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ON SEQUENCES OF MONOTONE FUNCTIONS

Abstract

Several kinds of convergence (including pointwise, monotone, a.c., uniform, ...) in the family of monotone functions are investigated.

Let $\mathcal{R}$ denote the set of all reals. Observe that the limit f of a converging sequence of monotone functions $f_n : I \mapsto \mathcal{R}$, where I is a nondegenerate interval, is a monotone function. Of course, there is a subsequence $(f_{n_k})_k$, where all functions f_{n_k} are decreasing or increasing and consequently, the function f is decreasing or respectively increasing as the limit of the subsequence $(f_{n_k})_k$.

Theorem 1. *If $f : [a,b] \mapsto \mathcal{R}$ is an increasing function (i.e. nondecreasing) then there are continuous increasing functions $f_n : [a,b] \mapsto \mathcal{R}$, $n = 1, 2, \ldots$, such that $f_n(a) = f(a)$, $f_n(b) = f(b)$ for $n \geq 1$ and $\lim_{n\to\infty} f_n = f$.*

Proof. Fix a positive integer n and observe that the set

$$A = \left\{x \in [a,b]; \operatorname{osc} f(x) \geq \frac{1}{n}\right\}$$

is empty or finite. We can assume that A is nonempty. Let

$$A = \{x_1, \ldots, x_k\}, \quad x_1 < \cdots < x_k.$$

There are closed intervals $I_i = [a_i, b_i]$, $i \leq k$, such that

$b_{i-1} < a_i < x_i < b_i < a_{i+1}$ for $i = 2, \ldots, k-1$;

if $a < x_1$ then $a < a_1 < x_1 < b_1$;

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if $a = x_1$ then $a = a_1 = x_1 < b_1$;

if $x_k < b$ then $a_k < x_k < b_k < b$;

if $x_k = b$ then $a_k < x_k = b_k = b$;

$b_i - a_i < \frac{1}{n}$ for $i \leq k$;

$f(b_i) - f(a_i) < \operatorname{osc} f(x_i) + \frac{1}{n}$ for $i \leq k$.

Since $\operatorname{osc} f(x) < \frac{1}{n}$ for each point $x \in [a,b] \setminus A$, there are points $c_{i,j} \in (b_{i-1}, a_i)$, $i = 2, \ldots, k$, $j \leq j(i)$, such that

$$b_{i-1} = c_{i,1} < c_{i,2} < \cdots < c_{i,j(i)-1} < c_{i,j(i)} = a_i \text{ for } i = 2, \ldots, k;$$
$$f(c_{i,j+1}) - f(c_{i,j}) < \frac{1}{n} \text{ for } i = 2, \ldots, k.$$

Analogously, if $a < a_1$ ($b_k < b$) there are points $c_{1,j}$, $j \leq j(1)$, ($c_{k+1,j}$, $j \leq j(k+1)$,) such that

$$a = c_{1,1} < \cdots < c_{1,j(1)} = a_1$$
$$(b_k = c_{k+1,1} < \cdots < c_{k+1,j(k+1)} = b) \text{ for } j \leq j(1) \quad (j \leq j(k+1));$$
$$f(c_{1,j+1}) - f(c_{1,j}) < \frac{1}{n}$$
$$\Big(f(c_{k+1,j+1}) - f(c_{k+1,j}) < \frac{1}{n}\Big) \text{ for } j \leq j(1) \quad (j \leq j(k+1)).$$

Define on the interval $[a,b]$ the following continuous increasing function

$$f_n(x) = \begin{cases} f(x) & \text{for} & x \in \{a, b, c_{i,j}\}, \;\; i \leq k+1, \;\; j \leq j(i) \\ f(x) & \text{for} & x = x_i, \;\; i \leq k \\ \text{linear} & \text{otherwise on} & [a,b]. \end{cases}$$

We will prove that $\lim_{n\to\infty} f_n = f$.

