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AN EXTENSION OF A THEOREM OF ASH ON GENERALIZED DIFFERENTIABILITY

Abstract

Let $A = \{b_0, b_1, \ldots, b_{k+\ell}; a_0, a_1, \ldots, a_{k+\ell}\}$ be a system of $2(k+\ell+1)$ real numbers such that $b_i \neq b_j$ for $i \neq j$, satisfying $\sum_{i=0}^{k+\ell} a_i b_i^p = 0$ for $p = 0, 1, \ldots, k-1$ and $\sum_{i=0}^{k+\ell} a_i b_i^k = L \neq 0$. It is proved that if f is measurable, and if $\sum_{i=0}^{k+\ell} a_i f(x + b_i h) = O(|h|^\lambda)$ as $h \to 0$, where $\lambda > k-1$, at each point x on a measurable set E then the Peano derivative $f_{([\lambda])}$ exists finitely *a.e.* on E. This will extend a result of Ash [1]. It is further proved that if p is a positive integer $\leq k-1$ and if the upper and lower approximate Peano derivatives of f of order p are finite on a set E then $f_{(p)}$ exists *a.e.* on E.

1 Introduction

Throughout the paper $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{N}^+$ will denote the set of real numbers, the set of all non-negative integers, and the set of all positive integers respectively. The Lebesgue measure of a measurable set E will be denoted by $\mu(E)$, and the Lebesgue outer measure of a set H will be denoted by $\mu^*(H)$.

We shall consider $f : \mathbb{R} \to \mathbb{R}$. Recall that f is said to have Peano derivative (resp. approximate Peano derivative) at x of order k if there exist real numbers α_i, $1 \leq i \leq k$, depending on x and f only, such that

$$f(x+t) = f(x) + \sum_{i=1}^{k} \frac{t^i \alpha_i}{i!} + \frac{t^k \epsilon(x,t;f)}{k!}$$

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where

$$\lim_{t\to 0}\epsilon(x,t;f)=0 \quad \Big(\text{resp.}\quad \operatorname*{lim\,ap}_{t\to 0}\,\epsilon(x,t;f)=0\Big)\,.$$

The number α_k is called the Peano derivative (resp. approximate Peano derivative) of f at x of order k and is denoted by $f_{(k)}(x)$ (resp. $f_{(k),a}(x)$). For convenience, we take $\alpha_0=f(x)=f_{(0)}(x)=f_{(0),a}(x)$.

Suppose that f has Peano derivative (resp. approximate Peano derivative) at x of order k. For $t\neq 0$ write

$$w_{k+1}(x,t;f)=w_{k+1}(x,t)=(k+1)!\,\frac{f(x+t)-\sum_{i=0}^{k}\frac{t^i\alpha_i}{i!}}{t^{k+1}}\,.$$

The upper (resp. approximate upper) Peano derivative of f at x of order $k+1$ is defined by

$$\overline{f}_{(k+1)}(x)=\limsup_{t\to 0}w_{k+1}(x,t)$$

$\big($ respectively

$$\overline{f}_{(k+1),a}(x)=\operatorname*{limsup\,ap}_{t\to 0}w_{k+1}(x,t)\Big)\,.$$

The lower derivatives $\underline{f}_{(k+1)}(x)$ and $\underline{f}_{(k+1),a}(x)$ are defined analogously. If

$$\overline{f}_{(k+1)}(x)=\underline{f}_{(k+1)}(x)\quad\Big(\text{respectively}\ \ \overline{f}_{(k+1),a}(x)=\underline{f}_{(k+1),a}(x)\Big)$$

then the common value is called the Peano derivative (resp. approximate Peano derivative) of f at x (possibly infinite) of order $k+1$.

Definition 1.1. Let $k\in\mathbb{N}^+$, $\ell\in\mathbb{N}$ and $L\in\mathbb{R}\setminus\{0\}$. Let

$$A=\Big\{b_0,b_1,\dots,b_{k+\ell};a_0,a_1,\dots,a_{k+\ell}\Big\} \tag{1.1}$$

be a system of real numbers such that $b_i\neq b_j$ for $i\neq j$, $i,j=0,1,\dots,k+\ell$, and

$$\sum_{i=0}^{k+\ell}a_ib_i^p \begin{cases} =0 & \text{for}\quad p=0,1,\dots,k-1\\ =L & \text{for}\quad p=k\,.\end{cases} \tag{1.2}$$

For a fixed system A in (1.1) satisfying (1.2), and for a function $f:\mathbb{R}\to\mathbb{R}$ we shall write

$$\Phi_k(x,h)=\Phi_k(x,h;f)=\Phi_k(x,h;f;A)=\sum_{i=0}^{k+\ell}a_if(x+b_ih)\,. \tag{1.3}$$

The generalized Riemann derivative of f at x of order k with respect to the system A is defined by

$$\mathrm{GRD}_k\, f(x) = \mathrm{GRD}_k\, f(x, A) = \frac{k!}{L} \lim_{h\to 0} \frac{\Phi_k(x, h; f; A)}{h^k}\,,$$

if this limit exists. It can be shown that if the Peano derivative $f_{(k)}(x)$ exists finitely then $\mathrm{GRD}_k\, f(x, A)$ exists for every system A in (1.1) satisfying (1.2) and equals $f_{(k)}(x)$. The upper and lower derivatives $\overline{\mathrm{GRD}}_k\, f(x)$ and $\underline{\mathrm{GRD}}_k\, f(x)$ are defined in the obvious way. Thus A may be called the basis of a kth order generalized derivative. The number ℓ is called its excess.

The following lemma is immediate.

Lemma 1.1. *Let $\ell \in \mathbb{N}^+$ and let there be m $(\le \ell)$ zeros among the a_i's in* (1.1). *Let A_0 be obtained from A by omitting those a_i's which are* 0 *and those b_i's which correspond to those a_i's. Then A_0 is a basis having excess only $l-m$ and*

$$\Phi_k(x, h; f; A) = \Phi_k(x, h; f; A_0)$$

and therefore the kth derivative with respect to A is the same as the kth derivative with respect to A_0.

If $\ell = 0$ and the b_i's are given then the a_i's are uniquely determined by (1.2). In fact (b_i^p), $0 \le i \le k$, $0 \le p \le k$, being a Van der Monde matrix, its determinant is given by

$$\det\bigl(b_i^p\bigr) = \prod_{i<j} (b_j - b_i)$$

and so if (C_r^k) is the cofactor of b_r^k in $\det(b_i^p)$ then (C_r^k) is also Van der Monde and

$$\det\bigl(C_r^k\bigr) = (-1)^{k+r} \prod_{i<j}{}' (b_j - b_i)$$

where b_r never occurs in Π'. Thus

$$\frac{\det(b_i^p)}{\det(C_r^k)} = \prod_{i=0, i\ne r}^{k} (b_r - b_i)$$

and

$$a_r = L\Bigl(\prod_{i=0, i\ne r}^{k} (b_r - b_i)\Bigr)^{-1}, \quad 0 \le r \le k\,. \tag{1.4}$$

If in particular $L = k!$ then the system (1.1) with (1.2) is considered by Ash [1] and it covers a wide class of kth derivatives. The advantage of taking L is that we can also accommodate the derivative $\tilde{D}_k$ considered in [5, pp. 9–11]. Indeed, if

$$L = 2^{k-1} \prod_{i=1}^{k-1} \left(2^{k-1} - 2^{i-1}\right) \quad \text{and} \quad \ell = 0\,,$$

and if $b_0 = 0$, $b_i = 2^{i-1}$, $1 \le i \le k$, then the kth derivative with respect to this system is the derivative $\tilde{D}_k$.

