi) = 0$. Furthermore, $\lim_{t\to 0} J_1(\xi,t) = 0$. Hence it suffices to choose ξ large enough so that $\lim_{t\to 1} J_1(\xi,t) \ge 0$. This proves (3.10).

Consequently, by (3.9) and (3.10),

$$F_1(x) \le c \left(\log \frac{e}{|x|}\right)^{\gamma+1}. \tag{3.11}$$

Combining (3.7), (3.8), and (3.11) we get (iii).

Prove (iv). Fix γ with $-n < \gamma < -1$. As $\beta = -n$ we have $\alpha+\beta+n = \alpha < 0$. By (3.2) and (3.3),

$$F_1(x) \le c|x|^{\alpha} \left(\log \frac{e}{|x|}\right)^{\gamma+1} \tag{3.12}$$

and

$$F_2(x) \le \frac{c|x|^{\alpha}}{\alpha+n} \left(\log \frac{e}{|x|}\right)^{\gamma}, \tag{3.13}$$

respectively. The estimate (3.4) implies that

$$F_3(x) \le c \int_{|x|}^1 r^{\alpha-1} \left(\log \frac{e}{r}\right)^{\gamma} dr. \tag{3.14}$$

Similarly as in the proof of (iii) we shall show that there is $c_2 > 0$ such that

$$G_2(t) \le c_2 H_2(t), \qquad 0 < t < 1, \tag{3.15}$$

where

$$G_2(t) = \int_t^1 r^{\alpha-1} \left(\log \frac{e}{r}\right)^\gamma \, dr, \qquad 0 < t < 1,$$
$$H_2(t) = t^\alpha \left(\log \frac{e}{t}\right)^{\gamma+1}, \qquad 0 < t < 1.$$

The function $J_2(\xi, t) = \xi H_2(t) - G_2(t)$ is increasing in ξ. Fix $\xi > 0$ and consider $J_2(\xi, .)$. Clearly $J_2(\xi, 1) > 0$ and

$$\frac{\partial J_2(\xi,t)}{\partial t} = t^{\alpha-1} \left(\log \frac{e}{t}\right)^\gamma \left[\alpha\xi \log \frac{e}{t} - \xi(\gamma+1) + 1\right]$$

and we see that $\partial J_2/\partial t(\xi, t) < 0$ for t small. The function $M(t) = \alpha\xi \log(e/t) - \xi(\gamma+1) + 1$ is increasing and it vanishes at $t_\xi = \exp[1 - (\gamma+1)/\alpha + 1/(\alpha\xi)]$. Therefore it suffices to find $\xi_0 > 0$ such that $t_{\xi_0} \geq 1$; then M will be negative in $(0, 1)$ together with $\partial J_2/\partial t(\xi_0, t)$. This gives

$$c_2 = \xi_0 \geq \frac{1}{|\alpha| - |\gamma+1|}$$

(observe that $-\alpha + \gamma + 1 > n - 1 + 1 - n = 0$). From (3.14) we arrive at

$$F_3(x) \leq \frac{c}{|\alpha| - |\gamma+1|} \, |x|^\alpha \left(\log \frac{e}{|x|}\right)^{\gamma+1}. \tag{3.16}$$

The proof of (iv) is now complete by summing up the estimates (3.12), (3.13), and (3.16).

It remains to prove (v). Let α, β, γ satisfy the assumptions. By (3.2) and (3.4) we have

$$F_1(x) \leq c 2^{-(\alpha+\beta+n)} \frac{1}{\beta+n} \, |x|^{\alpha+\beta+n} \leq c_1, \qquad x \in B, \tag{3.17}$$

and

$$\begin{aligned} F_3(x) &\leq \frac{c_3}{|\gamma+1|} \left(\left(\log \frac{e}{2|x|}\right) - 1\right) \\ &\leq \frac{c_3}{|\gamma+1|}, \qquad x \in B. \end{aligned} \tag{3.18}$$

Further,

$$
\begin{aligned}
F_2(x) &\le 2^{-\beta}|x|^\beta \int\limits_{B_x^2} |x-y|^\alpha \, dy \\
&\le c2^{-\beta}|x|^\beta \int\limits_0^{3|x|} r^{\alpha+n-1}\, dr \\
&\le \frac{c_2}{\alpha+n}, \qquad x \in B.
\end{aligned} \tag{3.19}
$$

Putting together (3.17), (3.18), (3.19), we get (v). □

Theorem 3.2. *Let* $If(x) = \int\limits_B |x-y|^{1-n} f(y)\, dy, \qquad f \ge 0, \quad f \in L^1_{\text{loc}}.$

(i) *If* $-n < \delta \le 0$, *then*

$$
\Big(\int\limits_B |If(x)|^q |x|^\delta \, dx\Big)^{1/q} \le cq^{1/n'} \Big(\int\limits_B |f(x)|^n \, dx\Big)^{1/n} \tag{3.20}
$$

for large q's, with c independent of f and q.

(ii) *If* $0 < \delta < n(n-1)$, *then*

$$
\begin{aligned}
&\Big(\int\limits_B \big| |x|^{\delta/n} If(x)|^q |x|^{-n} \, dx\Big)^{1/q} \\
&\qquad \le cq^{1/n'} \Big(\int\limits_B |f(x)|^n |x|^\delta \, dx\Big)^{1/n}
\end{aligned} \tag{3.21}
$$

for large q's, with c independent of f and q.

(iii) *If* $\delta < n-1$, *then*

$$
\begin{aligned}
&\Big(\int\limits_B \Big| \Big(\log \frac{e}{|x|}\Big)^{\delta/n - 1/n'} If(x)\Big|^q |x|^{-n} \Big(\log \frac{e}{|x|}\Big)^{-1} dx\Big)^{1/q} \\
&\qquad \le cq^{1/n'} \Big(\int\limits_B |f(x)|^n \Big(\log \frac{e}{|x|}\Big)^\delta \, dx\Big)^{1/n}
\end{aligned} \tag{3.22}
$$

for $q > n$, with c independent of f and q.

