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A TOPOLOGICAL INTERPRETATION OF $\mathfrak{t}$

This paper is dedicated to Prof. Rabbi Haim Judah

Abstract

Hurewicz found connections between some topological notions and the combinatorial cardinals $\mathfrak{b}$ and $\mathfrak{d}$. Recław gave topological meaning to the definition of the cardinal $\mathfrak{p}$. We extend the picture with a topological interpretation of the cardinal $\mathfrak{t}$. We compare our notion to the one related to $\mathfrak{p}$, and to some other classical notions. This sheds new light on the famous open problem whether $\mathfrak{p} = \mathfrak{t}$.

1 Introduction

Cardinals associated with infinitary combinatorics play an important role in set theory. Some earlier works ([8], [13], [1], [12], and [9]) have pointed out a strong connection between these cardinals and classes of spaces having certain topological properties. In this paper, we continue this line of research in a way which enables us to give a topological meaning to an open problem from infinitary combinatorics.

1.1 Preliminaries

Let $\omega = \{0, 1, 2, \dots\}$ and $2 = \{0, 1\}$ be the usual discrete spaces. ω^ω and 2^ω are equipped with the product topology. Identify 2^ω with $P(\omega)$ via characteristic functions. $[\omega]^\omega$ is the set of infinite elements of $P(\omega)$, with $O_n = \{a : n \in a\}$ and $O_{\neg n} = \{a : n \notin a\} = O_n^c$ $(n \in \omega)$ as a clopen subbase.

For $a, b \subseteq \omega$, $a \subseteq^* b$ if $a \setminus b$ is finite. $X \subseteq [\omega]^\omega$ is *centered* if every finite $F \subseteq X$ has an infinite intersection. $a \in [\omega]^\omega$ is an *almost-intersection* (a.i.)

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of X if it is infinite, and for all $b \in X$, $a \subseteq^* b$. $X \subseteq [\omega]^\omega$ is a *power* if it is centered, but has no a.i.. $\mathfrak{p}$ is the minimal size of a power. $X \subseteq [\omega]^\omega$ is a *tower* if it is linearly ordered by $\subseteq^*$, and has no a.i..[1] $\mathfrak{t}$ is the minimal size of a tower.

$\leq^*$ is the partial order defined on ω^ω by eventual dominance ($f \leq^* g$ iff $\forall^\infty n\ (f(n) \leq g(n))$). $\mathfrak{b}$ is the minimal size of an unbounded family, and $\mathfrak{d}$ is the minimal size of a dominating family, with respect to $\leq^*$.

The Main Problem

Let $\mathfrak{c}$ denote the size of the continuum. The following holds.

Theorem 1.1 ([4, Theorem 3.1.a]). $\mathfrak{p} \leq \mathfrak{t} \leq \mathfrak{b} \leq \mathfrak{d} \leq \mathfrak{c}$.

For each pair except $\mathfrak{p}$ and $\mathfrak{t}$, it is well known that a strict inequality is consistent.

Problem 1.2. *Is $\mathfrak{p} < \mathfrak{t}$ consistent with ZFC ?*

This is one of the major and oldest problems of infinitary combinatorics. Allusions for this problem can be found in Rothberger's works (see, e.g., [15, Lemma 7]). It is only known that $\mathfrak{p} = \omega_1 \to \mathfrak{t} = \omega_1$ [4, Theorem 3.1.b], hence also $\mathfrak{t} \leq \omega_2 \to \mathfrak{p} = \mathfrak{t}$.

1.2 γ-spaces

Throughout this paper, by *space* we mean a zero-dimensional, separable, metrizable topological space. As any such space is homeomorphic to a subset of the irrationals, our results can be thought of as dealing with sets of reals.

The definition of a γ-space is due to Gerlits and Nagy [6]. Let X be a space. A collection of open sets $\mathcal{G}$ is an *ω-cover* of X if for every finite $F \subseteq X$ there is a $G \in \mathcal{G}$ such that $F \subseteq G$. $\langle G_n : n < \omega \rangle$ is a *γ-sequence* for X if $\forall x \in X \forall^\infty n\, (x \in G_n)$. An open cover $\mathcal{G}$ of X is a *γ-cover* of X if it contains a γ-sequence for X. Clearly every γ-cover is an ω-cover. X is a *γ-space* if every ω-cover of X is a γ-cover of X. For convenience, we may assume that the ω-covers $\mathcal{G}$ of X are countable and clopen (replacing each element from $\mathcal{G}$ by all finite unions of basic clopen sets contained in it), and that for all finite $F \subseteq X$, there are *infinitely many* $G \in \mathcal{G}$ with $F \subseteq G$. (Using a partition of ω into ω many infinite sets, one can create a sequence consisting of ω copies of the original cover $\mathcal{G}$. The resulted sequence has the required property.) And so we will, from now on.