Now suppose that $\ell = 0$. If $b_i = i + C$, where C is a constant, then from (1.4) we have

$$a_i = \frac{L}{k!} (-1)^{k-i} \binom{k}{i} \,. \tag{1.5}$$

If on the other hand $\ell > 0$ then the b_i's and the $k+1$ equations in (1.2) cannot determine the a_i's uniquely. It is clear that if A is Riemann's symmetric system, i.e., $\ell = 0$, $L = k!$ and $b_i = i - \frac{k}{2}$ (and so a_i are as in (1.5)) then Φ_k becomes Riemann's symmetric difference of order k given by

$$\Delta_k(x, h; f) = \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} f(x + ih - kh/2)\,. \tag{1.6}$$

Marcinkiewicz and Zygmund proved in a deep theorem [5, Theorem 1] that if f is measurable and

$$\Delta_k(x, h; f) = O(h^k)\,, \quad \text{as } h \to 0 \tag{1.7}$$

for each x on a measurable set E, then the Peano derivative $f_{(k)}$ exists finitely *a.e.* on E. We have extended in [6] the theorem of Marcinkiewicz and Zygmund cited above replacing k at the right of (1.7) by any real number $\lambda > k - 1$. More precisely, our theorem is:

Theorem 1.2. *Let $k \in \mathbb{N}^+$ and $\lambda \in \mathbb{R}$ be such that $\lambda > k - 1$. Let f be measurable. If*

$$\Delta_k(x, h; f) = O\big(|h|^\lambda\big)\,, \quad \text{as} \;\; h \to 0\,,$$

for each point x in a set $E \subset \mathbb{R}$, then $f_{([\lambda])}$ exists finitely a.e. on E, where $[\lambda]$ denotes the greatest integer not exceeding λ.

Ash [1] generalized the theorem of Marcinkiewich and Zygmund for any general system A in (1.1) satisfying (1.2) with $L = k!$. In the present paper we consider a general system A in (1.1) satisfying (1.2), and consider the generalized difference $\Phi_k(x, h; f; A)$ instead of $\Delta_k(x, h; f)$ and prove the analogue of Theorem 1.2. This will be an extension of Theorem 1 of [1].

Remark. In addition to [3] we wish to mention that there seems to be a difficulty in the assumption

$$\Big|\sum_{i=0}^{k+\ell} A_i f_2(x+a_i t)\Big| \le M|t|^k \quad \text{if } |t|<\delta \text{ for all } x\in\Pi \tag{1.8}$$

in [1, p. 189]. It may be noted that the similar assumption

$$\big|\omega(x,t)\big| < M \quad \text{for } x\in\Pi,\ |t|<d$$

in [11, Vol. II, p. 75] can now be proved by taking

$$G_n = \Big\{x\,:\, x\in E_{k-1}\,;\ \big|\omega_k(x,t)\big| \le n \text{ for } 0<|t|<\frac{1}{n}\Big\},$$

where E_{k-1} is the set where $f_{(k-1)}$ exists, and noting the measurability of G_n for all n (cf. [7, p. 771]), and choosing $G_m \subset E$, $\mu(E\setminus G_m) < \frac{\epsilon}{2}$ and a perfect set $\Pi\subset G_m$, $\mu(G_m\setminus\Pi) < \frac{\epsilon}{2}$ and setting $M = m+1 = \frac{1}{d}$. This approach will not work for (1.8) since the sets

$$S_n = \Big\{x\,:\, \Big|\sum_{i=0}^{k+\ell} A_i f(x+a_i t)\Big| \le n|t|^k \quad \text{for} \quad |t| < \frac{1}{n}\Big\}$$

need not be measurable even for sufficiently large n. We show this in Example 1.5 which is an extension of Example 1 of [3]. We need a set S of measure 0 such that $(S+S)/2$ is non-measurable. For the proof of the existence of such a set the authors of [3] suggested a method and refered to a source not available to the readers. We give a proof in Theorem 1.4.

For any two sets A, B, $-A$ is the set of all x such that $-x\in A$, and for a fixed $\tau\in\mathbb{R}$, $A+\tau$ is the set of all points $x+\tau$ such that $x\in A$, and τA is the set of all τx such that $x\in A$, and $A+B$ is the set of all points $x+y$ such that $x\in A$ and $y\in B$. For the definition of a Hamel basis we refer to [10, p. 411]. We need the following lemma which is a generalization of a result of Sierpinski [9] and is proved by Rubel [8]. We give a proof for completeness.

Lemma 1.3. *There exists a bounded set E of Lebesgue measure 0, but $E+E$ is non-measurable.*

Proof. Let C be the Cantor ternary set in $[0,1]$. Let $r\in[0,1]$ and let $0.a_1a_2\ldots$, where $a_i = 0$, 1 or 2, be the ternary expansion of $r/2$. Define c_i and c_i' for each i such that $(c_i,c_i') = (0,0)$ if $a_i = 0$, $(c_i,c_i') = (2,0)$ if $a_i = 1$, and

$(c_i, c_i') = (2,2)$ if $a_i = 2$. Then $c = 0.c_1c_2\ldots$ and $c' = 0.c_1'c_2'\ldots$ are points of C and

$$\frac{r}{2} = \frac{c+c'}{2} \quad \text{giving} \quad r = c + c'\,.$$

Thus $[0,1] \subset C + C$. Hence $C \pm C \pm C \pm \ldots = \mathbb{R}$ and therefore C contains a Hamel basis H. Let

$$E_0 = H \cup (-H) \cup \{0\}\,; \quad E_{n+1} = E_n + E_n \ \text{ for } \ n = 0,1,2,\ldots\,.$$

Then

$$\mathbb{R} = \bigcup\nolimits_{n=0}^{\infty} \bigcup\nolimits_{m=1}^{\infty} \frac{1}{m} E_n\,. \tag{1.9}$$

For, if $r \in \mathbb{R}$, then there are $h_1, h_2, \ldots, h_p \in H$ and rationals $\rho_1, \rho_2, \ldots, \rho_p$ such that

$$r = \sum_{i=1}^{p} \rho_i h_i = \frac{1}{d} \sum_{i=1}^{p} e_i h_i\,,$$

where $\rho_i = \frac{e_i}{d}$ with $|e_i| \in \mathbb{N}^+$, $d \in \mathbb{N}^+$, and so

$$r \in \frac{1}{d} E_n \quad \text{if} \quad 2^n \geq \sum_{i=1}^{p} |e_i|\,.$$

Hence all sets E_n cannot be of measure 0. Let n_0 be the smallest of n for which E_n has positive outer measure. Since E_0 is of measure 0, $n_0 \geq 1$. If possible, let E_{n_0} be measurable. Since $E_{n_0} = -E_{n_0}$, $E_{n_0+1} = E_{n_0} - E_{n_0}$. So by [4, p. 68], E_{n_0+1} contains an open interval I containing the origin. Let $h \in H$. Then we can find an integer $j \geq 2$ such that $\frac{h}{j} \in I$ and hence $\frac{h}{j} \in E_{n_0+1} = E_{n_0} + E_{n_0}$. Since every element of E_{n_0} is a linear combination of elements of E_o and hence of H with integral coefficients, $\frac{h}{j}$ is a linear combination of elements of H with integral coefficients. But this is a contradiction since H, being a Hamel basis, is a linearly independent set with rational coefficients. Therefore, E_{n_0} is not measurable. Putting $E = E_{n_0-1}$ the proof is complete. □

Theorem 1.4. *For any bounded interval I there is a set $S \subset I$ of measure 0, but $\frac{S+S}{2}$ is non-measurable.*

Proof. Let $I = [a,b]$, $\alpha = \inf E$, $\beta = \sup E$, where E is the set of Lemma 1.3. Let

$$S = \left\{ \frac{(b-a)(x-\alpha)}{\beta - \alpha} + a \,:\, x \in E \right\}.$$

Then S satisfies the requirements. □

Example 1.5. *There exists a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that for each $n \in \mathbb{N}^+$ the set*

$$E_n = \left\{ x \,:\, \frac{|f(x+h) - 2f(x) + f(x-h)|}{h^2} \leq n \quad \textit{for} \quad 0 < |h| < \frac{1}{n} \right\}$$

is non-measurable.