(iv) *Let $\eta < 0$. Then*

$$\left(\int_B |If(x)|^q \left(\log \frac{e}{|x|}\right)^{\eta-1} |x|^{-n}\, dx\right)^{1/q} \le cq^{1/n'} \left(\int_B |f(x)|^n \left(\log \frac{e}{|x|}\right)^{n-1} dx\right)^{1/n}, \tag{3.23}$$

for $q > n$, with c independent of f and q.

PROOF. Let

$$\Lambda(y) = \int_B |y-z|^{r(1-n)} \varphi(z)\, dz,$$

and

$$\Psi(x) = \int_B |x-y|^{r(1-n)} v(y)^{-n'/n} \Lambda(y)^{n'/q}\, dy.$$

Then (2.5) becomes

$$C = \sup_{x\in B} \left(\frac{w(x)}{\varphi(x)}\right)^{n'/q} \Psi(x) < \infty.$$

Step 1. We prove (3.20) for $\delta = 0$. It is $w \equiv v \equiv 1$ and putting $\varphi \equiv 1$ we have

$$\Lambda(y) = \int_B |y-z|^{r(1-n)}\, dz$$

which is of type (3.1) where $\beta = \gamma = 0$ and

$$\alpha = r(1-n) = \frac{qn'}{q+n'}(1-n) = -\frac{qn}{q+n'}. \tag{3.24}$$

If $q > n$, then

$$-n < \alpha = -\frac{n}{1+n'/q} < -\frac{n}{1+n'/n} = 1-n.$$

By (3.24),

$$\alpha + n = \alpha + \beta + n = n - \frac{qn}{q+n'} = \frac{nn'}{q+n'} > 0.$$

An application of Lemma 3.1 (i) yields

$$\Lambda(y) \le c_1 c^{nn'/(q+n')} \left(\frac{1}{n} + \frac{2(q+n')}{nn'}\right) \le c'q, \qquad y \in B,$$

thus

$$\Psi(x) \le c q^{n'/q} \int_B |x-y|^{r(1-n)}\, dy, \qquad x \in B.$$

Since $q^{n'/q} \le e^{n'/e}$ the same argument as above implies that

$$\Psi(x) \le cq, \qquad x \in B, \quad q > n,$$

therefore,

$$\sup_{x\in B} \left(\frac{w(x)}{\varphi(x)}\right)^{n'/q} \Psi(x) \le cq, \qquad x \in B, \quad q > n. \tag{3.25}$$

Step 2. We prove (3.20) for $-n < \delta < 0$. We have $w(x) = |x|^\delta$, $v \equiv 1$, and we put $\varphi(x) = |x|^\delta$. The integral

$$\Lambda(y) = \int_B |y-z|^{r(1-n)} |z|^\delta\, dz$$

is of type (3.1) with $\gamma = 0$, $\beta = \delta > -n$, and α given by (3.24). If q is sufficiently large, say, $q \ge q_0$, then $\alpha + \beta + n = \frac{nn'}{q+n'} + \delta < 0$. Furthermore, $\beta + n = \delta + n$ and $\alpha + n = \frac{nn'}{q+n'} > 0$. Applying Lemma 3.1. (ii) we get

$$\begin{aligned}\Lambda(y) &\le c\left(\frac{1}{\delta+n} + \frac{q+n'}{nn'} + \frac{1}{|\delta + nn'/(q+n')|}\right)|y|^{\delta + nn'/(q+n')}\\ &\le c' q |y|^{\delta+nn'/(q+n')}.\end{aligned}$$

Hence

$$\Psi(x) \le c \int_B |x-y|^{r(1-n)} |y|^{\frac{nn'}{q+n'}\cdot\frac{n'}{q} + \frac{\delta n'}{q}}\, dy.$$

The last integral is of type (3.1) with α given by (3.24), $\gamma = 0$, and

$$\beta = \beta(q) = \frac{nn'}{q+n'}\cdot\frac{n'}{q} + \frac{\delta n'}{q}.$$

Further,

$$\begin{aligned}\alpha + \beta + n &= -\frac{qn}{q+n'} + n + \frac{nn'}{q+n'}\cdot\frac{n'}{q} + \frac{\delta n'}{q}\\ &= \frac{n'(n+\delta)}{q} > 0.\end{aligned}$$

Hence Lemma 3.1 (i) can be applied and we get $\Psi(x) \le cq, \qquad x \in B, \quad q \ge q_0$. As a result,

$$\sup_{x\in B} \left(\frac{w(x)}{\varphi(x)}\right)^{n'/q} \Psi(x) \le cq, \quad q \ge q_0.$$

Thus we have proved (3.20).

Step 3. We prove (3.21). This time

$$w(x) = |x|^{-n+\delta q/n} \quad \text{and} \quad v(x) = |x|^{\delta} \text{ with } 0 < \delta < n(n-1),$$

and we put $\varphi(x) = |x|^{\varepsilon}$, $x \in B$, where $-n < \varepsilon < \min(-1, \delta - n)$. Then $\Lambda(y) = \int\limits_B |y-z|^{r(1-n)}|y|^{\varepsilon}\,dy$ is of type (3.1) with $\gamma = 0$, $\beta = \varepsilon$, and α as in (3.24). We have $\alpha + n = \frac{nn'}{q+n'} > 0$, $\beta + n = \varepsilon + n > 0$, and if $q > n$, then

$$\alpha + \beta + n = \frac{nn'}{q+n'} + \varepsilon < \frac{nn'}{n+n'} + \varepsilon = 1 + \varepsilon < 0.$$

According to Lemma 3.1 (ii) we get $\Lambda(y) \le cq|y|^{\varepsilon + nn'/(q+n')}$ which gives

$$\Psi(x) \le c \int\limits_B |x-y|^{r(1-n)} |y|^{\frac{nn'}{q+n'}\cdot\frac{n'}{q} + \frac{\varepsilon n'}{q} - \frac{\delta n'}{n}}\,dy. \tag{3.26}$$