If $x \in \{a, b, x_i; i \leq k\}$ then $f_n(x) = f(x)$. Moreover, if $x \in [c_{i,j}, c_{i,j+1}]$ then $|f_n(x) - f(x)| < \frac{1}{n}$. So, if

$$x \in [a,b] \;\; \wedge \;\; \operatorname{dist}(x, A) = \inf\big\{|u - x|; u \in A\big\} \geq \frac{1}{n}$$

then

$$|f_n(x) - f(x)| < \frac{1}{n}.$$

For each point $x \in [a,b]$ which is a discontinuity point of the function f there is a positive integer n such that $\operatorname{osc} f(x) > \frac{1}{n}$. Consequently, for every $k > n$ we have $f_k(x) = f(x)$.

Now we suppose that $x \in [a,b]$ is a continuity point of the function f. Fix a positive real η and a positive integer n with $\frac{2}{n} < \eta$. Let

$$B = \left\{ x \in [a,b]; \operatorname{osc} f(x) \geq \frac{1}{n} \right\}.$$

Since $x \in [a,b] \setminus B$ and B is a closed set, there is a positive integer $k > n$ with

$$\left(x - \frac{1}{k}, x + \frac{1}{k} \right) \cap B = \emptyset.$$

Fix an integer $m > k$. If

$$E = \left\{ y \in [a,b]; \operatorname{osc} f(y) \geq \frac{1}{m} \right\}$$

then

$$|f_m(x) - f(x)| < \frac{1}{m} < \eta$$

if $\operatorname{dist}(x, E) \geq \frac{1}{m}$ and

$$|f_m(x) - f(x)| < \operatorname{osc} f(y) + \frac{1}{m} < \frac{1}{n} + \frac{1}{m} < \frac{2}{n} < \eta$$

for some $y \in E \setminus B$, if $\operatorname{dist}(x, E) < \frac{1}{m}$.

So, $\lim_{n\to\infty} f_n(x) = f(x)$ and the proof is completed. □

It is well known that the limit of a decreasing (increasing) sequence of continuous functions is upper (lower) semicontinuous.

Theorem 2. *If $f : [a,b] \mapsto \mathcal{R}$ is an upper semicontinuous increasing function then there are continuous increasing functions $f_n : [a,b] \mapsto \mathcal{R}$, $n \geq 1$, such that $f_n \geq f_{n+1} > f$ for $n \geq 1$ and $f = \lim_{n\to\infty} f_n$.*

In the proof of the above theorem we apply the following sandwich lemma:

Lemma 1. *Let $f : [a,b] \mapsto \mathcal{R}$ be an upper semicontinuous increasing (decreasing) function and let $g : [a,b] \mapsto \mathcal{R}$ be a continuous function such that $f(x) < g(x)$ for each $x \in [a,b]$. Then there is a continuous increasing (decreasing) function $h : [a,b] \mapsto \mathcal{R}$ such that $f(x) < h(x) < g(x)$ for all $x \in [a,b]$.*

Proof of Lemma 1. We suppose that the function f is increasing. The proof for a decreasing function f is analogous. From the upper semicontinuity of f follows that f is continuous from the right hand. Let

$$r = \inf\{ g(x) - f(x); x \in [a,b] \}.$$

Since the function $g-f$ is positive and lower semicontinuous, the real r is positive. Define the set

$$A=\left\{x\in[a,b];\operatorname{osc} f(x)\geq\frac{r}{5}\right\}$$

and we observe that it is empty or finite. We can assume that A is nonempty. Let

$$A=\{x_1,\dots,x_k\},\quad x_1<\dots<x_k.$$

There are closed intervals $I_i=[a_i,x_i]$, $i\leq k$, such that

$$\begin{gathered}a<a_i<x_i<a_{i+1}<x_{i+1}\leq b \text{ for } i=1,\dots,k-1;\\ f(x_i)-f(a_i)<\operatorname{osc} f(x_i)+\frac{r}{5} \text{ for } i\leq k;\\ |g(x)-g(x_i)|<\frac{r}{5} \text{ for } x\in I_i, i\leq k.\end{gathered}$$

