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CONVERGENCE OF AUTOMORPHISMS AND SEMICONTINUITY OF AUTOMORPHISM GROUPS

Abstract

We study the compactness of the automorphism group of a domain in $\mathbb{C}^n$, and in particular the convergence properties of mappings. We supply an application to the semicontinuity of automorphism groups under perturbation of the underlying domain. Relevant examples are provided.

1 Introduction

A *domain* Ω in $\mathbb{C}^n$ is a connected, open set. An *automorphism* of Ω is a biholomorphic self-map. The collection of automorphisms forms a group under the binary operation of composition of mappings. The standard topology on this group is uniform convergence on compact sets, or the compact-open topology. We denote the automorphism group by $\mathrm{Aut}(\Omega)$. When Ω is a bounded domain, the group $\mathrm{Aut}(\Omega)$ is a real (never a complex) Lie group.

Although domains with *transitive automorphism group* are of some interest, they are relatively rare (see [9, Section III.3]). A geometrically more natural condition to consider, and one that gives rise to a more robust and broader class of domains, is that of having *non-compact automorphism group*. Clearly a domain has non-compact automorphism group if there are automorphisms $\{\varphi_j\}$ which have no subsequence that converges to an automorphism. The

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following proposition of Henri Cartan is of particular utility in the study of these domains:

Proposition 1.1. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain. Then Ω has non-compact automorphism group if and only if there are a point $X \in \Omega$, a point $P \in \partial\Omega$, and automorphisms φ_j of Ω such that $\varphi_j(X) \to P$ as $j \to \infty$.*

We refer the reader to [14, p. 65] for discussion and proof of Cartan's result.

We say that a domain $\Omega \subseteq \mathbb{C}^n$ has C^k boundary, $k \geq 1$ an integer, if it is possible to write

$$\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$$

for a function ρ that is C^k and which satisfies $\nabla\rho \neq 0$ on $\partial\Omega$. This definition is equivalent to a number of other natural definitions of C^k boundary for a domain (see the Appendices in [11]). Below we shall define a topology on the collection of domains with C^k boundary.

Domains with compact automorphism group exhibit certain rigidities which are of interest for our studies. We begin this paper by showing that, for certain smoothly bounded domains with compact automorphism group, the convergence of automorphisms will take place in a much stronger topology than the standard one specified in the first paragraph. This fact has intrinsic interest, but is also of considerable use for further studies in complex function theory. It is even new in the context of one complex variable.

As an application of the ideas in the last paragraph, we offer a new result about the semicontinuity of the automorphism group under perturbation of the underlying domain. This generalizes results of [7]. We also offer a direct generalization of the result of [7, Theorem 0.1] to finite type domains. Some of the proof techniques presented here are new.

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2 Convergence of holomorphic mappings

Throughout this section, and in subsequent parts of the paper, we shall use the concept of finite type as developed by Kohn/Catlin/D'Angelo. See [11, Section 11.5] for an explication of these ideas. For completeness we supply the relevant definitions here.

Definition 2.1. Let $\Omega = \{z \in \mathbb{C}^2 : \rho(z) < 0\}$ be a smoothly bounded domain and $P \in \partial\Omega$. Let m be a non-negative integer. We say that $\partial\Omega$ is of finite type m at P if the following condition holds: there is a non-singular complex analytic disc φ tangent to $\partial\Omega$ at P (with $\varphi(0) = P$ and $\varphi'(0) \neq 0$) such that, for small ζ,

$$|\rho \circ \phi(\zeta)| \leq C|\zeta|^m .$$

But there is no non-singular disc ψ tangent to $\partial\Omega$ at P such that $\psi(0) = P$, $\psi'(0) \neq 0$, and, for small ζ,

$$|\rho \circ \phi(\zeta)| \leq C|\zeta|^{(m+1)}.$$

We note that the definition just given, which is sometimes called *geometric finite type*, is equivalent to another definition involving commutators of vector fields (and which is called *analytic finite type*). Details may be found in [11, Section 11.5].