[1] Note that, unlike the customary definition, we do not demand that a tower is well-ordered. However, by [4, Theorem 3.7], this does not change the value of $\mathfrak{t}$.

Let Γ denote the collection of all γ-spaces. Recław has given an elegant characterization of γ-spaces.

Theorem 1.3 (Recław [13, Theorem 3.2])**.** *X is a γ-space iff no continuous image*[2] *of X in $[\omega]^\omega$ is a power.*

This gives an alternative proof to the following. For a family $\mathcal{F}$ of spaces, let

$$\mathrm{non}(\mathcal{F}) = \min\{|X| : X \text{ is a space, } X \notin \mathcal{F}\}$$

Corollary 1.4 (Galvin, Miller, Taylor [5, p. 146])**.** $\mathrm{non}(\Gamma) = \mathfrak{p}$.

2 The Tower of τ

Let X be a topological space. An open cover $\mathcal{G}$ of X is T_1 if for all distinct $x, y \in X$ there is a $G \in \mathcal{G}$ such that $x \in G, y \notin G$. Therefore, $\mathcal{G}$ is *not* T_1 iff there are distinct $x, y \in X$ such that for all $G \in \mathcal{G}$, $x \in G \to y \in G$. Strengthening the "not T_1" property demanding that the above holds for *all* $x, y \in X$ would trivialise $\mathcal{G}$ to be $\{X\}$, or $\{X, \emptyset\}$. We therefore compensate by means of a "modulo finite" restriction.

For $\mathcal{G} = \langle G_n : n < \omega \rangle$, we write $x \overset{\mathcal{G}}{\rightsquigarrow} y$ for

$$\forall^\infty n \, (x \in G_n \to y \in G_n) \, .$$

$\mathcal{G}$ is a *τ-sequence* for X if

1. $\mathcal{G}$ is a *large cover*; i.e. every element of X is covered by infinitely many elements of $\mathcal{G}$,[3] and
2. for all $x, y \in X$, either $x \overset{\mathcal{G}}{\rightsquigarrow} y$, or $y \overset{\mathcal{G}}{\rightsquigarrow} x$.

An open cover $\mathcal{J}$ of X is a *τ-cover* of X if it contains a τ-sequence for X. It is easy to see that every γ-cover is a τ-cover, and every τ-cover is an ω-cover.

X is a *τ-space* if every *clopen* τ-cover of X is a γ-cover of X. Equivalently, if every countable clopen τ-cover of X is a γ-cover of X. Let $\mathcal{T}$ denote the collection of all τ-spaces.

Corollary 2.1. $\Gamma \subseteq \mathcal{T}$.

[2] A continuous image of X is the image of a continuous function with domain X.
[3] This requirement was added in order to avoid trivial cases.

We wish to transfer our covering notions into $[\omega]^\omega$, in order to obtain their combinatorial versions. In Recław's proof of Theorem 1.3, the following natural function $h = h_\mathcal{G}$ is considered. Given a countable large cover $\mathcal{G} = \{G_n : n \in \omega\}$ of X, define $h : X \to [\omega]^\omega$ by $h(x) = \{n : x \in G_n\}$. Now, let us see what h does to our topological notions. Assume that $\mathcal{G} = \langle G_n : n < \omega\rangle$ is an ω-cover of X. Then for all finite $F \subseteq X$, F is a subset of infinitely many G_n's. That is, $n \in \cap h[F]$ for infinitely many n's. This means that $h[X]$ is centered.

Next, assume that $\mathcal{G}$ is a γ-sequence for X. Then $\forall x \forall^\infty n\ (x \in G_n)$. That is, $\forall x \forall^\infty n\ (n \in h(x))$, or ω is an a.i. of $h[X]$. Therefore, $\mathcal{G}$ is a γ-*cover* of X iff the associated $h[X]$ has an a.i..

Finally, a large cover $\mathcal{G}$ is a τ-sequence for X iff for all $x, y \in X$, either $x \overset{\mathcal{G}}{\rightsquigarrow} y$, or $y \overset{\mathcal{G}}{\rightsquigarrow} x$. Now, $a \overset{\mathcal{G}}{\rightsquigarrow} b$ iff $\forall^\infty n\ (n \in h(a) \to n \in h(b))$ iff $h(a) \subseteq^* h(b)$. Therefore, $h[X]$ is linearly ordered by $\subseteq^*$.

We have showed the following.