Proof. Let $n \in \mathbb{N}^+$ and let

$$a_n = \frac{1}{n} - \frac{1}{8n^2}\,, \qquad \delta_n = \frac{1}{32n^2}\,.$$

Let I_n be the closed interval with center $2n-1$ and length $2\delta_n$. By Theorem 1.4 there is a set $S_n \subset I_n$ of measure 0 such that $\frac{S_n + S_n}{2}$ is non-measurable. Let

$$f_n = \frac{1}{2n} \chi_{(S_n - a_n) \cup (S_n + a_n)}\,,$$

where χ_E is the characteristic function of E. Since S_n is of measure 0, f_n is measurable. Applying similar arguments as in [3] with a, δ, S and $\left[\frac{1}{2} - \delta, \frac{1}{2} + \delta\right]$ being replaced by a_n, δ_n, S_n and I_n respectively, it can be shown that

$$\frac{S_n + S_n}{2} = \left\{ x \,:\, x \in I_n\,;\ \ \frac{|f_n(x+h) - 2f_n(x) + f_n(x-h)|}{h^2} > n\,, \right.$$
$$\left. \text{for some } h\,, \quad 0 < |h| < \frac{1}{n} \right\}.$$

Hence the set

$$\left\{ x \,:\, x \in I_n\,;\ \ \frac{|f_n(x+h) - 2f_n(x) + f_n(x-h)|}{h^2} \leq n\,,\ \text{for } 0 < |h| < \frac{1}{n} \right\}$$

is non-measurable. Let $f = \sum_{n=1}^{\infty} f_n$. Then for each $\nu \in \mathbb{N}^+$ the set

$$E_\nu = \left\{ x \,:\, \frac{|f(x+h) - 2f(x) + f(x-h)|}{h^2} \leq \nu\,,\ \text{for } 0 < |h| < \frac{1}{\nu} \right\}$$

is non-measurable. For, if possible, suppose E_ν is measurable. Then $E_\nu \cap I_\nu$ is measurable. But

$$E_\nu \cap I_\nu =$$
$$= \left\{ x \,:\, x \in I_\nu\,;\ \ \frac{|f_\nu(x+h) - 2f_\nu(x) + f_\nu(x-h)|}{h^2} \leq \nu\,,\ \text{for } 0 < |h| < \frac{1}{\nu} \right\}$$

which is non-measurable, giving a contradiction. □

We shall follow the approach of Ash [1] with essential modifications.

2 Auxiliary Results

We need the following results from [3]:

Lemma 2.1. *Let* 0 *be a point of outer density of* E*, let* $\alpha, \beta \in \mathbb{R}$ *with* $\beta \neq 0$ *and let* $\epsilon > 0$*. For each* $u > 0$ *set*

$$B_u = \left\{ v \in [u, 2u] \,:\, \alpha u + \beta v \in E \right\}.$$

Then there is a $\delta > 0$ *such that if* $0 < u < \delta$*, then* $\mu^*(B_u) > u(1 - \epsilon)$*.*

Theorem 2.2. *Let* f *be measurable and let* $n \in \mathbb{N}^+$*. Suppose that* α_i*,* β_i*,* $i = 0, 1, \ldots, n$ *are real numbers such that* $\beta_i \neq \beta_j$ *for* $i \neq j$ *and for some* $i \in \{0, 1, \ldots, n\}$*,* $\alpha_i \neq 0$*,* $\beta_i \neq 0$*. If*

$$\sum_{i=0}^{n} \alpha_i f(x + \beta_i t) = O(1)\,, \quad \text{as} \;\; t \to 0$$

for $x \in E \subset \mathbb{R}$*, then* f *is bounded in a neighborhood of almost every point* $x \in E$*.*

Theorem 2.3. *Let the hypotheses of Theorem 2.2 hold. If* $\alpha \geq 0$ *and*

$$\sum_{i=0}^{n} \alpha_i f(x + \beta_i t) = O\big(|t|^\alpha\big)\,, \quad \text{as} \;\; t \to 0$$

for all $x \in E \subset \mathbb{R}$*, then for each* $\beta \in \mathbb{R}$

$$\sum_{i=0}^{n} \alpha_i f\big(x + (\beta_i - \beta)t\big) = O\big(|t|^\alpha\big)\,, \quad \text{as} \;\; t \to 0$$

for almost every $x \in E$*.*

The theorem is true if "O" is replaced by "o".

The above results are respectively Lemma 1, Theorem 2 and Theorem 3 of [3].

Lemma 2.4. *Let* f *be measurable and let the Peano derivative* $f_{(k-1)}(x)$ *of* f *at* x *of order* $k - 1$ *exist for each* x *in a set* $E \subset \mathbb{R}$*. If*

$$f(x + h) - \sum_{i=0}^{k-1} \frac{h^i f_{(i)}(x)}{i!} = O(h^k)\,, \quad \text{as} \;\; h \to 0$$

for $x \in E$ *then* $f_{(k)}$ *exists a.e. on* E*.*

Proof. The proof is in [5, Lemma 7] and discussed in [6, Theorem MZ1] when E is measurable. When E is non-measurable, let

$$E_1 = \left\{ x \,:\, f_{(k-1)}(x) \text{ exists and } f(x+h) - \sum_{i=0}^{k-1} \frac{h^i f_{(i)}(x)}{i!} = O(h^k) \ \text{ as } \ h \to 0 \right\}.$$

Then since the upper and lower Peano derivatives are measurable, E_1 is measurable and so $f_{(k)}$ exists *a.e.* on E_1. Since $E \subset E_1$, the result follows. □

3 Main Results

The C_rP-integral, which is introduced by J. C. Burkill and used in the following lemma, can be found in [2]. Indeed, any integral will suffice if integrability of f implies measurability of f.