The definition of finite type in higher dimensions (due to J. P. D'Angelo) is more complex. We give it in three steps.

Definition 2.2. Let f be a scalar-valued holomorphic function of a complex variable and P a point of its domain. The *multiplicity* of f at P is defined to be the least positive integer k such that the k^{th} derivative of f does not vanish at P. If m is that multiplicity then we write $v_P(f) = v(f) = m$.

If ϕ is instead a vector-valued holomorphic function of a complex variable then its multiplicity at P is defined to be the minimum of the multiplicities of its entries. If that minimum is m then we write $v_P(\phi) = v(\phi) = m$.

Definition 2.3. Let $\phi : D \to \mathbb{C}^n$ be a holomorphic curve and ρ the defining function for a smoothly bounded domain Ω. Then the *pullback* of ρ under ϕ is the function $\phi^*\rho(\zeta) = \rho \circ \phi(\zeta)$.

Definition 2.4. Let Ω be a smoothly bounded domain in $\mathbb{C}^n$ and $\partial\Omega$ its boundary. Let $P \in \partial\Omega$. Let ρ be a defining function for Ω in a neighborhood of P. We say that P is a point of finite type (or finite 1-type) if there is a constant $C > 0$ such that

$$\frac{v(\phi^*\rho)}{v(\phi)} \leq C$$

whenever ϕ is a non-constant (possibly singular) one-dimensional holomorphic curve through P such that $\phi(0) = P$.

The infimum of all such constants C is called the *type* (or *1- type*) of P. It is denoted by $\Delta(M, P) = \Delta_1(M, P)$.

Again, the reference [11, Section 11.5] provides a thorough treatment, with examples, of the concept of point of finite type.

It is a basic fact—see, for instance, [2, Main Theorem, p. 103] and the discussion in [11, Section 11.5]—that any automorphism of a smoothly bounded, finite type domain Ω extends to be a C^∞ diffeomorphism of the closure of the domain Ω to itself.[1] Thus it is natural in the present context to equip the automorphism group with a different topology which we shall call the C^k topology. Fix k a positive integer. Let $\epsilon > 0$. If $\varphi_0 \in \mathrm{Aut}(\Omega)$ then a subbasic neighborhood of φ_0 is one of the form

$$\mathcal{U}_{k,\epsilon}(\varphi_0) \equiv \left\{ \varphi \in \mathrm{Aut}(\Omega) : \left| \frac{\partial^\alpha}{\partial z^\alpha} (\varphi - \varphi_0)(z) \right| < \epsilon \text{ for all } z \in \Omega \right.$$

$$\left. \text{and all multi-indices } \alpha \text{ with } |\alpha| \leq k \right\}.$$

It is easy to see that, with this topology, $\mathrm{Aut}(\Omega)$ is still a real Lie group (see [10, Section V.2]) when Ω is a bounded domain.

Our first result of this section is as follows:

Proposition 2.5. *Let $\Omega \subseteq \mathbb{C}^n$ be a bounded finite type domain with compact automorphism group in the C^k topology, $k > 0$ an integer. Let α be a multi-index such that $|\alpha| \leq k$. Then there is a positive, finite constant K_α such that*

$$\sup_{z \in \Omega} \left| \frac{\partial^\alpha}{\partial z^\alpha} \varphi(z) \right| \leq K_\alpha \tag{1.5.1}$$

for all $\varphi \in \mathrm{Aut}(\Omega)$.

The point here is that we have a uniform bound on the αth derivative of *all* automorphisms of Ω, that bound being valid *up to the boundary.* A result of this kind was proved in [7, Proposition 5.1] for the automorphism group of a *strongly pseudoconvex* domain considered in the compact-open topology. That proof was rather complicated, using Fefferman's asymptotic expansion for the Bergman kernel of a strongly pseudoconvex domain [4, Theorem 2] as well as the concept of Bergman representative coordinates [6, Section 4.2]. The proof presented here—for the C^k topology—is much simpler, and works in considerably greater generality.