Lemma 2.2. *Assume that $\mathcal{G}$ is a countable large cover of X.*

1. *$\mathcal{G}$ is an ω-cover of X iff $h_\mathcal{G}[X]$ is centered.*
2. *$\mathcal{G}$ is a γ-cover of X iff $h_\mathcal{G}[X]$ has an almost-intersection.*
3. *$\mathcal{G}$ is a τ-sequence for X iff $h_\mathcal{G}[X]$ is linearly ordered by $\subseteq^*$.*

Note that if $\mathcal{G}$ is a *clopen* cover, then $h = h_\mathcal{G}$ is continuous, since for all n, $h^{-1}[O_n] = G_n$, and $h^{-1}[O_{\neg n}] = G_n^c$. Therefore, 2.2(1) and 2.2(2) yield Recław's Theorem 1.3, and 2.2(2) and 2.2(3) yield the following.

Theorem 2.3. *X is a τ-space iff no continuous image of X in $[\omega]^\omega$ is a tower.*

PROOF. ($\Leftarrow$) Assume that $\mathcal{J}$ is a clopen τ-cover of X and let $\mathcal{G} \subseteq \mathcal{J}$ be a τ-sequence for X. Then by Lemma 2.2(3), $h_\mathcal{G}[X]$ is linearly ordered by $\subseteq^*$. As $h_\mathcal{G}$ is continuous, $h_\mathcal{G}[X]$ cannot be a tower, and hence has an a.i.. Applying Lemma 2.2(2), we get that $\mathcal{G}$ is a γ-cover of X, and hence so is $\mathcal{J}$.

($\Rightarrow$) Assume that X is a τ-space, and $f : X \to [\omega]^\omega$ is continuous. Assume that $f[X]$ is linearly ordered by $\subseteq^*$. Then $\langle O_n : n < \omega\rangle$ is a clopen τ-sequence for $f[X]$. Therefore $\mathcal{G} = \langle f^{-1}[O_n] : n \in \omega\rangle$ is a clopen τ-sequence for X; hence a γ-cover of X. By 2.2(2) $h_\mathcal{G}[X]$ has an a.i.; hence is not a tower. But $h_\mathcal{G} = f$, as for all $x \in X$,

$$n \in h_\mathcal{G}(x) \iff x \in f^{-1}[O_n] \iff f(x) \in O_n \iff n \in f(x).$$

Therefore, $f[X]$ is not a tower. $\square$

The reader might have noticed that in the last proof we have indirectly used the following lemma.

Lemma 2.4. *Every continuous image of a τ-space is a τ-space.*

PROOF. A continuous preimage of a clopen τ-cover is a clopen τ-cover. $\square$

We get a topological characterzation of $\mathfrak{t}$.

Corollary 2.5. $\mathrm{non}(\mathcal{T}) = \mathfrak{t}$.

For a family $\mathcal{F}$ of spaces, let

$$\mathrm{add}(\mathcal{F}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{F} \ \& \ \bigcup \mathcal{A} \notin \mathcal{F}\}.$$

Theorem 2.6. $\mathrm{add}(\mathcal{T}) = \mathfrak{t}$.

PROOF. We need the following lemma.

Lemma 2.7. *Assume that $X \subseteq [\omega]^\omega$ is linearly ordered by $\subseteq^*$, and for some $\kappa < \mathfrak{t}$, $X = \bigcup_{i<\kappa} X_i$ where each X_i has an a.i.. Then X has an a.i..*

PROOF. If for each $i < \kappa$ X_i has an a.i. $x_i \in X$, then an a.i. of $\{x_i : i < \kappa\}$ is also an a.i. of X. Otherwise, there exsists $i < \kappa$ such that X_i has no a.i. $x \in X$. That is, for all $x \in X$ there exists a $y \in X_i$ such that $x \not\subseteq^* y$; thus $y \subseteq^* x$. Therefore, an a.i. of X_i is also an a.i. of X. $\square$

Now we can use Theorem 2.3. Assume that $X = \bigcup_{i<\kappa<\mathfrak{t}} X_i$, where each X_i is a τ-space. If $f : X \to [\omega]^\omega$ is continuous and $f[X]$ is linearly ordered by $\subseteq^*$, then each $f[X_i]$ has an a.i. and therefore by the lemma, $f[X]$ has an a.i.. Therefore, X is a τ-space. $\square$

A similar result cannot be obtained for γ-spaces. CH implies that γ-spaces are not even closed under taking *finite* unoions. We will use an argument of Galvin and Miller [5, p. 151] to show this.