We verify that the parameters in (3.26) satisfy the assumptions in (ii) of Lemma 3.1. As before, we have $\alpha + n = nn'/(q+n')$. Further,

$$\beta + n = \frac{nn'}{q+n'}\cdot\frac{n'}{q} + \frac{\varepsilon n'}{q} - \frac{\delta n'}{n} + n$$

which tends to the positive limit $n - \delta n'/n$ for $q \to \infty$. Finally,

$$\begin{aligned}\alpha + \beta + n &= \frac{nn'}{q} + \frac{nn'}{q+n'}\cdot\frac{n'}{q} + \frac{\varepsilon n'}{q} - \frac{\delta n'}{n}\\ &= \frac{nn'}{q} + \frac{\varepsilon n'}{q} - \frac{\delta n'}{n}\end{aligned}$$

which is negative iff $q > n(n+\varepsilon)/\delta$. But $\varepsilon < \delta - n$ so that $\alpha + \beta + n < 0$ for each $q > n$. Consequently, $\Psi(x) \le cq^{nn'/q + \varepsilon n'/q - \delta n'/n}$, $q \ge q_0$, and

$$\begin{aligned}C &= \sup_{x\in B}\left(\frac{w(x)}{\varphi(x)}\right)^{n'/q}\Psi(x)\\ &\le cq|x|^{\frac{n'}{q}\left(\frac{\delta q}{n} - n - \varepsilon\right) + \frac{nn'}{q} + \frac{\varepsilon n'}{q} - \frac{\delta n'}{n}}\\ &= cq.\end{aligned} \tag{3.27}$$

The inequality (3.21) now follows.

Step 4. We prove (3.22). We have

$$w(x) = |x|^{-n} \left(\log \frac{e}{|x|}\right)^{(\delta/n - 1/n')q - 1} \quad \text{and} \quad v(x) = \left(\log \frac{e}{|x|}\right)^{\delta}$$

and this time we take

$$\varphi(x) = |x|^{-n} \left(\log \frac{e}{|x|}\right)^{\varepsilon},$$

with $-n < \varepsilon < -1$. Then $\Lambda(y) = \int_B |y-z|^{r(1-n)} |z|^{-n} \left(\log \frac{e}{|z|}\right)^{\varepsilon} dz$, and because $\alpha + n = nn'/(q+n')$ and $-n < \varepsilon < -1$ we can use Lemma 3.1 (iv) to obtain $\Lambda(y) \leq cq|y|^{r(1-n)} \left(\log \frac{e}{|y|}\right)^{\varepsilon+1}$. This yields

$$\begin{aligned} \Psi(x) \leq & c \int_B |x-y|^{r(1-n)} |y|^{r(1-n)n'/q} \\ & \times \left(\log \frac{e}{|y|}\right)^{n'(\varepsilon+1)/q - \delta n'/n} dy. \end{aligned} \tag{3.28}$$

The right hand side of (3.28) permits use of Lemma 3.1 (iii). Indeed, we have

$$\begin{aligned} \alpha + n &= \frac{nn'}{q+n'} < 1 \text{ for } q > n, \\ \beta + n &= \frac{qn'}{q+n'} \frac{(1-n)n'}{q} + n = \frac{nq}{q+n'} > 0, \\ \alpha + \beta + n &= \frac{nn'}{q+n'} + \frac{nq}{q+n'} - n = 0. \end{aligned}$$

Further, let $\gamma = (\varepsilon+1)n'/q - \delta n'/n$. Because $\delta < n-1$ we have $-\delta n'/n > -1$. Choosing ε sufficiently close to -1 so that $\gamma_0 = \varepsilon + 1 - \frac{n'\delta}{n} > -1$, we get, for $q > n$, that $-\frac{\delta n'}{n} > \gamma > \gamma_0 > -1$. Whence (3.28) yields $\Psi(x) \leq cq \left(\log \frac{e}{|x|}\right)^{\gamma+1}$

for $q > n$, and

$$\begin{aligned}
&\sup_{x\in B}\left(\frac{w(x)}{\varphi(x)}\right)^{n'/q}\Psi(x)\\
&\le cq\sup_{x\in B}\left(\frac{\left(\log\frac{e}{|x|}\right)^{(\delta/n-1/n')q-1}}{\left(\log\frac{e}{|x|}\right)^{\varepsilon}}\right)^{n'/q}\left(\log\frac{e}{|x|}\right)^{(\varepsilon+1)n'/q-\delta n'/n+1}\\
&\le cq\sup_{x\in B}\left(\log\frac{e}{|x|}\right)^{(\delta/n-1/n')n'-n'/q-\varepsilon n'/q+\varepsilon n'/q+n'/q-\delta n'/n+1}\\
&\le cq.
\end{aligned}$$

Thus we have proved (iii).
Step 5. Now we prove the remaining estimate (3.23). Without loss of generality let us assume that $1-n<\eta<0$. We have

$$v(x)=\left(\log\frac{e}{|x|}\right)^{n-1},\quad w(x)=|x|^{-n}\left(\log\frac{e}{|x|}\right)^{\eta-1},\quad x\in B.$$

Put $\varphi\equiv w$. Then $\Lambda(y)=\int\limits_B|y-z|^{r(1-n)}|z|^{-n}\left(\log\frac{e}{|z|}\right)^{\eta-1}dz$ is of type (3.1) with $\alpha=r(1-n)$, $\beta=-n$, and $\gamma=\eta-1$. Our choice of γ gives $-n<\gamma<-1$. Therefore we can apply Lemma 3.1. (iv) to get

$$\Lambda(y)\le\frac{c}{r(1-n)+n}\,|y|^{r(1-n)}\left(\log\frac{e}{|y|}\right)^{\eta},\qquad y\in B.$$

