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HARR NULL SETS IN THE SPACE OF AUTOMORPHISMS ON [0,1]

Abstract

In this paper we study the distinctness of the concepts of left transverse, right transverse and transverse in the space $\mathcal{H}[0,1]$ of automorphisms on $[0,1]$. We show that, in $\mathcal{H}[0,1]$, the existence of a Borel probability measure left transverse and right transverse to a set Y does not imply the set Y is Haar null and we exhibit an uncountable disjoint collection of Borel sets that are not Haar null.

1 Introduction

The theory of Haar null sets was developed by Christensen [7] as a substitute for sets of measure zero relative to a Haar measure in settings where no Haar measure exists. The theory has now found substantial applications (e.g., [1], [2], [3], [5], [17], [18]) in the study of derivatives of functions defined on a Banach space since these sets play the role of exceptional sets in a number of assertions. The concept has been studied too for its own merit (e.g., [2], [4], [10], [11], [12], [15], [16], [20], [21]) and, by now, the nature of Haar null sets in an Abelian Polish group is very well understood.

Less well understood is the corresponding notion in a non-Abelian Polish group. Christensen [8, pp. 123] sketched out some of the theory and gave the correct definition but did not develop the ideas further. Some development can be found in [10], [19] and [21].

Let G denote a Polish group, that is to say G is a topological group that is completely metrizable and separable. Since we do not assume that G is

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Abelian, we shall write the group operation as a multiplication. Also, following [13] and [14], we shall use the simpler term *shy* in place of the term *Haar null set*.

Definition 1.1. (**Christensen**) A universally measurable set $X \subseteq G$ is called *shy* if there exists a Borel probability measure μ such that $\mu(gXh) = 0$ for all $g, h \in G$. We also say that the measure μ is *transverse* to X.

In the simpler theory where G is Abelian, then this reduces to the requirement that $\mu(g + X) = 0$ for all $g \in G$, where the group operation is now written as addition. That this is indeed a simplification is witnessed by the number of open problems that remain in the non-Abelian setting in attempts to understand the nature of shy sets. Note that the requirement here of universal measurability can be taken in the sense that X is in the completion of any Borel probability measure defined on the group. In this article, however, all sets appearing will be shown to be Borel.

To better study this concept we introduce some weaker versions.

Definition 1.2. A universally measurable set $X \subseteq G$ is called *left shy* if there exists a Borel probability measure μ such that $\mu(gX) = 0$ for all $g \in G$. We also say that μ is *left transverse* to X.

Definition 1.3. A universally measurable set $X \subseteq G$ is called *right shy* if there exists a Borel probability measure μ such that $\mu(Xh) = 0$ for all $h \in G$. We also say that μ is *right transverse* to X.

Definition 1.4. A universally measurable set $X \subseteq G$ is called *left-and-right shy* if there exists a Borel probability measure μ such that $\mu(Xh) = 0$ and $\mu(hX) = 0$ for all $h \in G$. We also say that μ is *left-and-right transverse* to X.

Trivially we have the following implications.

shy $\Rightarrow$ *left-and-right shy* $\Rightarrow$ *left shy (right shy)*

but no immediate implications in the opposite direction.

The remainder of the paper is devoted to the problem of whether in a Polish group the concepts of left transverse, right transverse, left-and-right transverse, and transverse are distinct. We address, too, the related problem of whether the concepts of left shy, right shy, left-and-right shy, and shy are distinct. These problems are open in general. Even the specific problem asked by Jan Mycielski [16] has no answer at present.

> Let G be a non-Abelian Polish group. What conditions on the group G ensure that if X is a left shy set, then X must be shy?

It is not difficult to show that if G is locally compact, then this must be so, since for locally compact Polish groups the set of left-shy sets is precisely the set of universally measurable sets of zero measure with respect to any left or right Haar measure on the group. In this article we partially answer this question by showing that the answer is no in the particular case of the space of automorphisms of the unit interval $[0, 1]$.

All of our study is in this space where we are able to demonstrate that the concepts are quite distinct.

There is one problem in this regard that can be dispensed with quite simply similarly as in [16]. We present that here without proof.