Theorem 2.8 (Brendle [2, Theorem 4.1])**.** *$CH \to$ there is a γ-space of size $\mathfrak{c}(= \omega_1)$ all of whose subsets are γ-spaces.*

For $X \subseteq [0, 1]$, let $X + 1 = \{x + 1 : x \in X\}$.

Theorem 2.9 (Galvin, Miller [5, Theorem 5])**.** *Assume that $A \subseteq X \subseteq [0, 1]$, and $(X \setminus A) \cup (A + 1)$ is a γ-space. Then A is G_δ and F_σ in X.*

Now, consider a subspace A of Brendle's space X, such that A is neither G_δ nor F_σ. Then the union of $X \setminus A$ and $A+1$ (which are both γ-spaces) is not a γ-space.

The γ-property is very strict. Gerlits and Nagy [6, Corollary 6] proved that γ-spaces are C''. In particular, it is consistent that all γ-spaces are countable. However, large τ-spaces do exist in ZFC. In fact, we have the following.

Theorem 2.10 (Shelah)**.** *2^ω is a τ-space.*

PROOF. Towards a contradiction, assume that $f : 2^\omega \to [\omega]^\omega$ is continuous such that $f[2^\omega]$ is a tower. Let

$$T = \{s \in 2^{<\omega} : f[\ [s]\ ] \text{ has no a.i.}\}.$$

T is a perfect tree. Assume that for some $s \in T$, there are no incomparable extensions s_0, s_1 such that both $f[\ [s_0]\ ]$ and $f[\ [s_1]\ ]$ have no a.i.. Then for all $\tilde{s}$ extending s, at least one of $f[\ [\tilde{s}\hat{\ }\langle 0\rangle]\ ]$ and $f[\ [\tilde{s}\hat{\ }\langle 1\rangle]\ ]$ has an a.i.. Let $\sigma \in 2^\omega$ extend s such that for all $n \geq |s|$, $f[\ [(\sigma \restriction n)\hat{\ }\langle 1-\sigma(n)\rangle]\ ]$ has an a.i.. $f[\ [s]\ ] = \bigcup_{n<\omega} f[\ [(\sigma \restriction n)\hat{\ }\langle 1-\sigma(n)\rangle]\ ] \cup \{f(\sigma)\}$ is a union of ω many sets having an a.i., contradicting Lemma 2.7.

We now show that $f[2^\omega]$ cannot be linearly ordered by $\subseteq^*$. Define two branches β and ξ in T as follows. Start with incomparable $b_0, c_0 \in T$. Pick $x_0 \in [b_0]$. As $f[\ [c_0]\ ]$ has no a.i., we can find a $y_0 \in [c_0]$ such that $f(x_0) \not\subseteq^* f(y_0)$. Choose an $n_0 \in f(x_0) \setminus f(y_0)$. Since f is continuous, we can find b_1, an initial segment of x_0, such that $f[\ [b_1]\ ] \subseteq O_{n_0}$. Similarly, find c_1, an initial segment of y_0, such that $f[\ [c_1]\ ] \subseteq O_{\neg n_0}$.

Now we reverse the roles, and find $x_1 \in [b_1]$, $y_1 \in [c_1]$, $n_1 > n_0$ such that $n_1 \in f(y_1) \setminus f(x_1)$. Then we take b_2 and c_2, initial segments of y_1 and x_1 respectively, such that $f[\ [b_2]\ ] \subseteq O_{\neg n_1}$ and $f[\ [c_2]\ ] \subseteq O_{n_1}$.

We continue by induction. Finally, let $\beta = \bigcup_i b_i = \lim_i x_i$, and $\xi = \bigcup_i c_i = \lim_i y_i$. Since f is continuous, the sets $\{n_{2k} : k \in \omega\}$ and $\{n_{2k+1} : k \in \omega\}$ witness that neither $f(\beta) \subseteq^* f(\xi)$ nor $f(\xi) \subseteq^* f(\beta)$. □

This theorem implies that the inclusion in Corollary 2.1 is proper. We will modify it to get a large class of τ-spaces which are not γ-spaces.

Theorem 2.11. *ω^ω is a τ-space.*

PROOF. Identify ω^ω with $2^\omega \setminus F$ (where F are the eventually zero sequences), and work in $2^\omega \setminus F$ instead of 2^ω.

1. In the proof that T is perfect, we need not care whether $\sigma \in 2^\omega \setminus F$ or not.

2. When choosing the initial segment b_{i+1} of x_i, use the fact that $x_i \notin F$ to make sure that b_{i+1} ends with a "1" (a similar treatment for c_{i+1}). This will make β and ξ belong to $2^\omega \setminus F$.