Let $x_0=a$. Since $\operatorname{osc} f(x)<\frac{r}{5}$ for each point $x\in[a,b]\setminus A$, there are points $c_{i,j}\in[x_{i-1},a_i]$, $i=1,\dots,k$, $j\leq j(i)$, such that

$$\begin{gathered}x_{i-1}=c_{i,1}<c_{i,2}<\dots<c_{i,j(i)-1}<c_{i,j(i)}=a_i \text{ for } i=1,\dots,k;\\ f(c_{i,j+1})-f(c_{i,j})<\frac{r}{5} \text{ for } i=1,\dots,k \text{ and } j\leq j(i)-1.\end{gathered}$$

Analogously, if $x_k<b$ there are points $c_{k+1,j}$, $j\leq j(k+1)$, such that

$$\begin{gathered}x_k=c_{k+1,1}<\dots<c_{k+1,j(k+1)}=b;\\ f(c_{k+1,j+1})-f(c_{k+1,j})<\frac{r}{5} \text{ for } j<j(k+1).\end{gathered}$$

Define on the interval $[a,b]$ a continuous increasing function in the following way:

$$g_1(x)=\begin{cases} f(x) & \text{for} \quad x\in\{a,b,c_{i,j}\},\ \ i\leq k+1,\ \ j\leq j(i)\\ f(x) & \text{for} \quad x=x_i,\ \ i\leq k\\ \text{linear} & \text{otherwise on} \quad [a,b].\end{cases}$$

If $x\in\{a,b,x_i;i\leq k\}$ then $g_1(x)=f(x)$. Moreover, if $x\in[c_{i,j},c_{i,j+1}]$ then $|g_1(x)-f(x)|<\frac{r}{5}$.

Let

$$h(x)=g_1(x)+\frac{r}{4},\quad x\in[a,b].$$

Then h is a continuous increasing function and for

$$x\in[a,b]\setminus\bigcup_{i\leq k}I_i$$

the inequalities

$$f(x) < g_1(x) + \frac{r}{5} < h(x) < f(x) + r \le g(x)$$

are true. If $x \in I_i$ for some $i \le k$ then

$$f(x) \le f(x_i-) = f(a_i) + \frac{r}{5} < g_1(a_i) + \frac{r}{4} = h(a_i) \le h(x)$$
$$= f(x_i) + \frac{r}{4} \le g(x_i) - r + \frac{r}{4} < g(x_i) - \frac{r}{5} < g(x).$$

So, the function h satisfies to all requirements. □

Proof of Theorem 2. Since the function f is upper semicontinuous, there are continuous functions $g_n : [a,b] \mapsto \mathcal{R}$ such that

$$f(x) < g_{n+1}(x) < g_n(x), \quad x \in [a,b], \quad n \ge 1,$$

and $f = \lim_{n\to\infty} g_n$ ([1]).

By Lemma 1 there is a continuous increasing function $f_1 : [a,b] \mapsto \mathcal{R}$ with $f < f_1 < g_1$. Let $h_2 = \min(f_1, g_2)$. By Lemma 2 there is a continuous increasing function $f_2 : [a,b] \mapsto \mathcal{R}$ with $f < f_2 < h_2 = \min(f_1, g_2)$. Next by induction, for each positive integer $n > 2$ there is a continuous increasing function $f_n : [a,b] \mapsto \mathcal{R}$ with $f < f_n < \min(f_{n-1}, g_n)$. Consequently, the sequence $(f_n)_n$ satisfies all requirements and the proof is completed. □

Remark 1. *If the function f is upper semicontinuous and increasing (decreasing) then there are continuous increasing (decreasing) functions $g_n : [a,b] \mapsto \mathcal{R}$ such that $g_n(a) = f(a)$, $g_n(b) = f(b)$, $g_n \ge g_{n+1}$ for $n \ge 1$ and $\lim_{n\to\infty} g_n = f$.*

Without loss of the generality we can suppose that

$$b = u = \inf\{x \in [a,b]; f(x) = f(b)\}$$

and

$$a = v = \sup\{x \in [a,b]; f(x) = f(a)\},$$

since in the contrary case we can consider the reduced function $f/[u,v]$.