This yields

$$\begin{aligned}
\Psi(x)\le&\left(\frac{c}{r(1-n)+n}\right)^{n'/q}\\
&\times\int\limits_B|x-y|^{r(1-n)}|y|^{r(1-n)n'/q}\left(\log\frac{e}{|y|}\right)^{n'\eta/q-1}dy
\end{aligned}$$

for $x\in B$, with the integral on the right hand side of type (3.1) where

$$\begin{aligned}
\alpha&=r(1-n)=-\frac{qn}{q+n'},\\
\beta&=r(1-n\,\frac{n'}{q})=-\frac{nn'}{q+n},
\end{aligned}$$

and $\alpha+\beta+n=0$. Making use of Lemma 3.1 (v) we arrive at

$$\begin{aligned}\Psi(x) &\le c\left(\frac{q+n'}{nn'}\right)^{n'/q}\left(\frac{q+n'}{nn'}+\frac{q}{|\eta|n'}\right)\\ &\le cq, \qquad x\in B.\end{aligned}$$

As $w\equiv\varphi$ we are done. □

Remark 3.3. The validity of (3.20)–(3.22) for large q's is enough to establish embedding into the appropriate weighted Orlicz spaces with a Young function equivalent to $\Phi(t)=\exp t^{n'}-1$. It is of independent interest. However, that

(i) if $-n<\delta\le 0$, then

$$\left(\int_B |f(x)|^q|x|^\delta\,dx\right)^{1/q} \le cq^{1/n'}\left(\int_B |\nabla f(x)|^n\,dx\right)^{1/n} \tag{3.29}$$

for all $f\in C_0^\infty(B)$ and all $q\ge 1$, with c independent of f and q;

(ii) if $0<\delta<n(n-1)$, then

$$\left(\int_B \big|\,|x|^{\delta/n}f(x)\big|^n|x|^{-n}\,dx\right)^{1/n} \le c\left(\int_B |\nabla f(x)|^n|x|^\delta\,dx\right)^{1/n} \tag{3.30}$$

for all $f\in C_0^\infty(B)$, with c independent of f;

(iii) If $\delta\le 0$, then

$$\begin{aligned}&\left(\int_B \left|\left(\log\frac{e}{|x|}\right)^{\delta/n-1/n'} f(x)\right|^n |x|^{-n}\left(\log\frac{e}{|x|}\right)^{-1}dx\right)^{1/n}\\ &\le c\left(\int_B |\nabla f(x)|^n\left(\log\frac{e}{|x|}\right)^{\delta}dx\right)^{1/n}\end{aligned} \tag{3.31}$$

for all $f\in C_0^\infty(B)$, with c independent of f.

The proof of (3.29) goes in a straightforward manner from (3.20). We know that (3.20) holds for exponents greater or equal to q_0. Fix $q<q_0$. Then, by

Hölder's inequality,

$$\int_B |f(x)|^q |x|^\delta \, dx \le \left(\int_B |f(x)|^{q_0} |x|^\delta \, dx\right)^{q/q_0} \left(\int_B |x|^\delta \, dx\right)^{1-q/q_0}$$
$$\le c^q \left(\int_B |\nabla f(x)|^n \, dx\right)^{q/n}.$$

The inequality (3.30) can be reduced to the one-dimensional Hardy's inequality (see, e.g. [C])

$$\left(\int_0^1 |f(t)|^n t^{\delta-1} \, dt\right)^{1/n} \le c\left(\int_0^1 |f'(t)|^n t^{\delta+n-1} \, dt\right)^{1/n},$$

for all $f \in C^\infty([0,1])$ such that $f(1) = 0$.

As to (3.31), it is also sufficient to consider the one-dimensional case

$$\left(\int_0^1 |f(t)|^n \, \frac{1}{t} \, \left(\log \frac{e}{t}\right)^{\delta-n} \, dt\right)^{1/n} \le c\left(\int_0^1 |f'(t)|^n t^{n-1} \left(\log \frac{e}{t}\right)^{\delta} \, dt\right)^{1/n}$$

for $f \in C^\infty([0,1])$, $f(1) = 0$. According to [Ma1], p. 72 or [Ma2], Subsection 1.3.1 this inequality holds iff

$$\sup_{0<t<1} \left(\int_0^t \left(\log \frac{e}{s}\right)^{\delta-n} \, \frac{ds}{s}\right)^{1/n} \left(\int_t^1 \left[s^{n-1} \left(\log \frac{e}{s}\right)^{\delta}\right]^{-n'/n} \, ds\right)^{1/n'} < \infty,$$

which is indeed the case.

Remark 3.4. The inequalities (3.29) and (3.31) do *not* hold if $q < n$. Namely, as to (3.29), we would have

$$\left(\int_0^1 |g(t)|^q t^{-1+\delta q/n} \, dt\right)^{1/q} \le c\left(\int_0^1 |g'(t)|^n t^{\delta+n-1} \, dt\right)^{1/n} \tag{3.32}$$

for all $g \in C^\infty([0,1])$, $g(1) = 0$ (put $f(x) = g(|x|)$ in (3.29)). But (3.32) is false (see, e.g. [Ma1], p. 76 or [Ma2], Subsection 1.3.2). The argument for (3.31) with $q < n$ is analogous.

4 Weighted Estimates for Gradients

In this section we shall establish necessary conditions for the weighted inequalities between the function and its gradient, supported in the unit ball. Combining results from this and previous section we shall arrive at necessary and sufficient conditions.

Recall that $n \geq 3$. Fix $q_0 \geq 1$ and consider first the inequality

$$\left(\int_B \big|\, |x|^{\varepsilon} f(x)\big|^q \, |x|^{\eta} \, dx\right)^{1/q} \leq c_q \left(\int_B |\nabla f(x)|^n |x|^{\delta} \, dx\right)^{1/n} \tag{4.1}$$

for $f \in C_0^\infty(B)$, $q \geq q_0$, where δ, ε, and η are real valued parameters.