□

Corollary 2.12. *Every analytic set of reals is a τ-space.*

PROOF. Every analytic set of reals is a continuous image of ω^ω. □

Remark 2.13.

1. One cannot prove in ZFC that all projective sets of reals are τ-spaces. Since the reals have a projective well-ordering in the constructible universe L, a straightforward inductive construction will yield a projective tower.
2. Due to a theorem of Suslin (see, e.g., [11, Corollary 2C.3]), every uncountable analytic set contains a perfect set, and hence is not a γ-space. (It is not even strongly null.)

As in the case of γ-spaces [5, p. 147], the property of being a τ-space need not be hereditary for subspaces of the same size.

We will work in $P(\omega)$.

Theorem 2.14. $\mathfrak{t} = \mathfrak{c} \rightarrow$ *there is a space $X \subseteq [\omega]^\omega$ s.t.*

1. *$|X| = \mathfrak{c}$,*
2. *$X \cup [\omega]^{<\omega}$ is a τ-space, and*
3. *X is not a τ-space.*

PROOF. First, note that (1) follows from (2) and (3), using Corollary 2.5.

We will use a modification of the Galvin-Miller construction (see [5, Theorem 1]). For $y \in [\omega]^\omega$, define $y^* = \{x : x \subseteq^* y\}$. We need the following lemma.

Lemma 2.15 (Galvin, Miller [5, Lemma 1.2])**.** *Assume that $\mathcal{G}$ is an open ω-cover of $[\omega]^{<\omega}$. Then for all $x \in [\omega]^\omega$ there exists a $y \in [x]^\omega$ such that $\mathcal{G}$ γ-covers y^*.*

Let $\langle \mathcal{G}_i : i < \mathfrak{c} \rangle$ enumerate all countable families of clopen sets in $P(\omega)$, and let $\langle y_i : i < \mathfrak{c} \rangle$ enumerate all elements $y \in [\omega]^\omega$ such that both y and $\omega \setminus y$ are infinite.

Construct, by induction, $\langle x_i : i < \mathfrak{c} \rangle \subseteq [\omega]^\omega$ such that $i < j \rightarrow x_j \subseteq^* x_i$. For a limit i, use $i < \mathfrak{t}$ to get x_i. For successor $i = k + 1$, x_i is constructed as follows.

Case 1 $\mathcal{G}_k$ is a τ-cover of $B_k = \{x_j : j \leq k\} \cup [\omega]^{<\omega}$. By Theorem 2.5, as $|B_k| < \mathfrak{t}$, $\mathcal{G}_k$ is a γ-cover of B_k. In particular, $\mathcal{G}_k$ γ-covers $[\omega]^{<\omega}$. By the lemma, there exists an $x_{k+1} \in [x_k]^\omega$ such that $\mathcal{G}_k$ γ-covers x^*_{k+1}.

Case 2 $\mathcal{G}_k$ is not a τ-cover of $\{x_j : j < k\} \cup [\omega]^{<\omega}$. Since this case is not interesting, we may take $x_{k+1} = x_k$.

After x_i is chosen (either for limit or successor i), modify it as follows. If $x_i \subseteq^* y_i$, leave it as is. Otherwise, replace it by $x_i \setminus y_i$. This does the construction.

Define $X = \{x_i : i < \mathfrak{c}\}$. Then $X \cup [\omega]^{<\omega}$ is a τ-space. By the construction, if $\mathcal{G}_k$ is a τ-cover of $X \cup [\omega]^{<\omega}$, then it γ-covers $\{x_j : j \leq k\} \cup x^*_{k+1}$. But

$$X \cup [\omega]^{<\omega} \subseteq \{x_j : j \leq k\} \cup x^*_{k+1}.$$

This does (2).

(3) X is a tower. Let $a \in [\omega]^\omega$. We will show that a is not an a.i. of X. Take $a_0 \subseteq a$ such that both a_0 and $\omega \setminus a_0$ are infinite. Now, some x_i satisfies either $x_i \subseteq^* a_0$, or $x_i \subseteq^* \omega \setminus a_0$. Therefore, $a \not\subseteq^* x_i$. By Lemma 2.4 (considering the identity function on $[\omega]^\omega$), X is not a τ-space. □

Corollary 2.16. $\mathfrak{t} = \mathfrak{c} \to$ *τ-spaces are not closed under Borel images.*

Proof. Let X be given by the theorem. Consider any function $f : X \cup [\omega]^{<\omega} \to [\omega]^\omega$ such that $f \restriction X$ is the identity function, and $f[[\omega]^{<\omega}] \subseteq X$. As $[\omega]^{<\omega}$ is countable, f is Borel. $X \cup [\omega]^{<\omega}$ is a τ-space, but X, its Borel image, is not a τ-space. □

3 Comparing τ-spaces to Other Classical Classes

Hurewicz and Menger

We give Hurewicz' topological interpretations of $\mathfrak{b}$ and $\mathfrak{d}$.