We will prove the remark for the case of an increasing function f, because the case of a decreasing f is analogous. Let $(a_n)_n$ and $(b_n)_n$ be sequences such that

$$a < a_{n+1} < a_n < \cdots < a_1 < b_1 < \cdots < b_n < b_{n+1} < b,$$

and

$$a = \lim_{n\to\infty} a_n, ;\ b = \lim_{n\to\infty} b_n.$$

By Theorem 2, there is a decreasing sequence of continuous increasing functions $f_n : [a,b] \mapsto \mathcal{R}$ with $f = \lim_{n\to\infty} f_n$ and $f_n > f$ for $n = 1, 2, \ldots$. Find a strictly increasing sequence $(n_k)_k$ of positive integers such that

$$\lim_{k\to\infty} n_k = \infty;$$
$$\frac{f_{n_k}(a_k) - f(a_k)}{a_k - a} < \min_{i<k} \frac{f_{n_i}(a_i) - f(a_i)}{k(a_i - a)} \text{ for } k > 1;$$
$$\frac{f_{n_k}(b_k) - f(b_k)}{b - b_k} < \min_{i<k} \frac{f_{n_i}(b_i) - f(b_i)}{k(b - b_i)} \text{ for } k > 1.$$

For $k \geq 1$ let

$$h_k(x) = \frac{f_{n_k}(a_k) - f(a_k)}{a_k - a}(x - a) \text{ for } x \in [a, a_k],$$
$$h_k(x) = \frac{f_{n_k}(b_k) - f(b_k)}{b - b_k}(b - x) \text{ for } x \in [b_k, b]$$

and

$$g_k(x) = \begin{cases} f(x) + h_k(x) & \text{for } x \in [a, a_k] \\ f_{n_k}(x) & \text{for } x \in [a_k, b_k] \\ f(x) + h_k(x) & \text{for } x \in [b_k, b]. \end{cases}$$

The sequence $(g_k)_k$ satisfies all requirements and the proof is completed. □

Remark 2. *If a function $f : [a,b] \mapsto \mathcal{R}$ is increasing (decreasing) and lower semicontinuous then there is a increasing sequence of continuous increasing (decreasing) functions $f_n : [a,b] \mapsto \mathcal{R}$ such that $f_n(a) = f(a)$, $f_n(b) = f(b)$ for $n \geq 1$ and $\lim_{n\to\infty} f_n = f$.*

PROOF. It suffices to apply Remark 1 to the function $(-f)$. □

We will write that a.c. $\lim_{n\to\infty} f_n = f$ ([2, 3]) if for each point x there is a positive integer $n(x)$ such that for $n > n(x)$ the equality $f_n(x) = f(x)$ is true.

Since monotone functions have only countable sets of discontinuity points, we prove the following theorem:

Theorem 3. *Suppose that functions $f, f_n : [a,b] \mapsto \mathcal{R}$ satisfy the following conditions:*

- $f =$ a.c. $\lim_{n\to\infty} f_n$;
- *for each integer $n \geq 1$ the set $D(f_n)$ of all discontinuity points of the function f_n is countable.*

Then for each nonempty closed set $F \subset [a,b]$ there are an open interval I and a positive integer k such that $I \cap F \neq \emptyset$ and for each point $x \in (F \cap I) \setminus \bigcup_n D(f_n)$ and for each integer $n > k$ the equality $f(x) = f_n(x)$ is true.

Proof. Since a.c. $\lim_{n\to\infty} f_n = f$, for each point $x \in [a,b]$ there is a positive integer $n(x)$ such that $f(x) = f_n(x)$ for all integers $n > n(x)$. For each integer $m \geq 1$ let

$$A_m = \big\{x \in [a,b]; n(x) = m\big\}.$$

Let $F \subset [a,b]$ be a nonempty closed set. If the set F has an isolated point then the condition of our theorem is satisfied. So, we can assume that F is a perfect set. Since

$$F = \bigcup_m (A_m \cap F),$$

by the Baire category theorem there is an integer $k \geq 1$ such that the set $A_k \cap F$ is of the second category in F. Consequently, there is an open interval I such that $I \cap F \neq \emptyset$ and for every open interval $J \subset I$ with $J \cap F \neq \emptyset$ the set $J \cap F \cap A_k$ is of the second category in F. Since the set