Lemma 4.1. *Suppose that* (4.1) *holds.*

(i) *Let $\eta > -n$. Then $\varepsilon \geq 0$ if $\delta \leq 0$ and $\varepsilon \geq \delta/n$ if $\delta > 0$.*

(ii) *Let $\eta = -n$. Then $\varepsilon > 0$ if $\delta \leq 0$ and $\varepsilon \geq \delta/n$ if $\delta > 0$.*

(iii) *Let $\eta < -n$. Then $\varepsilon > 0$ if $\delta \leq 0$ and $\varepsilon > \delta/n$ if $\delta > 0$.*

Proof. Suppose that (4.1) holds for some $\delta, \varepsilon, \eta \in \mathbb{R}^1$. Putting $f(x) = \varphi(|x|)$, $\varphi \in C^\infty$, $\varphi(1) = 0$, we get

$$\left(\int_0^1 |\varphi(t)|^q t^{\varepsilon q + \eta + n - 1} \, dt\right)^{1/q} \leq c_q \left(\int_0^1 |\varphi'(t)|^n t^{\delta + n - 1} \, dt\right)^{1/n} \tag{4.2}$$

for all $q \geq q_0$. Let

$$B_q(t) = \left(\int_0^t s^{\varepsilon q + \eta + n - 1} \, ds\right)^{1/q} \left(\int_t^1 s^{(\delta + n - 1)(-n'/n)} \, ds\right)^{1/n'}.$$

Then (4.2) implies (cf. [Ma1], p. 72 or [Ma2], Subsection 1.3.1) that

$$B_q = \sup_{0<t<1} B_q(t) < \infty, \qquad q \geq q_0, \tag{4.3}$$

and it follows that, necessarily, $\varepsilon \geq 0$. Indeed, if $\varepsilon < 0$, then

$$\lim_{q \to \infty} (\varepsilon q + \eta + n - 1) = -\infty$$

which contradicts to (4.3).

Proof of (i). We already know that ε must be non-negative. Therefore it suffices to show that $\varepsilon \geq \delta/n$ for $\delta > 0$. Fix $\delta > 0$. We have

$$B_q(t) = ct^{\varepsilon+(\eta+n)/q-\delta/n}(1 - t^{\delta n'/n})^{1/n'}, \tag{4.4}$$

whence by (4.3),

$$\varepsilon - \frac{\delta}{n} + \frac{\eta + n}{q} \geq 0 \qquad \text{for all } q \geq q_0, \tag{4.5}$$

and this gives $\varepsilon \geq \delta/n$.

Proof of (ii). In this case we assume $\eta = -n$. If $\varepsilon = 0$, then

$$\int_0^t s^{\varepsilon q+\eta+n-1}\, ds = \infty \tag{4.6}$$

and (4.3) does not hold. Therefore $\varepsilon > 0$. If $\delta > 0$, then $\varepsilon \geq \delta/n$ because of (4.5).

Finally to prove (iii). Let $\eta < -n$. Again, $\varepsilon = 0$ can be excluded because (4.6) holds, too. Finally, if $\delta > 0$, then we have (4.4) and (4.5) as before. Because $\eta + n < 0$ we get $\varepsilon > \delta/n$. $\square$

Theorem 3.2 (i), (ii) and Remark 3.3 yield the estimates

$$\left(\int_B |f(x)|^q |x|^\eta\, dx\right)^{1/q} \leq cq^{1/n'} \left(\int_B |\nabla f(x)|^n\, dx\right)^{1/n}, \quad q \geq 1, \tag{4.7}$$

for all $f \in C_0^\infty(B)$ provided $-n < \eta \leq 0$, and

$$\begin{aligned} &\left(\int_B \big|\, |x|^{\delta/n} f(x)\big|^q |x|^{-n}\, dx\right)^{1/q} \\ &\leq cq^{1/n'} \left(\int_B |\nabla f(x)|^n |x|^\delta\, dx\right)^{1/n}, \quad q \geq n, \end{aligned} \tag{4.8}$$

for all $f \in C_0^\infty(B)$ provided $0 < \delta < n(n-1)$. We observe that this is an immediate consequence of the well-known inequality $|f(x)| \leq c|I(\nabla f)(x)|$ (see, for instance, [GT], Chapter 7).

Theorem 4.2. *Let δ, ε, $\eta \in \mathbb{R}^1$ and $\delta < n(n-1)$. Then there is $q_0 \geq 1$ such that* (4.1) *holds for every $f \in C_0^\infty(B)$ and every $q \geq q_0$ if and only if any of* (i), (ii), (iii) *from Lemma 4.1 holds.*

Moreover, if c_q^* denotes the best possible constant in (4.1)*, then*

$$c_q^* \le c q^{1/n'}, \qquad q \ge q_0. \tag{4.9}$$

The constant c in (4.9) *may depend on δ, ε, η, n and q_0, but it is independent of q.*

PROOF. The "only if" part is a consequence of Lemma 4.1. We prove the "if" part. Let $\eta > -n$, $\varepsilon \ge 0$, and $\delta \le 0$. Then $|x|^\delta \ge 1$, $x \in B$, and

$$|x|^{\varepsilon q+\eta} \le \begin{cases} |x|^\eta & \text{if } \eta < 0, \\ 1 & \text{if } \eta \ge 0, \end{cases}$$

for all $x \in B$ and (4.7) yields the estimates (4.1) and (4.9). Let $\eta > -n$, $0 < \delta < n(n-1)$, $\varepsilon \ge \delta/n$. Then $|x|^{\varepsilon q+\eta} \le |x|^{\delta q/n-n}$, $x \in B$, and (4.1) and (4.9) follow from (4.8). Let $\eta = -n$, $\varepsilon > 0$, $\delta \le n\varepsilon$. First, consider the case $\varepsilon < n-1$. Putting $\delta_\varepsilon = n\varepsilon$, we have $0 < \delta_\varepsilon < n(n-1)$ and $\delta \le \delta_\varepsilon$. Consequently, (4.8) holds with δ_ε instead of δ. This also proves (4.1) and (4.9) because $|x|^{\delta_\varepsilon} \le |x|^\delta$, $x \in B$. Now, let $\varepsilon \ge n-1$. Choose $\widetilde{\varepsilon}$ with $0 < \widetilde{\varepsilon} < n-1$ and $\widetilde{\varepsilon} \ge \delta/n$. By the above considerations, (4.1) holds with $\widetilde{\varepsilon}$ replacing ε. Further, $|x|^{\varepsilon q-n} \le |x|^{\widetilde{\varepsilon} q-n}$, $x \in B$. Thus we get (4.1) and (4.9).