X has the *Hurewicz property* if for every sequence of open covers $\mathcal{G}_n$, there is a sequence of finite $\tilde{\mathcal{G}}_n \subseteq \mathcal{G}_n$ such that the sets $\cup\tilde{\mathcal{G}}_n$ form a γ-cover of X. X has the *Menger property* if for every sequence of open covers $\mathcal{G}_n$, there is a sequence of finite $\tilde{\mathcal{G}}_n \subseteq \mathcal{G}_n$ such that the sets $\cup\tilde{\mathcal{G}}_n$ cover X. Let $\mathcal{H}$ and *MEN* denote the classes of spaces having the Hurewicz and Menger properties, respectively. Clearly $\mathcal{H} \subseteq$ *MEN*.

Theorem 3.1 (Hurewicz [8, §5]). *Let X be a space.*

1. *X has the Hurewicz property iff every continuous image of X in ω^ω is bounded. In particular,* $\text{non}(\mathcal{H}) = \mathfrak{b}$.

2. *X has the Menger property iff every continuous image of X in ω^ω is not dominating. In particular,* $\mathrm{non}(MEN) = \mathfrak{d}$.

We get that none of these two notions is provably comparable to $\mathcal{T}$.

Corollary 3.2.

1. $\mathcal{T} \not\subseteq MEN$*, and*
2. $\mathfrak{t} < \mathfrak{b} \to \mathcal{H} \not\subseteq \mathcal{T}$

Proof. (1) By Theorem 2.11, $\omega^\omega \in \mathcal{T}$, and by Theorem 3.1(2), $\omega^\omega \notin MEN$.
(2) follows from Corollary 2.5 and Theorem 3.1(1). □

Indeed, τ-spaces could be pretty far from having the Menger property. According to a theorem of Hurewicz [7, Theorem 20], an analytic set of reals having the Menger property must be F_σ. Corollary 2.12 could be contrasted with this. However, these classes need not be orthogonal. Gerlits and Nagy [6, p. 155] proved that, given a sequence of ω-covers $\mathcal{G}_n$ of a γ-space X, there exists a sequence $G_n \in \mathcal{G}_n$ such that $\{G_n : n \in \omega\}$ γ-covers X. We therefore have the next assertion.

Corollary 3.3. $\Gamma \subseteq \mathcal{H} \cap \mathcal{T}$.

λ-spaces

X is a λ-*space* if every countable subset of X is G_δ. Let Λ denote the collection of λ-spaces. λ-spaces are perfectly meager (see [10, Theorem 5.2]). Therefore, by Remark 2.13(2), no uncountable analytic set is a λ-space. This again could be contrasted with Corollary 2.12.

On the other hand, we have the following.

Theorem 3.4. *There is a λ-space of size $\mathfrak{t}$ which is not a τ-space.*

Our theorem follows from the following two lemmas.

Lemma 3.5 ([16, Theorem 1])**.** $\mathrm{non}(\Lambda) = \mathfrak{b}$.

Lemma 3.6. *Every tower of size $\mathfrak{b}$ is a λ-space.*

Proof. We use the standard argument (see [4, Theorem 9.1]). Before getting started, note that for all $y \in [\omega]^\omega$, $y^* = \bigcup_{s\in[\omega]^{<\omega}}\{x : x \subseteq y \cup s\}$ is F_σ.

Assume that $X = \{x_i : i < \mathfrak{b}\}$ is a tower with $i < j \to x_j \subseteq^* x_i$. For $\alpha < \mathfrak{b}$, set $X_\alpha = \{x_i : i < \alpha\}$. Then each X_α is G_δ in X. (Its complement in X is F_σ.) Assume that $F \subseteq X$ is countable. As $\mathfrak{b}$ is regular, there exists

$\alpha < \mathfrak{b}$ such that $F \subseteq X_\alpha$. As $|X_\alpha| < \mathfrak{b}$, X_α is a λ-space. Hence, F is G_δ in X_α; i.e., there is a G_δ set $A \subseteq X$ such that $F = X_\alpha \cap A$. As X_α is also G_δ, F is G_δ in X. □

With some set theoretic assumptions, we can have an example of size $\mathfrak{b}$. In fact, our $\mathfrak{b}$-example will have some additional properties related to our study. $X \subseteq \mathbb{R}$ is a *λ'-space* if for all countable $F \subseteq \mathbb{R}$, $X \cup F$ is a λ-space. For $D \subseteq \mathbb{R}$, X is *κ-concentrated* on D if for all open $U \supseteq D$, $|X \setminus U| < \kappa$.