$$E = \bigcup_n D(f_n)$$

is countable, the set

$$B = (I \cap F \cap A_k) \setminus E$$

is dense in $I \cap F$. The restricted functions $f_n/([a,b] \setminus E)$, $n \geq 1$, are continuous and for $m, n > k$ and $x \in B$ the equalities

$$f_n(x) = f_m(x) = f(x)$$

are true. So, for $m, n > k$ and for $x \in (I \cap F) \setminus E$ we obtain $f_m(x) = f_n(x) = f(x)$ and the proof is finished. □

Corollary 1. *If functions $f_n : [a,b] \mapsto \mathcal{R}$ are continuous and increasing (decreasing) and a.c. $\lim_{n\to\infty} f_n = f$ then the function f is increasing (decreasing) and in the class B_1^* (i.e. for every nonempty closed set $F \subset [a,b]$ there is an open interval I such that $I \cap F \neq \emptyset$ and the restricted function $f/(F \cap I)$ is continuous [2, 3]).*

Proof. This corollary is an evident consequence of the last theorem. □

Theorem 4. *Suppose that the function $f : [a,b] \mapsto \mathcal{R}$ is increasing (decreasing) and in the class B_1^*. Then there is a sequence of continuous increasing (decreasing) functions $f_n : [a,b] \mapsto \mathcal{R}$ with $f =$ a.c. $\lim_{n\to\infty} f_n$.*

PROOF. Observe that there are nonempty closed sets F_n, $n \geq 1$, such that

$$[a,b] = \bigcup_n F_n,$$

$$F_n \subset F_{n+1}, \quad n \geq 1,$$

and the restricted functions f/F_n are continuous ([2]). For each integer $n \geq 1$ the functions f/F_n can be extended to a continuous increasing (decreasing) function $f_n : [a,b] \mapsto \mathcal{R}$ such that $f_n(a) = f(a)$ and $f_n(b) = f(b)$. Evidently,

$$f = \text{a.c.} \lim_{n\to\infty} f_n$$

and the proof is completed. □

Theorem 5. *Let $f : [a,b] \mapsto \mathcal{R}$ be a function. The following conditions are equivalent:*

(a) f is increasing (decreasing);

(b) There are increasing (decreasing) functions $f_n : [a,b] \mapsto \mathcal{R}$ such that $f_n(a) = f(a)$, $f_n(b) = f(b)$ and the sets $D(f_n)$ of all discontinuity points of f_n, $n \geq 1$, are finite and $\lim_{n\to\infty} V(f_n - f, a, b) = 0$, where $V(f_n - f, a, b)$ denotes the total variation of $f_n - f$ on $[a,b]$;

(c) There is a sequence of increasing (decreasing) functions $f_n : [a,b] \mapsto \mathcal{R}$ which uniformly converges to f on $[a,b]$ and for which $f_n(a) = f(a)$, $f_n(b) = f(b)$ and the sets $D(f_n)$, $n \geq 1$, are finite.

PROOF. The implication $(c) \Rightarrow (a)$ is evident. Since for each point $x \in [a,b]$ we have

$$|f_n(x) - f(x)| \leq V(f_n - f, a, b),$$

we obtain the implication $(b) \Rightarrow (c)$. So, it suffices to prove the implication $(a) \Rightarrow (b)$. Fix an increasing function f and a positive real η. Observe that the set $D(f)$ is countable. We may assume that $D(f)$ is nonempty. Let