PROOF. Suppose to the contrary that, for some fixed multi-index α, there is no bound K_α. Then there are a sequence φ_j of automorphisms of Ω and points $P_j \in \Omega$ such that

$$\left| \frac{\partial^\alpha}{\partial z^\alpha} \varphi_j(P_j) \right| \to +\infty\,.$$

[1] In fact the standard condition to guarantee such an extension to a diffeomorphism of the closures is Bell's Condition R—see [11, Section 11.5]. Condition R is guaranteed by a subelliptic estimate for the $\overline{\partial}$-Neumann problem, and that condition is known to hold on domains of finite type.

But $\mathrm{Aut}(\Omega)$ is compact, so there is a subsequence φ_{j_k} that converges in the C^k topology to a limit automorphism φ_0. Let

$$L_0 \equiv \sup_{z\in\Omega} \left| \frac{\partial^\alpha}{\partial z^\alpha} \varphi_0(z) \right| ,$$

which is finite because Ω is finite type.

Let $\epsilon > 0$. Choose K so large that

$$\left| \frac{\partial^\alpha}{\partial z^\alpha} \varphi_{j_k}(P_{j_k}) \right| > L_0 + 2\epsilon$$

for $k > K$. Choose M so large that

$$\left| \frac{\partial^\alpha}{\partial z^\alpha} \varphi_{j_m}(z) - \frac{\partial^\alpha}{\partial z^\alpha} \varphi_0(z) \right| < \epsilon$$

for all $m > M$, $z \in \Omega$. It then follows that, for $\ell > \max(K, M)$,

$$\left| \frac{\partial^\alpha}{\partial z^\alpha} \varphi_0(P_{j_\ell}) \right| > L_0 + \epsilon .$$

This is impossible. □

The next result relates our different topologies on the automorphism group in an important new way.

Proposition 2.6. *Let k be a positive integer. Let Ω be a smoothly bounded domain on which*

$$\left| \frac{\partial^\alpha}{\partial z^\alpha} \varphi(z) \right| \le K_\alpha \tag{1.6.1}$$

for all $\varphi \in \mathrm{Aut}(\Omega)$, all $z \in \Omega$, and all multi-indices α such that $|\alpha| \le k$. Then any sequence φ_j of automorphisms that converges uniformly on compact sets to a limit automorphism φ_0 in fact converges in the C^{k-1} topology to φ_0.

Remark 2.7. As the previous result shows, the converse of this proposition is true as well for finite type domains.

Proof of the Proposition. From (1.6.1), there is a constant K_1 so that

$$|\nabla \varphi_j(z)| \le K_1$$

for all $\varphi \in \mathrm{Aut}(\Omega)$, all j, and all $z \in \Omega$. Let $\epsilon > 0$. Choose a compact set $K \subseteq \Omega$ so large that if $w \in \Omega \setminus K$ then there is a line segment ℓ_w connecting

w to an element $k_w \in K$ (and parametrized by $\gamma_w(t) = (1-t)w + tk_w$) which has length less than ϵ/K_1.

Now choose j so large that

$$|\varphi_j(z) - \varphi_0(z)| < \epsilon \tag{1.6.2}$$

for all $z \in K$. Choose a point $w \in \Omega \setminus K$. Then

$$\begin{aligned} |\varphi_j(w) - \varphi_0(w)| &\leq |\varphi_j(w) - \varphi_j(k_w)| + |\varphi_j(k_w) - \varphi_0(k_w)| + |\varphi_0(k_w) - \varphi_0(w)| \\ &\equiv I + II + III\,. \end{aligned}$$

Now we know that $II < \epsilon$ by (1.6.2). For I, notice that

$$\begin{aligned} |\varphi_j(w) - \varphi_j(k_w)| &= \left| \int_0^1 \frac{d}{dt} \left[\varphi_j \circ \ell_w(t)\right] \, dt \right| \\ &\leq K_1 \cdot \frac{\epsilon}{K_1} \\ &= \epsilon\,. \end{aligned}$$

A similar estimate obtains for III.