Considering the proof that an (ω_1, ω_1)-gap is a λ'-space (see [10, p. 215]), one might wonder whether our proof can be strengthened to make every $\mathfrak{b}$-tower X a λ'-space. In fact, following the steps of the proof carefully one gets that for all countable $F \subseteq [\omega]^\omega$, $X \cup F$ is a λ-space. The problem is with $[\omega]^{<\omega}$: If X, when viewed as a subset of ω^ω, is unbounded, then $[\omega]^{<\omega}$ is not G_δ in $X \cup [\omega]^{<\omega}$ [4, Lemma 9.3].

Theorem 3.7. *Assume that there exists a tower of size $\mathfrak{b}$. Then there is a space X of size $\mathfrak{b}$ such that*

1. *X is a λ-space,*
2. *X is $\mathfrak{b}$-concentrated on a countable set,*
3. *X is not a λ'-space,*
4. *X does not have the Hurewicz property, and*
5. *X is not a τ-space.*

Proof. We work in $P(\omega)$. Identify $[\omega]^\omega$ with $\omega^{\uparrow\omega}$, the strictly increasing elements of ω^ω. Let $X = \{x_i : i < \mathfrak{b}\} \subseteq \omega^{\uparrow\omega}$ be such that the following holds.

(♣) It is unbounded, $\leq^*$-increasing, and has size $\mathfrak{b}$.

The existence of such a set follows from [4, Theorem 3.3]. Let $A = \{a_i : i < \mathfrak{b}\}$ be a tower, and define $Y = \{y_i : i < \mathfrak{b}\}$ as follows. For each $i < \mathfrak{b}$, let $h \in \omega^\omega$ bound $\{y_k : k < i\} \cup \{x_i\}$, and take a $y_i \subseteq^* a_i$ such that $h \leq^* y_i$, $y_i(n) = \min\{k \in a_i : y_i(n-1), h(n) < k\}$. Y, like X, has the property (♣). Rothberger [14, Theorem 4] has proved that (♣) implies (1) and (3) (see also [4, Lemma 9.3]). By an observation of Miller [10, Theorem 5.7], (♣) implies that Y is $\mathfrak{b}$-concentrated on $[\omega]^{<\omega}$. By Theorem 3.1(1), (4) is also satisfied.

(5) Y is a tower: Any a.i. of Y would also be an a.i. of A. □

Our theorem has a cute corollary.

Corollary 3.8. $\mathfrak{t} = \mathfrak{b} \vee \mathfrak{b} < \mathfrak{d} \rightarrow$ *there exists an X as in Theorem 3.7.*

This follows from the following observation.

Lemma 3.9 ([17, Theorem 1]). *$\mathfrak{b} < \mathfrak{d} \to$ there is a tower of size $\mathfrak{b}$.*

For completeness, we give a proof of this lemma.

Proof. Let $X \subseteq \omega^{\uparrow\omega}$ have property (♣), and let $h \in \omega^\omega$ witness that X is not dominating. For each $x \in X$ define $a_x \in [\omega]^\omega$ by $a_x = \{n : x(n) < h(n)\}$. Then $\{a_x : x \in X\}$ is linearly ordered by $\subseteq^*$. Assume that it has an a.i. a. Then h' defined by $h'(n) = h(\min\{k \in a : n \leq k\})$ bounds X. A contradiction. □

Despite the large difference between τ and λ spaces, these classes need not be orthogonal. Their intersection could contain a space of size $\mathfrak{c}$: By [5, Theorem 2], a G_δ γ-subspace of a space is also an F_σ subspace of that space. Every co-countable subspace of Brendle's space (see Theorem 2.8) is G_δ and therefore F_σ. Therefore, every countable subspace of Brendle's space is G_δ.

Corollary 3.10. *$CH \to$ there is a γ(in particular, τ)-space of size $\mathfrak{c}$ which is also a λ-space.*

4 The Selection Principle S_1

Unlike γ-spaces, τ-spaces do not fit into the framework defined in [9]. We recall the basic definitions.

A space X has property $S_1(x, y)$ (x, y range over $\{\omega, \gamma, \tau, \dots\}$) if, given a *sequence* of x-covers $\mathcal{G}_n$, one can select from each $\mathcal{G}_n$ an element G_n such that $\{G_n : n \in \omega\}$ is a y-cover. As mentioned in section 3, Gerlits and Nagy proved that the γ-property is equivalent to the $S_1(\omega, \gamma)$ property. Using this notation, we have the following.