The last case to investigate is (iii) from Lemma 4.1, that is, $\eta < -n$ and $\delta \le 0$ or $\delta > 0$. If $\delta \le 0$, let $0 < \widetilde{\varepsilon} < \varepsilon$. If $\delta > 0$, let $\widetilde{\varepsilon} = \delta/n$. In either case,

$$|x|^{\varepsilon q+\eta} \le |x|^{\widetilde{\varepsilon} q-n}, \qquad x \in B, \quad q \text{ large}. \tag{4.10}$$

The triple δ, $\widetilde{\varepsilon}$ (replacing ε), and $-n$ (replacing η) satisfies the conditions in (ii) of Lemma 4.1. Thus (4.1) holds with these parameters. The desired assertion follows now easily from (4.10). □

Remark 4.3. The assumption $\delta < n(n-1)$ from Theorem 4.2 comes in when establishing the sufficient condition – Corollary 2.4. Clearly, the function v is supposed to satisfy $v^{-n'/n} \in L_1(B)$ and the choice $v(x) = |x|^\delta$ gives exactly $\delta < n(n-1)$.

We now turn our attention to the inequality

$$\begin{aligned} &\left(\int_B \left|\left(\log\frac{e}{|x|}\right)^{\varepsilon} f(x)\right|^q \left(\log\frac{e}{|x|}\right)^{\eta-1} |x|^{-n}\,dx\right)^{1/q} \\ &\le c_q \left(\int_B |\nabla f(x)|^n \left(\log\frac{e}{|x|}\right)^{\delta} dx\right)^{1/n} \end{aligned} \tag{4.11}$$

for $f \in C_0^\infty(B)$, $q \ge q_0$, and real parameters δ, ε, η.

Lemma 4.4. *Suppose that* (4.11) *holds.*

(i) *Let* $\eta < 0$. *Then* $\varepsilon \le 0$ *if* $\delta \ge n-1$ *and* $\varepsilon \le \delta/n - 1/n'$ *if* $\delta < n-1$.

(ii) *Let* $\eta = 0$. *Then* $\varepsilon < 0$ *if* $\delta \ge n-1$ *and* $\varepsilon \le \delta/n - 1/n'$ *if* $\delta < n-1$.

(iii) *Let* $\eta > 0$. *Then* $\varepsilon < 0$ *if* $\delta \ge n-1$ *and* $\varepsilon < \delta/n - 1/n'$ *if* $\delta < n-1$.

PROOF. Suppose that (4.11) holds for some $\delta, \varepsilon, \eta \in \mathbb{R}^1$, $q_0 \ge 1$. Take $f(x) = \varphi(|x|)$ with φ from the proof of Lemma 4.1. Then

$$\begin{aligned} &\left(\int_0^1 |\varphi(t)|^q \, \frac{1}{t} \left(\log \frac{e}{t}\right)^{\varepsilon q + \eta - 1} dt\right)^{1/q} \\ &\le c_q \left(\int_0^1 |\varphi'(t)|^n t^{n-1} \left(\log \frac{e}{t}\right)^{\delta} dt\right)^{1/n} \end{aligned} \tag{4.12}$$

for all $q \ge q_0$. Let

$$\begin{aligned} B_q(t) = &\left(\int_0^t \left(\log \frac{e}{s}\right)^{\varepsilon q + \eta - 1} \frac{ds}{s}\right)^{1/q} \\ &\times \left(\int_t^1 \left[\left(\log \frac{e}{s}\right)^{\delta}\right]^{-n'/n} \frac{ds}{s}\right)^{1/n'}. \end{aligned} \tag{4.13}$$

By (4.12) (cf. [Ma1], p. 72 or [Ma2], Subsection 1.3.1),

$$B_q = \sup_{0<t<1} B_q(t) < \infty, \quad q \ge q_0. \tag{4.14}$$

If $\varepsilon > 0$ and q is large, then

$$\int_0^t \left(\log \frac{e}{s}\right)^{\varepsilon q + \eta - 1} \frac{ds}{s} = \infty. \tag{4.15}$$

Hence (4.14) and, consequently, also (4.11) cannot hold and from now on we can restrict our considerations to $\varepsilon \le 0$.

Prove (i). Given $\eta < 0$, all we have to show is $\varepsilon \le \delta/n - 1/n'$ whenever $\delta < n-1$. Let $\delta < n-1$. Calculating the integrals in (4.13), we get

$$B_q(t) = c \left(\log \frac{e}{t}\right)^{\varepsilon + \eta/q + 1/n' - \delta/n} \left[1 - \left(\log \frac{e}{t}\right)^{\delta n'/n - 1}\right]^{1/n'}. \tag{4.16}$$

By (4.14) this yields

$$\varepsilon + \frac{n}{q} + \frac{1}{n'} - \frac{\delta}{n} \leq 0, \qquad q \geq q_0, \tag{4.17}$$

and sending q to ∞ gives $\varepsilon \leq \delta/n - 1/n'$.

Prove (ii). Let $\eta = 0$. If $\varepsilon = 0$, then we have (4.15) which cannot hold simultaneously with (4.14) and (4.11). Going along the lines of the proof of (i) we get $\varepsilon \leq \delta/n - 1/n'$ provided $\delta < n-1$.

Finally prove (iii). Let $\eta > 0$ and $\varepsilon \geq 0$. Then (4.15) holds again and it suffices to prove that $\varepsilon < \delta/n - 1/n'$ for each $\delta < n-1$. If $\delta < n-1$, then we have (4.16) and (4.17). This time $\eta > 0$. Therefore (4.17) yields the claim. $\square$

Before presenting the theorem on necessary and sufficient conditions, we explicitly formulate two embedding inequalities related to Theorem 3.2 (iii) and (iv). Namely, for all $f \in C_0^\infty(B)$ we have

$$\begin{aligned}
&\left(\int_B \left| \left(\log \frac{e}{|x|} \right)^{\delta/n - 1/n'} f(x) \right|^q |x|^{-n} \left(\log \frac{e}{|x|} \right)^{-1} dx \right)^{1/q} \\
&\leq c q^{1/n'} \left(\int_B |\nabla f(x)|^n \left(\log \frac{e}{|x|} \right)^{\delta} dx \right)^{1/n}, \quad q \geq n,
\end{aligned} \tag{4.18}$$

where $\delta < n-1$, and, similarly for all $f \in C_0^\infty(B)$,

$$\begin{aligned}
&\left(\int_B |f(x)|^q \left(\log \frac{e}{|x|} \right)^{\eta - 1} |x|^{-n} \, dx \right)^{1/q} \\
&\leq c q^{1/n'} \left(\int_B |\nabla f(x)|^n \left(\log \frac{e}{|x|} \right)^{n-1} dx \right)^{1/n}, \quad q > n,
\end{aligned} \tag{4.19}$$

where $\eta < 0$.