In summary,

$$|\varphi_j(w) - \varphi_0(w)| < 3\epsilon\,.$$

This gives the uniform convergence estimate that we want for all points of Ω. That proves the result for $k = 1$.

Of course similar estimates may be applied to $|(\partial^\alpha/\partial z^\alpha)\varphi_j(w) - (\partial^\alpha/\partial z^\alpha)\varphi_0(w)|$ for any $|\alpha| < k$. Thus we get convergence in the C^{k-1} topology. □

Corollary 2.8. *Let $\Omega \subseteq \mathbb{C}^n$ be a smoothly bounded domain on which automorphisms satisfy uniform bounds on derivatives as in (1.6.1). Let $\varphi_j \in$ $\mathrm{Aut}(\Omega)$ be a sequence of automorphisms that converges uniformly on compact sets to a limit automorphism φ_0. Then in fact $\varphi_j \to \varphi_0$ uniformly on $\overline{\Omega}$.*

PROOF. This is a special case of the preceding result. □

Remark 2.9. Let Ω be a strongly pseudoconvex domain with real analytic boundary which is not biholomorphic to the ball. Then the results on uniform bounds of derivatives of automorphisms are particularly easy to prove. For $\mathrm{Aut}(\Omega)$ must be compact (see [15, Main Theorem, p. 253]). It is further known—see [8]—that there is an open neighborhood U of $\overline{\Omega}$ such that every automorphism (and its inverse, of course) analytically continues to U. It then follows directly from Cauchy estimates that, if α is a multi-index, then

$$\left| \frac{\partial^\alpha}{\partial z^\alpha} \varphi(z) \right| \leq K_\alpha$$

for all $\varphi \in \mathrm{Aut}(\Omega)$ and all $z \in \Omega$.

It is possible to use Bergman representative coordinates (see [6, Section 4.2]) in a new fashion to obtain the uniform-bounds-on-derivatives result for finite type domains in $\mathbb{C}^2$ *in the compact-open topology.* More precisely,

Theorem 2.10. *Let $\Omega \subseteq \mathbb{C}^2$ be a smoothly bounded, finite type domain in $\mathbb{C}^2$ with compact automorphism group in the compact-open topology. Let α be a multi-index. Then there is a constant $K_\alpha > 0$ so that*

$$\left| \frac{\partial^\alpha}{\partial z^\alpha} \varphi(z) \right| \leq K_\alpha$$

for all $\varphi \in \mathrm{Aut}(\Omega)$ and all $z \in \Omega$.

Proof. For a fixed $w \in \Omega$, let δ_w denote the Dirac delta mass at w. Then of course

$$K(z, w) = P(\delta_w)(z) \tag{1.10.1}$$

for all $z \in \Omega$, where K is the Bergman kernel for Ω and P the Bergman projection.

Now, by a well-known formula of Kohn (see [12, Section 7.9]),

$$P = I - \overline{\partial}^* N \overline{\partial}\,.$$

Here N is the $\overline{\partial}$-Neumann operator. It follow that P is hypoelliptic up to the boundary (again see [12, Sections 7.8, 7.9]).

Let U be a tubular neighborhood of $\partial\Omega$. Let $L \subset\subset \Omega$ be a compact set so that $\partial L \subseteq U$. Now pick $w \in \partial L$. So there will be an $r > 0$, with r greater than the radius of U, so that $K(\,\cdot\,, w)$ is smooth on $\overline{\Omega} \cap B(w, r)$.