Remark 4.1. $S_1(\omega, \gamma) \subseteq S_1(\tau, \gamma) \subseteq S_1(\gamma, \gamma)$.

Proof. As noted in §3, every γ-cover is a τ-cover, and every τ-cover is an ω-cover. □

Obviously, $S_1(\tau, \gamma) \subseteq \mathcal{T}$. By [9, Theorem 2.3], 2^ω does not belong to the class $S_1(\gamma, \gamma)$, and therefore not to the class $S_1(\tau, \gamma)$, either.

Corollary 4.2. *$S_1(\tau, \gamma) \neq \mathcal{T}$.*

We now study the $S_1(\tau, \gamma)$ property. Let us begin with saying that the τ-covering notion fits nicely into the framework of [9]. (In fact, it suggests many interesting notions, but we will stick to $S_1(\tau, \gamma)$ in this paper.) For example, it can be added to [9, Theorem 3.1]. In particular, we have the following.

Theorem 4.3. $S_1(\tau, \gamma)$ *is closed under taking closed subsets and continuous images.*

There are more properties, which follow from Remark 4.1 We quote some of them.

Theorem 4.4.

1. ([9, Corollary 5.6]) *Every element of* $S_1(\tau, \gamma)$ *is perfectly meager (i.e., has meager intersection with every perfect set).*

2. ([9, Theorem 5.7]) *If* $X \in S_1(\tau, \gamma)$*, then for every* G_δ *set* G *containing* X*, there exists an* F_σ *set* F *such that* $X \subseteq F \subseteq G$*.*

Remark 4.5. If we omit the metrizability assumption on the spaces, then $S_1(\tau, \gamma)$ is not closed under cartesian products, nor under finite unions: Todorčević [18] showed that there exist nonmetrizable $X, Y \in S_1(\omega, \gamma)$ such that $X \cup Y \notin S_1(\gamma, \omega)$. (In fact, he showed that they do not even have the Menger property.)

Theorem 4.6 (Daniels [3, Lemma 9])**.** $S_1(\omega, \gamma)$ *is closed under taking finite powers.*

Question 4.7. Is $S_1(\tau, \gamma)$ closed under taking finite powers?

One can see that if $\mathcal{G}$ is a τ-cover of X, then $\{G^n : G \in \mathcal{G}\}$ is a τ-cover of X^n. But this is not enough for answering this question.

Theorem 4.8. $\mathrm{non}(S_1(\tau, \gamma)) = \mathfrak{t}$.

Proof. Assume that $|X| < \mathfrak{t}$ and let $\mathcal{G}_n$ be τ-covers of X. We wish to conclude that the $\mathcal{G}_n$'s are γ-covers of X. Corollary 2.5 is not enough for our purposes, since the τ-covers need not be clopen. However, Theorem 2.2 gives the desired result. Now, by [9, Theorem 4.7], $\mathrm{non}(S_1(\gamma, \gamma)) = \mathfrak{b}$. As $|X| < \mathfrak{b}$, $X \in S_1(\gamma, \gamma)$. Therefore, one can extract a γ-cover of X from the $\mathcal{G}_n$'s. This proves $\mathfrak{t} \leq \mathrm{non}(S_1(\tau, \gamma))$.

The other direction follows from the fact that $S_1(\tau, \gamma) \subseteq \mathcal{T}$, together with Corollary 2.5. □

In particular, it is consistent that $S_1(\tau, \gamma) \neq S_1(\gamma, \gamma)$.

Corollary 4.9. $\mathfrak{t} < \mathfrak{b} \to S_1(\tau, \gamma) \neq S_1(\gamma, \gamma)$.

We therefore have the following.

Question 4.10. Does $S_1(\omega, \gamma) = S_1(\tau, \gamma)$?

As the consistency of $\mathfrak{p} < \mathfrak{t}$ would imply a negative answer, this question seems to be closely related to the main problem whether $\mathfrak{p} = \mathfrak{t}$.

Remark 4.11.

1. Due to Theorem 4.4(1), the (in fact, any) Luzin set used in [9] to distinguish $S_1(\omega, \omega)$ from $S_1(\omega, \gamma)$ will also distinguish it from $S_1(\tau, \gamma)$.
2. By Theorem 4.6, a negative answer to Question 4.7 would imply a negative answer to Question 4.10.
3. Showing $S_1(\tau, \gamma) \not\subseteq S_1(\omega, \omega)$ is consistent would also yield a negative answer.

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