Theorem 4.5. *Let $\delta, \varepsilon, \eta \in \mathbb{R}^1$. Then there is q_0 such that* (4.11) *holds for every $f \in C_0^\infty(B)$ and every $q \geq q_0$ if and only if any of* (i), (ii), (iii) *from Lemma 4.4 holds.*

If c_q^* denotes the best constant in (4.11)*, then*

$$c_q^* \leq c q^{1/n'}. \tag{4.20}$$

The constant c in (4.20) *may depend on δ, ε, η, n and q_0, but it is independent of q.*

PROOF. The necessity is a consequence of Lemma 4.4. We prove the sufficiency. First, let $\eta < 0$, $\varepsilon \le 0$, and $\delta \ge n-1$. Then

$$\left(\log \frac{e}{|x|}\right)^{n-1} \le \left(\log \frac{e}{|x|}\right)^{\delta}, \quad x \in B, \tag{4.21}$$

and

$$\left(\log \frac{e}{|x|}\right)^{\varepsilon q+\eta-1} \le \left(\log \frac{e}{|x|}\right)^{\eta-1}, \quad x \in B. \tag{4.22}$$

Combining (4.19), (4.21) and (4.22), we get (4.11) and (4.20).

If $\eta < 0$, $\delta < n-1$ and $\varepsilon \le \delta/n - 1/n'$, the proof of (4.11) and (4.20) is analogous, using now (4.18).

The second case corresponds to (ii) from Lemma 4.4. This time $\eta = 0$, $\varepsilon < 0$ and $\delta \ge n(\varepsilon + 1/n')$. Put $\delta_\varepsilon = n(\varepsilon + 1/n')$. Then $\varepsilon = \delta_\varepsilon/n - 1/n'$, $\delta_\varepsilon < n-1$. Further,

$$\left(\log \frac{e}{|x|}\right)^{\delta_\varepsilon} \le \left(\log \frac{e}{|x|}\right)^{\delta}, \quad x \in B. \tag{4.23}$$

According to Theorem 3.2 (iii) (cf. (4.18)) the inequalities (4.11) and (4.20) hold for the parameter triple δ_ε, ε, 0. Then (4.23) implies (4.11) for δ, ε, 0.

The last case is that of Lemma 4.4 (iii). We have $\eta > 0$. If $\delta \ge n-1$, choose $\varepsilon < \widetilde{\varepsilon} < 0$, and if $\delta < n-1$, put $\widetilde{\varepsilon} = \delta/n - 1/n'$. In either case,

$$\left(\log \frac{e}{|x|}\right)^{\varepsilon q+\eta-1} \le \left(\log \frac{e}{|x|}\right)^{\widetilde{\varepsilon} q-1}, \quad x \in B, \tag{4.24}$$

for large q's. The triple δ, $\widetilde{\varepsilon}$, 0, meets the assumptions of the previous case, that is, it corresponds to (ii) from Lemma 4.4. Hence (4.11) and (4.20) hold with the parameters δ, $\widetilde{\varepsilon}$, 0 and the desired assertion follows now from (4.24). $\square$

Remark 4.6. The inequality (4.11) does *not* hold with any $|x|^{-\mu}$ with $\mu > n$ replacing $|x|^{-n}$ on the left hand side. This can be seen by an analysis of the proof of Lemma 4.4 (the one-dimensional integrals become infinite regardless of the choice of the remaining parameters). In this sense the inequality (4.11) is sharp.

We now formulate consequences of Theorem 4.2 and 4.5 in terms of the limiting embedding of the weighted Sobolev spaces in question into exponential Orlicz spaces generated by Young functions

$$\Phi_k(t) = \sum_{j=k}^{\infty} \frac{t^{n'j}}{j!}, \qquad t \in \mathbb{R}^1, \tag{4.25}$$

where k is a non-negative integer. The proof is standard, using Taylor's expansion of the exponential function.

Theorem 4.7. *Let δ, ε, $\eta \in \mathbb{R}^1$ and $\delta < n(n-1)$. Then there is $k \in \mathbf{N}$ such that the embedding*

$$W_0^{1,n}(B, |x|^\delta) \hookrightarrow L_{\Phi_k}(B, |x|^\varepsilon, |x|^\eta) \tag{4.26}$$

holds if and only if any of the conditions (i), (ii), (ii) *from Lemma 4.1 is satisfied.*

PROOF. We prove the "if" part. Let $f \in C_0^\infty(B)$ be such that

$$\int_B |\nabla f(x)|^n |x|^\delta \, dx = 1.$$

According to Theorem 4.2 there is $k \in \mathbf{N}$ and $c > 0$ such that

$$\int_B | \, |x|^\varepsilon f(x)|^{n'j} |x|^\eta \, dx \le (cn'j)^j, \qquad j \ge k.$$

This implies

$$m_{\Phi_k}(f/\lambda, |x|^\varepsilon, |x|^\eta) \le \sum_{j=k}^\infty \frac{1}{j!} \left(\frac{cn'j}{\lambda} \right)^j < \infty \tag{4.27}$$

if $\lambda > 0$ is large enough. Oobserve that c, k and λ in (4.27) are independent of f. This proves (4.26). The "only if" part follows by reversing the above arguments. □