Lemma 3.1. *Let f be C_rP-integrable in every finite interval on $\mathbb{R}$ for some $r \in \mathbb{N}^+$. Let*

$$\Phi_k(x,h;f;A) = \sum_{i=0}^{k+\ell} a_i f(x+b_i h) = O\big(|h|^\lambda\big)\,, \quad \text{as} \quad h \to 0\,, \tag{3.1}$$

where $\lambda \geq 0$ at each point x on a set $E \subset \mathbb{R}$. Then there is $s \in \mathbb{N}$ such that

$$\Delta_{k+s}(x,h;F_s) = O\big(|h|^{\lambda+s}\big)\,, \quad \text{as} \quad h \to 0\,, \tag{3.2}$$

for almost all $x \in E$ where F_s is the sth indefinite C_rP-integral of f, i.e.,

$$\begin{aligned} &F_0(x) = f(x)\,; \quad F_1(x) = \int_0^x f(t)\,dt\,; \\ &F_s(x) = \frac{1}{(s-1)!}\int_0^x (x-t)^{s-1} f(t)\,dt\,, \quad \text{for} \quad s \geq 2\,. \end{aligned} \tag{3.3}$$

Proof. We note that, since f is C_rP-integrable, it is measurable [2, Proposition 4.7]. We may suppose that $b_0 < b_1 < \ldots < b_{k+\ell}$. By Theorem 2.3 we may further suppose that $b_0 = 1$. We consider the following cases:

Case I. Let $\ell = 0$, $b_i \in \mathbb{N}^+$ for $i = 1,2,\ldots,k$. Then $b_k = s+k+1$ for some $s \in \mathbb{N}$. If $s = 0$ then $b_i = i+1$ for $i = 0,1,\ldots,k$, and so the a_i's are given by (1.5). Hence from (3.1) and Theorem 2.3 (with $\alpha_i = a_i$, $\beta_i = i+1$, $\beta = \frac{k}{2}+1$, $\alpha = \lambda$), we get (3.2) for $s = 0$.

If $s > 0$ there are s gaps in $b_0, b_1, \ldots, b_k$. Let n_1 be the smallest positive integer in (b_0, b_k) such that $n_1 \notin \{b_0, b_1, \ldots, b_k\}$. Applying Theorem 2.3 in

(3.1) with $\alpha_i = a_i$, $\beta_i = b_i$, $\beta = n_1$ and $\alpha = \lambda$, we have

$$\sum_{i=0}^{k} a_i f\big(x + (b_i - n_1)h\big) = O\big(|h|^{\lambda}\big)\,, \quad \text{as} \quad h \to 0\,,$$

for almost all $x \in E$, and integrating with respect to h from 0 to t, $|t|$ being sufficiently small, we have

$$\begin{aligned}\sum_{i=0}^{k} \frac{a_i}{b_i - n_1} F_1\big(x + (b_i - n_1)t\big) - \Big(\sum_{i=0}^{k} \frac{a_i}{b_i - n_1}\Big) F_1(x) &= \\ = O\big(|t|^{\lambda+1}\big)\,, \quad \text{as} \quad t \to 0\,,&\end{aligned} \tag{3.4}$$

for almost all $x \in E$. By (1.2) all the a_i's cannot be 0, and so applying Theorem 2.3 in (3.4) with $\beta = -n_1$, $\alpha = \lambda + 1$ and $\alpha_i = \frac{a_i}{b_i - n_1}$, $\beta_i = b_i - n_1$ for $i = 0, 1, \ldots, k$ and $\alpha_{k+1} = -\sum_{i=0}^{k} \frac{a_i}{b_i - n_1}$, $\beta_{k+1} = 0$, we have

$$\begin{aligned}\sum_{i=0}^{k} \frac{a_i}{b_i - n_1} F_1\big(x + b_i t\big) - \Big(\sum_{i=0}^{k} \frac{a_i}{b_i - n_1}\Big) F_1\big(x + n_1 t\big) &= \\ = O\big(|t|^{\lambda+1}\big)\,, \quad \text{as} \quad t \to 0\,,&\end{aligned} \tag{3.5}$$

for almost all $x \in E$. It is easy to check that the system

$$A_1 = \Big\{ b_0, b_1, \ldots, b_k, n_1, \frac{a_0}{b_0 - n_1}; \frac{a_1}{b_1 - n_1}, \ldots, \frac{a_k}{b_k - n_1}, -\sum_{i=0}^{k} \frac{a_i}{b_i - n_1} \Big\}$$

satisfies the condition (1.2) with k replaced by $k + 1$. Hence from (3.5) we observe that

$$\Phi_{k+1}(x, h; F_1; A_1) = O\big(|h|^{\lambda+1}\big)\,, \quad \text{as} \quad h \to 0\,,$$

for almost all $x \in E$. The numbers $b_0, b_1, \ldots, b_k, n_1$ have one fewer gap than $b_0, b_1, \ldots, b_k$ and also the excess is still 0. So, if $s = 1$, the proof is completed as in the first paragraph replacing k by $k + 1$ and A by A_1. Otherwise, choose the smallest positive integer n_2 in (b_0, b_k) such that $n_2 \notin \{b_o, b_1, \ldots, b_k, n_1\}$, and repeating this process $s - 1$ more times, we obtain the numbers $b_0, b_1, \ldots, b_k, n_1, \ldots, n_s$ which have no gap, and we obtain the system A_s such that

$$\Phi_{k+s}(x, h; F_s; A_s) = O\big(|h|^{\lambda+s}\big)\,, \quad \text{as} \quad h \to 0\,,$$

for almost all $x \in E$. The proof is completed as in the first paragraph.

CASE II. Let $\ell \in \mathbb{N}^+$, $b_i \in \mathbb{N}^+$ for $i = 1, 2, \ldots, k+\ell$. By employing the process of filling in the gaps employed in Case I, we may suppose that $b_i = i+1$, $i = 1, 2, \ldots, k+\ell$. It may be noted that the process of filling never increases the excess. Hence (3.1) reduces to

$$\sum_{i=1}^{r} a_{i-1} f(x+ih) = O\big(|h|^\lambda\big)\,, \quad \text{as} \ \ h \to 0 \ \ \text{for} \ \ x \in E\,, \tag{3.6}$$

where $r = k+\ell+1$. Applying Theorem 2.3 in (3.6) with $\beta = r+1$, $\alpha = \lambda$, we have

$$\sum_{i=1}^{r} a_{i-1} f\big(x + \big(i-(r+1)\big)h\big) = O\big(|h|^\lambda\big)\,, \quad \text{as} \ \ h \to 0\,, \tag{3.7}$$

for almost all $x \in E$. Integrating (3.6) and (3.7) with respect to h from 0 to t, we have

$$\begin{aligned} \sum_{i=1}^{r} \frac{a_{i-1}}{i} F_1(x+it) - \Big(\sum_{i=1}^{r} \frac{a_{i-1}}{i}\Big) F_1(x) = \\ = O\big(|t|^{\lambda+1}\big)\,, \quad \text{as} \ \ t \to 0\,, \end{aligned} \tag{3.8}$$

for almost all $x \in E$, and

$$\begin{aligned} \sum_{j=1}^{r} \frac{a_{j-1}}{j-r-1} F_1\big(x+(j-r-1)t\big) - \Big(\sum_{j=1}^{r} \frac{a_{j-1}}{j-r-1}\Big) F_1(x) = \\ = O\big(|t|^{\lambda+1}\big)\,, \quad \text{as} \ \ t \to 0\,, \end{aligned} \tag{3.9}$$

for almost all $x \in E$. Applying Theorem 2.3 to (3.9) with $\beta = -r$, $\alpha = \lambda+1$ and changing indices by setting $i = j-1$, we have

$$\begin{aligned} \sum_{i=0}^{r-1} \frac{a_i}{i-r} F_1\big(x+it\big) - \Big(\sum_{i=0}^{r-1} \frac{a_i}{i-r}\Big) F_1(x+rt) = \\ = O\big(|t|^{\lambda+1}\big)\,, \quad \text{as} \ \ t \to 0\,, \end{aligned} \tag{3.10}$$

for almost all $x \in E$. If possible, suppose that the coefficients of $F_1(x+it)$, $0 \le i \le r$, in (3.8) and (3.10) are proportional. Then there is $\rho \in \mathbb{R} \setminus \{0\}$ such that

$$\begin{aligned} -\sum_{i=1}^{r} \frac{a_{i-1}}{i} &= -\frac{\rho a_0}{r}\,; \\ \frac{a_{i-1}}{i} &= \frac{\rho a_i}{i-r}\,, \quad 1 \le i \le r-1\,; \\ \frac{a_{r-1}}{r} &= -\rho \sum_{i=0}^{r-1} \frac{a_i}{i-r}\,. \end{aligned} \tag{3.11}$$