$$D(f) = \{a_1, \ldots, a_k, \ldots\}.$$

Define $g(a) = 0$ and for $x \in (a,b]$ let

$$g(x) = \sum_{a_i < x} \operatorname{osc} f(a_i) + (f(x) - f(x-)).$$

Put

$$h(x) = f(x) - g(x) \ \text{ for } \ x \in [a,b].$$

Then the function h is increasing and continuous and $f = h + g$. Since

$$\sum_i \operatorname{osc} f(a_i) \le f(b) - f(a) < \infty,$$

there is a positive integer k with

$$\sum_{i>k} \operatorname{osc} f(a_i) < \frac{\eta}{2}.$$

Put $g_1(a) = 0$, $g_1(b) = g(b)$ and for $x \in (a, b)$ let

$$g_1(x) = \sum_{a_i<x;i\le k} \operatorname{osc} f(a_i) + (f(x) - f(x-)).$$

If

$$f_1(x) = h(x) + g_1(x) \text{ for } x \in [a, b],$$

then the function f_1 is increasing and

$$f_1(a) = f(a), \quad f_1(b) = f(b),$$

the set $D(f_1) \subset \{a_1, \dots, a_k, b\}$ is finite, and

$$V(f_1 - f, a, b) = 2\sum_{i>k} \operatorname{osc} f(a_i) < 2\frac{\eta}{2} = \eta.$$

This completes the proof for the increasing functions. If f is a decreasing function on $[a, b]$ then we can use the proved part to the function $(-f)$. So, the proof is completed. □

Now, denote by ω_1 the first uncountable ordinal number and consider a transfinite sequence of monotone functions $f_\alpha : [a, b] \mapsto \mathcal{R}$, $\alpha < \omega_1$. We will say that the sequence $(f_\alpha)_{\alpha<\omega_1}$ converges to a function f (then we write $\lim_\alpha f_\alpha = f$) if for each point $x \in [a, b]$ there is a countable ordinal $\alpha(x)$ such that $f(x) = f_\alpha(x)$ for $\alpha > \alpha(x)$ ([4]).

Theorem 6. *If a function $f : [a, b] \mapsto \mathcal{R}$ is the limit of a transfinite sequence of monotone functions f_α, $\alpha < \omega_1$, then there is a countable ordinal β such that $f = f_\alpha$ for $\alpha > \beta$.*

PROOF. The assumptions imply the monotonicity of the function f. Let $A \subset [a, b]$ be a countable set containing $D(f) \cup \{a, b\}$ which is dense in $[a, b]$. There is a countable ordinal β such that

$$f_\alpha(x) = f(x), \quad x \in A, \quad \alpha > \beta.$$

If $\alpha > \beta$ is a countable ordinal then $f_\alpha = f$. Of course, if there is a point $x \in [a, b]$ with $f_\alpha(x) \neq f(x)$ then $x \in [a, b] \setminus A$. Consequently, f is continuous at x and there is a positive real r such that $f_\alpha(x)$ is not in the interval $(f(x) - r, f(x) + r)$. Since the graph of the restricted function f/A is dense in the graph of f, there are points $u, v \in A$ with

$$f(x) - r < f(u) < f(x) < f(v) < f(x) + r.$$

But

$$f_\alpha(u) = f(u), \quad f_\alpha(v) = f(v)$$

and f_α is monotone, so

$$f_\alpha(x) \in (f(x) - r, f(x) + r),$$

a contrary. This completes the proof. □

Since each nondegenerate interval I is the union of closed intervals I_n, $n \geq 1$, such that $\operatorname{int}(I_n) \cap \operatorname{int}(I_m) = \emptyset$ for $n \neq m$, we obtain that

Remark 3. *Theorems* 1*,* 4*,* 5 *and* 6 *and Remarks* 1 *and* 2 *are true for monotone functions* $f : I \mapsto \mathcal{R}$ *with* $f_n(a+) = f(a+)$ *and* $f_n(b-) = f(b-)$.

References

[1] A. M. Bruckner, J. B. Bruckner and B. S. Thomson, *Real Analysis*, Prentice-Hall, New Jersey, 1994.

[2] A. Császár and M. Laczkovich, *Discrete and equal convergence*, Studia Sci. Math. Hungar. **10** (1975), 463–472.

[3] R. J. O'Malley, *Approximately differentiable functions. The r topology*, Pacific J. Math. **72** (1977), 207–222.

[4] W. Sierpiński, *Sur les suites transfinies convergentes de fonctions de Baire*, Fund. Math. **1** (1920), 132–141.