Now assume that $w \in U \cap \Omega$. Let $\widetilde{w}$ be the point of $\partial\Omega$ that is nearest to w. Then, because we are in complex dimension 2, (see [1, Theorem 3.1]) there is a holomorphic peak function[2] $f_{\widetilde{w}}$ for $\widetilde{w}$. We may replace $f_{\widetilde{w}}(z)$ with $[9 + f_{\widetilde{w}}(z)]/10$ so that our peak function does not vanish on $\overline{\Omega}$. Continue to

[2]The construction of peaking functions in [1, Theorem 3.1] is quite difficult and technical. It amounts to a delicate scaling procedure. An alternative approach to the matter, using entire functions that grow at a certain rate at infinity, appears in [5]. The paper [1] proves the peak point result for domains with real analytic boundary. The paper [5] proves the result for finite type domains.

denote the peak function by $f_{\widetilde{w}}$. Then we may write

$$\begin{aligned} K(z,w) &= P(\delta_w)(z) \\ &= \int_\Omega K(z,\zeta)\delta_w(\zeta) \\ &= \int_\Omega K(z,\zeta)\sum_j(\alpha_j)\cdot\left(\frac{1}{\eta_j^4\cdot\Omega_4}\right)\cdot\chi_{B(w,\eta_j)}(\zeta)\,dV(\zeta)\,. \end{aligned}$$

Here χ_S denotes the characteristic function of the set S. In the right-hand part of this last sequence of equalities, the α_j are positive numbers that sum to 1 and Ω_4 is the volume of the unit ball in $\mathbb{R}^4 \approx \mathbb{C}^2$ (see [11, Section 1.4]). [We are simply invoking here the mean value property of a holomorphic function on balls.] Also the η_j are an increasing sequence of finitely many positive radii with the largest of them equalling the distance τ of w to $\partial\Omega$.

Now this last equals

$$\int_\Omega K(z,\zeta)c\cdot f_{\widetilde{w}}^j(\zeta)\,dV(\zeta)+\mathcal{E}(z,w)=f_{\widetilde{w}}^j(z)+\mathcal{E}(z,w)\,,$$

where $c > 0$ is a constant, j (interpreted as a *power*) is a suitably chosen positive integer, and $\mathcal{E}(z,w)$ is an error term. Now we know that the first term in this last displayed expression *does not vanish* on Ω intersect a ball about w that has radius larger than τ and the error term is negligible in this regard—because the Bergman projection of $\sum_j \alpha_j \left(\frac{1}{\eta_j^4\cdot\Omega_4}\right)\chi_{B(w,\eta_j)}(\zeta)$ is, by inspection, approximated closely in the uniform topology by the dilated peaking function.

Thus Bergman representative coordinates (see [6, Section 4.2] for this concept), which are given by

$$b_{j,w}(z)=\frac{\partial}{\partial\bar\zeta_j}\log\frac{K(z,\zeta)}{K(\zeta,\zeta)}\bigg|_{\zeta=w}\,,$$

are well defined on $\beta_w\cap\Omega$ with $\beta_w = B(w,\tau')$ for some $\tau' > \tau$. And the size of $b_{j,w}$ may be taken to be uniformly bounded, independent of w, just by the noted regularity properties of the Bergman kernel. Of course $L\cap\beta_w \neq \emptyset$.

Now fix a multi-index α. Then certainly $|(\partial^\alpha/\partial z^\alpha)\varphi(z)|$ is bounded by some M_α for all $\varphi\in \mathrm{Aut}(\Omega)$ and all $z \in L$. But then the Bergman representative coordinates enable us to realize each automorphism as a linear map (namely, the Jacobian—again see [6, Section 4.2]) on $\beta_w\cap\Omega$. And the size of the coefficients of these linear maps depends only on the Jacobian of the automorphism at the center of the ball. Of course the center of the ball lies in a compact subset of Ω, so these Jacobians have uniformly bounded coefficients. The conclusion then is that $|(\partial^\alpha/\partial z^\alpha)\varphi(z)|$ is uniformly bounded on $L \cup \beta_w$. And the bound is independent of w. Remembering that w is an arbitrary element of ∂L, we see that $|(\partial^\alpha/\partial z^\alpha)\varphi(z)|$ is uniformly bounded on all of Ω, uniformly for all $\varphi \in \mathrm{Aut}(\Omega)$. □