It can be verified that the following two systems

$$\begin{aligned} B_1 &= \left\{0, 1, \ldots, r; -\sum_{i=1}^{r} \frac{a_{i-1}}{i}, a_0, \frac{a_1}{2}, \ldots, \frac{a_{r-1}}{r}\right\}, \\ B_2 &= \left\{0, 1, \ldots, r; \frac{a_0}{-r}, \frac{a_1}{1-r}, \ldots, \frac{a_{r-1}}{-1}, -\sum_{i=0}^{r-1} \frac{a_i}{i-r}\right\}, \end{aligned} \tag{3.12}$$

which correspond to (3.8) and (3.10) respectively, satisfy the conditions (1.2) with k replaced by $k+1$. In fact, it is easy for B_1. For B_2 note that

$$\sum_{i=0}^{r-1} \frac{a_i}{i-r}\, i^p - \Big(\sum_{i=0}^{r-1} \frac{a_i}{i-r}\Big)\, r^p =$$

$$= \sum_{i=0}^{r-1} a_i \sum_{\nu=0}^{p-1} i^{p-1-\nu} r^{\nu} = \sum_{\nu=0}^{p-1} r^{\nu} \sum_{j=1}^{r} a_{j-1}(j-1)^{p-1-\nu}$$

$$= \sum_{\nu=0}^{p-1} r^{\nu} \sum_{\mu=0}^{p-1-\nu} (-1)^{p-1-\nu-\mu} \binom{p-1-\nu}{\mu} \sum_{j=1}^{r} a_{j-1} j^{\mu},$$

and since the last sum is 0 for $\mu = 0, 1, \ldots, k-1$ and L for $\mu = k$, it is 0 if $p = 0, 1, \ldots, k$, and it is L if $p = k+1$, proving the assertion. Hence from (3.11), (3.12) and the last condition of (1.2), we have $\rho L = L$ showing that $\rho = 1$. Hence from (3.11)

$$a_i = -\frac{r-i}{i} a_{i-1} \text{ for } 1 \le i \le r-1,$$

and hence

$$\begin{aligned} \sum_{i=0}^{r-1} a_i (i+1)^k &= a_0 \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} (i+1)^k \\ &= a_0 \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \sum_{\nu=0}^{k} \binom{k}{\nu} i^{\nu} \\ &= a_0 \sum_{\nu=0}^{k} \binom{k}{\nu} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} i^{\nu} \\ &= (-1)^{r-1} a_0 \sum_{\nu=0}^{k} \binom{k}{\nu} \sum_{i=0}^{r-1} (-1)^{r-1-i} \binom{r-1}{i} i^{\nu}. \end{aligned} \tag{3.13}$$

Since

$$\sum_{i=0}^{m}(-1)^{m-i}\binom{m}{i} i^p = 0 \qquad \text{for} \;\; p = 0, 1, \ldots, m-1$$
$$= m! \quad \text{for} \;\; p = m\,,$$

(cf. (1.5) and (1.2)), and since $r-1 = k+\ell > k$, the inner summation at the right of (3.13) is 0 for $\nu = 0, 1, \ldots, k$, and so the right side of (3.13) is 0. But since $b_i = i+1$ and the system $\{b_0, b_1, \ldots, b_{k+\ell}; a_0, a_1, \ldots, a_{k+\ell}\}$ satisfies (1.2), the left hand side of (3.13) is $L \neq 0$, which is a contradiction. Therefore the coefficients of $F_1(x+it)$, $0 \le i \le r$, in (3.8) and (3.10) are not proportional. Therefore denoting the coefficients of $F(x+it)$, $0 \le i \le r$, in (3.8) and (3.10) by p_i and q_i respectively, we conclude that there is an $i_0 \in \{0, 1, \ldots, r\}$ such that p_{i_0} and q_{i_0} are not equal. Set

$$\gamma = q_{i_0}\bigl(q_{i_0} - p_{i_0}\bigr)^{-1}\,, \quad \delta = -p_{i_0}\bigl(q_{i_0} - p_{i_0}\bigr)^{-1}\,;$$

then $\gamma + \delta = 1$ and $p_{i_0}\gamma + q_{i_0}\delta = 0$. Since (3.8) and (3.10) can be written as

$$\Phi_{k+1}(x, h; F_1; B_i) = O\bigl(|t|^{\lambda+1}\bigr)\,, \quad \text{as} \;\; t \to 0\,, \;\; i = 1, 2\,,$$

for almost all $x \in E$, where B_1 and B_2 are given in (3.12), we have

$$\gamma\Phi_{k+1}(x, h; F_1; B_1) + \delta\Phi_{k+1}(x, h; F_1; B_2) = O\bigl(|t|^{\lambda+1}\bigr) \;\text{ as } t \to 0\,, \tag{3.14}$$

for almost all $x \in E$. Let B be obtained by adding γ times the elements of B_1 with δ times the corresponding elements of B_2. Since $\gamma + \delta = 1$, the first group of $r+1$ elements of B are $0, 1, \ldots, r$, and the second group of $r+1$ elements of B are $\gamma p_i + \delta q_i$, $0 \le i \le r$. Let B_0 be obtained by omitting from B those $\gamma p_i + \delta q_i$'s for which $\gamma p_i + \delta q_i = 0$ and the corresponding i's. Then by Lemma 1.1 and by (3.14)

$$\Phi_{k+1}(x, h; F_1; B_0) = \Phi_{k+1}(x, h; F_1; B) = O\bigl(|t|^{\lambda+1}\bigr) \;\text{ as } t \to 0\,,$$

for almost all $x \in E$, and therefore, since $\gamma p_{i_0} + \delta q_{i_0} = 0$, B_0 has excess $\le \ell - 1$. Repeating this process at most $\ell - 1$ more times, this case reduces to Case I.

CASE III. Let $\ell \in \mathbb{N}^+$ and b_i's be arbitrary reals for $1 \le i \le k+\ell$. (Note that we have assumed that $b_0 = 1 < b_1 < \ldots < b_{k+\ell}$.) Let b_{i_0} be the smallest of $b_1, b_2, \ldots, b_{k+\ell}$ which is not an integer. Let $n_1, n_2 \in \mathbb{N}^+ \setminus \{b_0, b_1, \ldots, b_{k+\ell}\}$, $n_1 \neq n_2$. Applying Theorem 2.3 with $\beta = n_j$, $j = 1, 2$, $\alpha = \lambda$ in (3.1), we have

$$\sum_{i=0}^{k+\ell} a_i f\bigl(x + (b_i - n_j)h\bigr) = O\bigl(|h|^{\lambda}\bigr)\,, \quad \text{as} \;\; h \to 0\,, \;\; j = 1, 2\,, \tag{3.15.j}$$

for almost all $x \in E$. Integrating (3.15.j) with respect to h from 0 to t, and then applying Theorem 2.3 with $\beta = -n_j$, $j = 1, 2$, $\alpha = \lambda + 1$, we have

$$\begin{aligned}\sum_{i=0}^{k+\ell} \frac{a_i}{b_i - n_j} F_1\big(x + b_i t\big) - \Big(\sum_{i=0}^{k+\ell} \frac{a_i}{b_i - n_j}\Big) F_1(x + n_j t) = \\ = O\big(|t|^{\lambda+1}\big)\,, \quad \text{as} \;\; t \to 0\,,\;\; j = 1, 2\,,\end{aligned} \tag{3.16.j}$$

for almost all $x \in E$. Set

$$p = \frac{b_{i_0} - n_1}{n_2 - n_1} \quad \text{and} \quad q = \frac{n_2 - b_{i_0}}{n_2 - n_1}\,.$$