Remark 4.8. In the case (i) of Lemma 4.1, that is, $\eta > -n$, the embedding (4.26) is equivalent to $W_0^{1,n}(B, |x|^\delta) \hookrightarrow L_{\Phi_1}(B, |x|^\varepsilon, |x|^\eta)$. Indeed, since $|x|^\eta$ is integrable over B the Orlicz spaces $L_{\Phi_k}(B, |x|^\varepsilon, |x|^\eta)$ and $L_{\Phi_1}(B, |x|^\varepsilon, |x|^\eta)$ coincide and have equivalent norms. This can be seen easily in view of equivalence of Φ_k and Φ_1 for any k, that is, taking into account that $\Phi_1(ct) \le \Phi_k(t) \le \Phi_1(t)$ for large t's and c independent of t and mimicking the arguments of [KR], Chapter II, Thm. 8.1. Note that, generally, $L_{\Phi_{k_1}}(B, w_1, w_2)$ and $L_{\Phi_{k_2}}(B, w_1, w_2)$ are equivalent if $w_2 \in L_1(B)$.

An analogous argument, invoking Lemma 4.4 and Theorem 4.5 leads to the following assertion.

Theorem 4.9. *Let δ, ε, $\eta \in \mathbb{R}^1$. Then there is $k \in \mathbf{N}$ such that the embedding*

$$\begin{aligned} &W_0^{1,n}\left(B, \left(\log \frac{e}{|x|}\right)^\delta\right) \\ &\hookrightarrow L_{\Phi_k}\left(B, \left(\log \frac{e}{|x|}\right)^\varepsilon, |x|^{-n} \left(\log \frac{e}{|x|}\right)^{\eta-1}\right) \end{aligned} \tag{4.28}$$

holds if and only if any of the conditions (i), (ii), (ii) *from Lemma 4.4 holds.*

Remark 4.10. If $\eta < 0$ (see Lemma 4.4. (i)), then the embedding in (4.28) is equivalent to

$$W_0^{1,n}\left(B, \left(\log\frac{e}{|x|}\right)^{\delta}\right) \hookrightarrow L_{\Phi_1}\left(B, \left(\log\frac{e}{|x|}\right)^{\varepsilon}, |x|^{-n}\left(\log\frac{e}{|x|}\right)^{\eta-1}\right).$$

This follows by an argument analogous to that in Remark 4.8.

Remark 4.11. It is also worth pointing out that the powers q on the left hand sides of (3.20), (3.21), and (3.22) are smaller than those gained from a straightforward application of Young's inequality to the convolution $|x|^{1-n} \star f$ in the non-weighted case. However, passing to the infinite series for the exponential function is still possible without affecting the growth of Φ in the target spaces.

We conclude this section with considering the exclusive case $n = 2$. We shall not pursue a detailed study of the corresponding logarithmic kernel here. Instead we present a simple weighted embedding relying on the non-weighted limiting embedding, and generalizing the classical non-weighted theorem. Again, the point of interest about the proof and the results is a deterioration of the sublimiting embedding due to the use of Hölder's inequality which, nevertheless, has no affect on the resulting target space.

Theorem 4.12. *Let* $n = 2$, $\varepsilon > 0$, $w \in L_{1+\varepsilon}(B)$, $\Phi(t) = \exp t^2 - 1$, $t \in \mathbb{R}^1$. *Then* $W_0^{1,2}(B) \hookrightarrow L_\Phi(B, 1, w)$.

Proof. Let $q \geq 1$, $r > 1$, $1/r + 1/r' = 1$, $r > 1 + 1/\varepsilon$, and $f \in W_0^{1,2}(B)$. Then

$$\begin{aligned}\int_B |f(x)|^q w(x)\,dx &\leq \left(\int_B |f(x)|^{qr}\,dx\right)^{1/r} \|w\|_{L_{1+\varepsilon}} \\ &\leq c\|f\|_{L_{qr}}^q \leq c(c')^{1/r} r^q q^q\end{aligned}$$

for some c and c' independent of f and w. Now it is easy to finish the proof. □

References

[C] Nguyen Phuong Các, *On an inequality of Hardy's type*, Chinese J. Math. 13(1985), 203–208.

[EK] D. E. Edmunds and M. Krbec, *Two limiting cases of Sobolev imbeddings*, Houston J. Math. 21(1995), 119–128.

[ET] D. E. Edmunds and H. Triebel, *Logarithmic Sobolev spaces and their applications to spectral theory*, Proc. London Math. Soc., **III 71** (1995), 333–371.

[FLS] N. Fusco, P. L. Lions and C. Sbordone, *Sobolev imbedding theorems in borderline cases*, Proc. Amer. Math. Soc., **124** (1996), 561–565.

[GT] D. Gilbarg and N. Trudinger, *Elliptic partial differential equations of second order*, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo 1983.

[G] P. Gurka, *Generalized Hardy's inequality*, Časopis Pěst. Mat. 109(1984), 194–203.

[KS] R. Kerman and E. T. Sawyer, *On weighted norm inequalities for positive linear operators*, Proc. Amer. Math. Soc. 105(1989), 589–593.

[KR] M. A. Krasnosel'skii and Ya. B. Rutitskii, *Convex functions and Orlicz spaces*, Noordhof, Groningen 1961. English transl. from the first Russian edition Gos. Izd. Fiz. Mat. Lit., Moskva 1958.

[Ma1] W. Mazy'a, *Einbettungssätze für Sobolewsche Räume I*, Teubner–Texte Math., Teubner, Leipzig 1979.

[Ma2] V. G. Mazy'a, *Sobolev spaces*, Springer-Verlag, Berlin–Heidelberg–New York–Tokyo 1985.

[Mo] J. Moser, *A sharp form of an inequality by N. Trudinger*, Indiana Univ. Math. J. 20(1971), 1077–1092.

[Tri] H. Triebel, *Approximation numbers and entropy numbers of embeddings of fractional Besov-Sobolev spaces in Orlicz spaces*, Proc. London Math. Soc. 66(1993), 589–618.

[Tru] N. Trudinger, *On imbeddings into Orlicz spaces and some applications*, J. Math. Mech. 17(1967), 473–483.