3 Topologies on domains

Let $\Omega = \{z \in \mathbb{C}^n : \rho(z) < 0\}$ be a domain with C^k boundary. Let $\epsilon > 0$. We define an ϵ-neighborhood of Ω in the C^k topology to be a set of the form

$$\mathcal{E}_{\Omega,\epsilon} = \left\{ \Omega^* \subseteq \mathbb{C}^n : \Omega^* = \{z \in \mathbb{C}^n : \rho^*(z) < 0\} \text{ and } \|\rho - \rho^*\|_{C^k} < \epsilon \right\}.$$

Note particularly that $\mathcal{E}_{\Omega,\epsilon}$ is a *set of domains*. Our semicontinuity results below will be formulated in terms of this topology on the collection of domains with C^k boundary. In particular, when we speak of a "small C^k perturbation of Ω," we mean a domain selected from $\mathcal{E}_{\Omega,\epsilon}$ with $\epsilon > 0$ small. For convenience, when Ω' is an element of $\mathcal{E}_{\Omega,\epsilon}$, then we say that Ω' has C^k distance less than ϵ from Ω.

4 The semicontinuity theorem

Now one of the main results of this paper is the following:

Theorem 4.1. *Let Ω be a smoothly bounded, finite type domain in $\mathbb{C}^2$ which has compact automorphism group in the compact-open topology. Let k an integer be sufficiently large. Then there is an $\epsilon > 0$ so that if Ω' is a smoothly bounded, finite type domain with C^k distance less than ϵ from Ω, then $\mathrm{Aut}(\Omega')$ can be realized as a subgroup of $\mathrm{Aut}(\Omega)$. By this we mean that there is a smooth diffeomorphism $\Phi : \Omega' \to \Omega$ so that*

$$\varphi \longmapsto \Phi \circ \varphi \circ \Phi^{-1}$$

is a univalent homomorphism of $\mathrm{Aut}(\Omega')$ into $\mathrm{Aut}(\Omega)$.

PROOF. The proof of this result is standard (see [7, Theorem 0.1], so we only sketch the steps.

Step 1: There is a Riemannian metric, smooth on $\overline{\Omega}$, which is invariant under any automorphism of Ω. We construct this metric simply by averaging the Euclidean metric with respect to Haar measure on the automorphism group of Ω. In order for the resulting metric to be smooth to the boundary, we must invoke the uniform bounds on automorphism derivatives that we proved in Section 2.

Step 2: The metric in Step 1 can be modified so that it is a product metric near the boundary, and still invariant. This is a standard construction from Riemannian geometry, and we omit the details.

Step 3: We may form the metric double $\widehat{\Omega}$ of $\overline{\Omega}$, and the resulting metric is smooth on $\widehat{\Omega}$.

Step 4: Any automorphism of Ω can now be realized as an isometry of $\widehat{\Omega}$.

Step 5: By a classical result of David Ebin [3, Section 1], there is a semicontinuity result for isometries of compact Riemannian manifolds. We may apply this result to the isometry group of $\widehat{\Omega}$. In particular, any smooth deformation Ω' of Ω gives rise to a smooth deformation $\widehat{\Omega'}$ of $\widehat{\Omega}$ and hence to a deformation of the invariant metric on $\widehat{\Omega}$. Thus we may compare the isometry group of the perturbed metric to the isometry group of the original metric.

Step 6: We may unravel the construction to see that Step 5 may be interpreted to say that the automorphism group of Ω' is a subgroup of the automorphism group of Ω, and we may extract the conjugation map Φ from the conjugation map provided by Ebin's theorem.