Then

$$p + q = 1 \quad \text{and} \quad p\frac{a_{i_0}}{b_{i_0} - n_1} + q\frac{a_{i_0}}{b_{i_0} - n_2} = 0\,. \tag{3.17}$$

Adding (3.16.1) multiplied by p with (3.16.2) multiplied by q we have

$$\Phi_{k+1}(x, h; F_1; C_1) = O\big(|h|^{\lambda+1}\big)\,, \quad \text{as} \;\; h \to 0\,,$$

for almost all $x \in E$, where

$$\begin{aligned}C_1 = \Bigg\{ & b_0, \dots, b_{i_0} - 1, b_{i_0+1}, \dots, b_{k+\ell}, n_1, n_2;\, p\frac{a_0}{b_0 - n_1} + q\frac{a_0}{b_0 - n_2}, \\ & \dots, p\frac{a_{i_0-1}}{b_{i_0-1} - n_1} + q\frac{a_{i_0-1}}{b_{i_0-1} - n_2}\,, p\frac{a_{i_0+1}}{b_{i_0+1} - n_1} + q\frac{a_{i_0+1}}{b_{i_0+1} - n_2}\,, \\ & \dots, p\frac{a_{k+\ell}}{b_{k+\ell} - n_1} + q\frac{a_{k+\ell}}{b_{k+\ell} - n_2}\,, -p\sum_{i=0}^{k+\ell}\frac{a_i}{b_i - n_1}\,, -q\sum_{i=0}^{k+\ell}\frac{a_i}{b_i - n_2}\Bigg\}\,.\end{aligned}$$

The system C_1 satisfies (1.2) with k replaced by $k+1$. Indeed, using (3.17) we have

$$\sum_{\substack{i=0\\ i\neq i_0}}^{k+\ell}\Big(p\frac{a_i}{b_i - n_1} + q\frac{a_i}{b_i - n_2}\Big) b_i^s - \sum_{i=0}^{k+\ell} p\frac{a_i}{b_i - n_1} n_1^s - \sum_{i=0}^{k+\ell} q\frac{a_i}{b_i - n_2} n_2^s$$

which is 0 if $s = 0$, and if $1 \le s \le k+1$, then this is

$$\begin{aligned} &= \sum_{i=0}^{k+\ell}\Big(p\frac{a_i(b_i^s - n_1^s)}{b_i - n_1} + q\frac{a_i(b_i^s - n_2^s)}{b_i - n_2}\Big) \\ &= \sum_{i=0}^{k+\ell}\Big(pa_i\sum_{j=0}^{s-1} b_i^{s-1-j} n_1^j + qa_i\sum_{j=0}^{s-1} b_i^{s-1-j} n_2^j\Big) \\ &= \sum_{j=0}^{s-1}\Big(pn_1^j + qn_2^j\Big)\sum_{i=0}^{k+\ell} a_i b_i^{s-1-j} = \sum_{i=0}^{k+\ell} a_i b_i^{s-1}\,,\end{aligned}$$

which is 0 if $1 \le s < k+1$ and is L if $s = k+1$. So, we have removed the non-integer b_{i_0} and got C_1. We next pick the smallest of $b_{i_0+1}, \ldots, b_{k+\ell}$, say b_{i_1}, which is not an integer, and choose $n_3, n_4 \in \mathbb{N}^+ \setminus \{b_0, b_1, \ldots, b_{k+\ell}, n_1, n_2\}$, $n_3 \neq n_4$, and repeat the above argument to get a system C_2 which contains n_1, n_2, n_3, n_4, instead of b_{i_0}, b_{i_1} of A such that

$$\Phi_{k+2}(x, h; F_2; C_2) = O\big(|h|^{\lambda+2}\big)\,, \quad \text{as} \ \ h \to 0\,,$$

for almost all $x \in E$. After repeating the process we get a system C_u, where $1 \le u \le k+\ell$, of $2(1+k+\ell+u)$ elements in which the first set of $1+k+\ell+u$ elements are all in $\mathbb{N}^+$, and for which

$$\Phi_{k+u}(x, h; F_u; C_u) = O\big(|h|^{\lambda+u}\big)\,, \quad \text{as} \ \ h \to 0\,,$$

for almost all $x \in E$. After rearranging the elements of C_u, this case now reduces to Case II. This completes the proof. □

Theorem 3.2. *Let f be measurable. If*

$$\Phi_k(x, h; f; A) = O\big(|h|^{\lambda}\big)\,, \quad \text{as} \ \ h \to 0\,,$$

where $\lambda > k-1$, at each point x in a measurable set $E \subset \mathbb{R}$, then $f_{([\lambda])}$ exists finitely a.e. on E, $[\lambda]$ being the greatest integer not exceeding λ.

Proof. We may suppose that E is bounded. By Theorem 2.2 there is a measurable set $E_1 \subset E$ such that $\mu(E_1) = \mu(E)$, and for each $x \in E_1$ there exist $\delta(x) > 0$ and $M(x)$ with

$$|f(t)| \le M(x) \quad \text{for} \quad t \in \big(x - \delta(x), x + \delta(x)\big)\,.$$

Let ϵ_1, ϵ_2 be arbitrary positive numbers. Then there is a closed set $E_2 \subset E_1$ such that $\mu(E_1 \setminus E_2) < \epsilon_1$, and so by the compactness of E_2 there exist open intervals $I_1, I_2, \ldots, I_n$ such that $E_2 \subset \cup_{i=1}^n I_i$ and f is bounded on $\cup_{i=1}^n I_i$. Clearly f is bounded on the closure $\overline{I} = \overline{\cup_i I_i}$. Let $\psi = f$ on $\overline{I}$ and $= 0$ outside $\overline{I}$. Then ψ is Lebesgue integrable and a fortiori C_rP-integrable on every finite interval in $\mathbb{R}$. Then by Lemma 3.1, there exist $s \in \mathbb{N}$ and a set $E_3 \subset E_2$ such that $\mu(E_3) = \mu(E_2)$ and

$$\Delta_{k+s}(x, t; \psi_s) = O\big(|t|^{\lambda+s}\big)\,, \quad \text{as} \ \ t \to 0\,,$$

for all $x \in E_3$, where ψ_s is the sth indefinite integral of ψ. Therefore, by Theorem 1.2, it follows that $(\psi_s)_{([\lambda]+s)}$ exists finitely on a set $E_4 \subset E_3$, where $\mu(E_4) = \mu(E_3)$. Let $E_5 \subset E_4$ be such that $\mu(E_5) = \mu(E_4)$ and $\psi_s^{(s)} = \psi$ on E_5. Now by [11, II; p. 77, Theorem 4.25], there is a perfect set $P \subset E_5$ such that $\mu(E_5 \setminus P) < \epsilon_2$ and there are functions G and H satisfying

(i) $\psi_s = G + H$,

(ii) $G^{([\lambda]+s)}$ exists continuously, and

(iii) $H_{(r)}(x) = 0$ for $x \in P$, $r = 0, 1, \ldots, [\lambda] + s$.

Let $g = G^{(s)}$. Then $g^{([\lambda])}$ exists continuously. So, $H^{(s)} = (\psi_s - G)^{(s)} = \psi - g$ on E_5. Let $h = H^{(s)}$ on E_5. Then $\psi = g + h$ on E_5. Since $H = 0$ on P and $H^{(s)}$ exists on P, $H^{(s)}(x) = 0$ for $x \in P$, and so $h(x) = 0$ for all $x \in P$. Since for all $x \in E_5$, ψ, g satisfy (note that $\psi = f$ on $\cup_{i=1}^{n} I_i$)

$$\sum_{i=0}^{k+\ell} a_i \psi(x + b_i t) = O\big(t^{[\lambda]}\big)\,, \quad \text{as} \quad t \to 0$$

and

$$\sum_{i=0}^{n} a_i g(x + b_i t) = O\big(t^{[\lambda]}\big)\,, \quad \text{as} \quad t \to 0\,,$$

we have for all $x \in E_5$

$$\sum_{i=0}^{n} a_i h(x + b_i t) = O\big(t^{[\lambda]}\big)\,, \quad \text{as} \quad t \to 0\,.$$

We now show that $h_{([\lambda])}$ exists finitely *a.e.* on P. Define for each $m \in \mathbb{N}^+$,

$$E_m^* = \left\{ x \,:\, x \in P\,;\ \Big| \sum_{i=0}^{k+\ell} a_i h(x + b_i t) \Big| \le m |t|^{[\lambda]}\,, \quad \text{for} \ \ 0 < |t| < \frac{1}{m} \right\}.$$

Then $P = \cup_{m=1}^{\infty} E_m^*$. Let m be fixed. Let $x_0 \in E_m^*$ be a point of outer density of E_m^*. We may suppose that $x_0 = 0$. Let η, $0 < \eta < \frac{1}{k+\ell+2}$, be arbitrary. Choose j, $0 \le j \le k + \ell$, such that $a_j \ne 0$, $b_j \ne 0$. By reordering the terms of A we may suppose that $a_0 \ne 0$, $b_0 \ne 0$. Then by Lemma 2.1 there is a δ_1, $0 < \delta_1 < 1$, such that if $0 < t < \delta_1$ then

$$\mu^*(B_i) > (1 - \eta)t \quad \text{and} \quad \mu^*(C) > (1 - \eta)t \quad \text{for} \ \ i = 1, 2, \ldots, k + \ell\,,$$

where

$$B_i = \Big\{ u \,:\, u \in [t, 2t];\ t + (b_i - b_0)u \in E_m^* \Big\}\,, \quad i = 1, 2, \ldots, k + \ell\,,$$

$$C = \Big\{ u \,:\, u \in [t, 2t];\ t - b_0 u \in E_m^* \Big\}\,.$$

Fix $t \in \big(0, \min(\delta_1, 1/2m)\big)$. Set

$$S_i = \Big\{u \,:\, u \in [t,2t];\ t + (b_i - b_0)u \in P\Big\}\,, \quad i = 1,2,\ldots,k+\ell\,,$$

$$D = \Big\{u \,:\, u \in [t,2t];\ \Big|\sum_{i=0}^{k+\ell} a_i h\big((t - b_0 u) + b_i u\big)\Big| \le m|u|^{[\lambda]}\Big\}\,.$$

Then the S_i's and D are measurable for $i = 1,2,\ldots,k+\ell$. and $C \subset D$, $B_i \subset S_i$, and so

$$\mu(D) > (1-\eta)t\,, \quad \mu(S_i) > (1-\eta)t\,, \quad \text{for } \ i = 1,2,\ldots,k+\ell\,.$$

Now, since

$$\mu\Big([t,2t] \setminus \Big(\big(\cap_{i=1}^{k+\ell} S_i\big) \cap D\Big)\Big) < (k+\ell+1)\eta t < t\,,$$

we have $\mu\big((\cap_{i=1}^{k+\ell} S_i) \cap D\big) > 0$. Hence there is an $u \in (\cap_{i=1}^{k+\ell} S_i) \cap D$, and so $t + (b_i - b_0)u \in P$, for all $i = 1,2,\ldots,k+\ell$, which gives

$$h\big(t + (b_i - b_0)u\big) = 0\,, \quad \text{for all } \ i = 1,2,\ldots,k+\ell\,.$$

Also, since $u \in D$,

$$\Big|\sum_{i=0}^{k+\ell} a_i h\big((t - b_0 u) + b_i u\big)\Big| \le m|u|^{[\lambda]}\,,$$

and hence

$$\big|a_0 h(t)\big| = \Big|\sum_{i=0}^{k+\ell} a_i h\big((t - b_0 u) + b_i u\big)\Big| \le m|u|^{[\lambda]} \le 2^{[\lambda]} m|t|^{[\lambda]}\,.$$

This shows that

$$h(t) = O\big(t^{[\lambda]}\big)\,, \quad \text{as } \ t \to 0\,.$$

Since $x_0 = 0$ is a point of outer density of E_m^*, it follows that

$$h(x+t) = O\big(t^{[\lambda]}\big)\,, \quad \text{as } \ t \to 0\,,$$

for almost all points x in E_m^*, and hence this also holds for almost all points x in P. Therefore by Lemma 2.4, $h_{([\lambda])}$ exists *a.e.* on P. Thus $\psi_{([\lambda])}$ exists *a.e.* on P. Since $P \subset E_5$ and $\mu(E_5 \setminus P) < \epsilon_2$, and since ϵ_2 is arbitrary, $\psi_{([\lambda])}$ exists *a.e.* on E_5. Since $E_5 \subset E_2 \subset \cup_{i=1}^n I_i$ and since $f = \psi$ on $\cup_{i=1}^n I_i$, which is an open set, $f_{([\lambda])}$ exists *a.e.* on E_5. Since $E_5 \subset E_2 \subset E_1 \subset E$, $\mu(E_5) = \mu(E_2)$, $\mu(E_1 \setminus E_2) < \epsilon_1$ and $\mu(E_1) = \mu(E)$, and since ϵ_1 is arbitrary, $f_{([\lambda])}$ exists *a.e.* on E. This completes the proof. $\square$

The above theorem is not true for $\lambda = k-1$ (see [6, Theorem 3.2]). However we have Theorem 3.3

Theorem 3.3. *Let $k, p \in \mathbb{N}^+$, $p \le k-1$ and let f be measurable. Let*

$$\Phi_k(x,u;f;A) = O\big(u^p\big)\,, \quad \text{as} \quad u \to 0\,,$$

for each point x in a set E. If $f_{(p),a}$ exists finitely on E, then $f_{(p)}$ exists a.e. on E. More generally, if

$$-\infty < \underline{f}_{(p),a} \le \overline{f}_{(p),a} < \infty \quad \text{on} \quad E\,,$$

then $f_{(p-1)}$ exists finitely and

$$-\infty < \underline{f}_{(p)} \le \overline{f}_{(p)} < \infty \quad \text{a.e.} \quad \text{on} \quad E\,.$$

To prove the theorem, we need the following lemma.

Lemma 3.4. *Let $k, p \in \mathbb{N}^+$ and let f be measurable. Let for all $m \in \mathbb{N}^+$,*

$$E_m = \Big\{x \,:\, f_{(p),a}(x) \text{ exists finitely and } \big|\Phi_k(x,u;f;A)\big| < m|u|^p \quad \text{for} \quad 0 < |u| < \frac{1}{m}\Big\}\,.$$

Then $f_{(p)}$ exists a.e. on E_m.