That completes the argument. □

Since we introduced the C^k metric for the space of automorphisms, it is worthwhile to formulate a result for that topology. We have:

Theorem 4.2. *Let Ω be a smoothly bounded, finite type domain in $\mathbb{C}^n$. Equip $\text{Aut}(\Omega)$ with the C^k topology, some integer $k \geq 0$. Assume that Ω has compact automorphism group in the C^k topology. Then there is an $\epsilon > 0$ so that if Ω' is a smoothly bounded, finite type domain with C^m distance less than ϵ from Ω (with $m \leq k$), then $\text{Aut}(\Omega')$ can be realized as a subgroup of $\text{Aut}(\Omega)$. By this we mean that there is a smooth mapping $\Phi : \Omega' \to \Omega$ so that*

$$\varphi \longmapsto \Phi \circ \varphi \circ \Phi^{-1}$$

is a univalent homomorphism of $\text{Aut}(\Omega')$ into $\text{Aut}(\Omega)$.

PROOF. The proof is just the same as that for the last theorem. The main point is to have a uniform bound for derivatives of automorphisms (Proposition 2.5), so that the smooth-to-the-boundary invariant metric can be constructed. □

5 Some examples

In this section we provide some examples which bear on the context of Theorems 3.1 and 3.2.

Example 5.1. Let

$$\Omega = B(0,2) \setminus \overline{B}(0,1)\,.$$

Then Ω is a bounded domain, but it is not pseudoconvex.

Of course any automorphism of Ω continues analytically to $B(0,2)$. But it also must preserve $S_1 \equiv \{z : |z| = 1\}$ and $S_2 \equiv \{z : |z| = 2\}$. It follows that $\mathrm{Aut}(\Omega) = U(n)$. Now an obvious Lie subgroup of $U(n)$ is $SU(n)$. But $SU(n)$ has precisely the same orbits as $U(n)$—in fact the orbit of any point in S_2 is S_2 itself and the orbit of any point in S_1 is S_1 itself. It follows that there is no domain that is "near" to Ω in any C^k topology and with automorphism group that is precisely $SU(n)$. Therefore an obvious sort of converse to Theorems 4.1, 4.2 fails in this case. That is to say, not every closed subgroup of the automorphism group of Ω arises as the automorphism group of a nearby domain.

We note, however, that with suitable hypotheses (including strong pseudoconvexity), there is a sort of converse to Theorem 4.1—see [13, Section 1].

Example 5.2. If we do not mandate that the domain Ω have smooth boundary, then Theorems 3.1 and 3.2 need not be true. As a simple example, consider

$$\Omega = \{z \in \mathbb{C}^n : 0 < |z| < 1\}\,.$$

Of course this Ω is not pseudoconvex and does not have a smooth defining function (so does not have smooth boundary by our reckoning). The automorphism group of Ω is $U(n)$. A "small" perturbation of Ω is $\Omega' = B = \{z \in \mathbb{C}^n : |z| < 1\}$. But the automorphism group of Ω' is much larger than $U(n)$ (it includes $U(n)$, but it also includes the Möbius transformations). So semicontinuity of automorphism groups fails.

6 Closing remarks

The idea of semicontinuity for automorphism groups is an important paradigm that has far-reaching applicability in geometry. In any situation where symmetries are considered, one may formulate the idea of semicontinuity. The basic idea is that symmetry is hard to create but easy to destroy: small perturbations can and will reduce symmetry, but it takes a large perturbation to create symmetry.

In the present paper we have taken a fundamental theorem of [GK1, Theorem 0.1] in the strongly pseudoconvex setting and extended it in various ways to the finite type setting. It would be interesting to know whether the result is true in complete generality. Even more interesting would be an example—say in the infinite type context—in which semicontinuity fails.

We hope to explore these matters further in future papers.

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