Proof. Without loss of generality we may assume that $a_0 \neq 0$, $b_0 \neq 0$ in $A = \big\{a_0, a_1, \ldots, a_{k+\ell}; b_0, b_1, \ldots, b_{k+\ell}\big\}$. Let $x_0 \in E_m$ be a point of outer density of E_m. We suppose that

$$x_0 = 0 = f(x_0) = f_{(1),a}(x_0) = \ldots = f_{(p),a}(x_0)\,.$$

Let $0 < \epsilon < 1$. Let

$$G = \Big\{x \,:\, |f(x)| \le \frac{\epsilon |x|^p}{p!}\Big\}\,.$$

Then G is measurable and $0 \in G$ is a point of density of G. Set $H = E_m \cap G$. Then 0 is a point of outer density of H. Let $0 < \eta < \frac{\epsilon}{2k+2\ell}$. Then by Lemma 2.1, there is a $\delta > 0$ such that, if $0 < u < \delta$ then

$$\mu^*(B) > (1-\eta)u\,, \quad \mu^*(C_j) > (1-\eta)u\,,$$

where

$$B = \Big\{v \in [u, 2u] \,:\, \frac{u+v}{2} \in H\Big\}\,,$$

$$C_j = \{v \in [u, 2u] \,:\, \lambda_j u + \mu_j v \in H\}\,, \quad \text{for } \ 1 \le j \le k + \ell\,,$$

where the λ_j's and μ_j's are given as follows:

$$\lambda_j = \frac{1 + \frac{b_j}{b_0}}{2}\,, \quad \mu_j = \frac{1 - \frac{b_j}{b_0}}{2} \quad \text{for } \ 1 \le j \le k + \ell\,.$$

Fix $u \in \big(0, \min(\delta, \frac{|b_0|}{m})\big)$. Let

$$S = \left\{v \in [u, 2u] \,:\, \Big|\Phi_k\Big(\frac{u+v}{2}, \frac{u-v}{2b_0}\Big)\Big| < m\Big|\frac{u-v}{2b_0}\Big|^p\right\}$$

and

$$T_j = \left\{v \in [u, 2u] \,:\, \big|f(\lambda_j u + \mu_j v)\big| \le \frac{\epsilon|\lambda_j u + \mu_j v|^p}{p!}\right\}, \quad \text{for } \ 1 \le j \le k + \ell\,.$$

Since f is measurable, S and T_j are measurable. Also $B \subset S$, $C_j \subset T_j$, and hence

$$\mu(S) > (1 - \eta)u\,, \quad \mu(T_j) > (1 - \eta)u\,.$$

Therefore

$$\mu\big(\cap_j(S \cap T_j)\big) > \big(1 - 2(k + \ell)\eta\big)u > (1 - \epsilon)u\,.$$

Hence

$$\big(\cap_j(S \cap T_j)\big) \cap (u, u + \epsilon u) \ne \emptyset\,.$$

Choose $v \in \big(\cap_j(S \cap T_j)\big) \cap (u, u + \epsilon u)$. Then

$$0 < v - u < \epsilon u < u\,,$$

and so

$$\Big|\Phi_k\Big(\frac{u+v}{2}, \frac{u-v}{2b_0}\Big)\Big| < m\Big|\frac{u-v}{2b_0}\Big|^p < m\Big|\frac{\epsilon u}{2b_0}\Big|^p\,,$$

which gives

$$\Big|\sum_{i=0}^{k+\ell} a_i f\Big(\frac{u+v}{2} + b_i\frac{u-v}{2b_0}\Big)\Big| < m\Big|\frac{\epsilon u}{2b_0}\Big|^p\,.$$

Hence

$$|a_0| \cdot |f(u)| < m\Big|\frac{\epsilon u}{2b_0}\Big|^p + \sum_{i=1}^{k+\ell} |a_i| \cdot \left|f\left(\frac{(1 + \frac{b_i}{b_0})u}{2} + \frac{(1 - \frac{b_i}{b_0})v}{2}\right)\right|\,.$$

Since $v \in T_i$ for $1 \le i \le k+\ell$,

$$
\begin{aligned}
|a_0| \cdot |f(u)| &< m\Big|\frac{\epsilon u}{2b_0}\Big|^p + \sum_{i=1}^{k+\ell} \frac{\epsilon |a_i| \cdot |\lambda_i u + \mu_i v|^p}{p!} \\
&\le m\Big|\frac{\epsilon u}{2b_0}\Big|^p + \frac{\epsilon}{p!}\sum_{i=1}^{k+\ell} |a_i|\big(|\lambda_i| + 2|\mu_i|\big)^p u^p \\
&\le \epsilon\Big[\frac{m}{|2b_0|^p} + \frac{1}{p!}\sum_{i=1}^{k+\ell} |a_i|\big(|\lambda_i| + 2|\mu_i|\big)^p\Big] u^p\,.
\end{aligned}
$$

This shows that $\frac{f(u)}{u^p} \to 0$ as $u \to 0^+$.

Similarly, it can be shown that $\frac{f(u)}{u^p} \to 0$ as $u \to 0^-$. This completes the proof of the lemma. □

Proof of the theorem. The sequence $\{E_m\}$, defined in Lemma 3.4, is nondecreasing and $E \subset \cup_{m=1}^{\infty} E_m$. By Lemma 3.4, $f_{(p)}$ exists *a.e.* on E_m, and so the first part follows.

For the last part we proceed exactly as in Lemma 3.4 and in the first part of the theorem, but with the following changes:

$$
E_m = \Big\{x \,:\, -m < \underline{f}_{(p),a}(x) \le \overline{f}_{(p),a}(x) < m \quad \text{and} \quad \big|\Phi_k(x,u;f;A)\big| < m|u|^p \quad \text{for} \quad 0 < |u| < \frac{1}{m}\Big\}\,,
$$

with the assumption that

$$
x_0 = 0 = f(x_0) = f_{(1),a}(x_0) = \ldots = f_{(p-1),a}(x_0)\,,
$$

and

$$
G_m = \Big\{x \,:\, |f(x)| \le \frac{m|x|^p}{p!}\Big\},
$$

$$
T_j = \Big\{v \in [u, 2u] \,:\, \big|f(\lambda_j u + \mu_j v)\big| \le \frac{m|\lambda_j u + \mu_j v|^p}{p!}\Big\}, \quad \text{for} \quad 1 \le j \le k+\ell\,,
$$

other sets in Lemma 3.4 remaining unchanged. Proceeding as in Lemma 3.4 ,

$$
|a_0| \cdot |f(u)| \le \Big[\frac{m\epsilon^p}{|2b_0|^p} + \frac{m}{p!}\sum_{i=1}^{k+\ell} |a_i|\big(|\lambda_i| + 2|\mu_i|\big)^p\Big] u^p\,,
$$

showing that $f(u) = O(u^p)$ as $u \to 0^+$, and similarly $f(u) = O(u^p)$ as $u \to 0^-$, and the rest is clear. □

Corollary 3.5. *Under the hypotheses of Theorem 3.3, if*

$$-\infty < \underline{f}_{(p),a} \leq \overline{f}_{(p),a} < \infty \quad on \;\; E$$

then $f_{(p)}$ *exists finitely a.e. on* E*.*

The proof follows from Theorem 3.3 and Theorem 2.2 of [6].

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