Theorem 1.5. *If $X \subseteq G$ is left shy and right shy, then X is left-and-right shy.*

2 The Space of Automorphisms

The space $\mathcal{H}[0, 1]$ is defined as the set of all homeomorphisms $h : [0, 1] \to [0, 1]$ that fix $h(0) = 0$ and $h(1) = 1$. Note that these are exactly the strictly increasing continuous functions leaving the endpoints fixed. This is a subspace of the complete metric space $C[0, 1]$ and we can impose that topology on the space. It is certainly metrizable and separable. That it is completely metrizable arises from the fact that it is a $\mathcal{G}_\delta$ in that space; see [6, pp. 468] for details. Equipped with the operation of composition of functions this space becomes, then, a non-Abelian, non-locally compact Polish group. It is a fruitful source of examples because, as first noted by Dieudonné [9] it is an example of a Polish group with no left invariant metric under which it is complete.

In our study of shy sets in this space we use two basic methods. First, in order to construct Borel probability measures that are transverse to certain sets we introduce a mapping $F : [1/2, 1] \to \mathcal{H}[0, 1]$ by $F(t) = x^t$. It is easy to see that F is continuous, and so $F([1/2, 1])$ is a compact set. Then we define a probability measure μ on $\mathcal{H}[0, 1]$ by writing

$$\mu(B) = 2\lambda_1(\{t \in [1/2, 1] : F(t) \in B\}) \tag{1}$$

for Borel sets B, where λ_1 denotes one-dimensional Lebesgue measure. We will use this example to show that a measure may be left transverse to a set without being right transverse, and that a measure may be left-and-right transverse without being transverse.

The second method is an elementary device that can be used to show that certain sets are not shy. We present this as a lemma.

Lemma 2.1. *If a set S in $\mathcal{H}[0,1]$ contains a two-sided translate [left translate, or right translate] of every compact set, then S is not shy [left shy, or right shy].*

Proof. For any Borel probability measure μ there is a compact set $K \subseteq \mathcal{H}[0,1]$ such that $\mu(K) > 0$. Since, by hypothesis, there exist functions $g,\ h \in \mathcal{H}[0,1]$ such that $g \circ k \circ h \in S$ for all $k \in K$, we have $K \subseteq g^{-1}Sh^{-1}$ and so $\mu(g^{-1}Sh^{-1}) \geq \mu(K) > 0$. Thus S is not shy. The non-left shy result and the non-right shy result can be shown in the same way. □

We need some properties of compact sets in $\mathcal{H}[0,1]$. It is easy to prove that the infimum and supremum of a compact subset of $\mathcal{H}[0,1]$ are again functions in $\mathcal{H}[0,1]$. This is needed in the proof of Theorem 5.4. For completeness we include a full characterization of compact subsets of $\mathcal{H}[0,1]$.

Theorem 2.2. *A set $K \subseteq \mathcal{H}[0,1]$ is compact iff*
(i) K is closed,
(ii) K is equicontinuous, and
(iii) for every non-empty closed set $K_0 \subseteq K$,

$$\inf_{h\in K_0} h(x) \text{ and } \sup_{h\in K_0} h(x)$$

are both in $\mathcal{H}[0,1]$.

Proof. Let $K \subseteq \mathcal{H}[0,1]$ be a compact set. Assertion (i) is immediate while assertion (ii) follows from the classical Arzelá-Ascoli theorem.

We now show that (iii) is true. For any non-empty closed set $K_0 \subseteq K$, let

$$g(x) = \sup_{f\in K_0} f(x) \text{ and } h(x) = \inf_{f\in K_0} f(x).$$

It is easy to see that both g and h leave $x = 0,\ 1$ fixed. Since $K_0 \subseteq K$ is closed, it is compact. For any $x_1, x_2 \in [0,1],\ x_1 < x_2$, there exist $f_1, f_2 \in K_0$ such that $f_1(x_1) = g(x_1)$ and $f_2(x_2) = h(x_2)$. Since f_1, f_2 are strictly increasing,

$$g(x_1) = f_1(x_1) < f_1(x_2) \leq \sup_{f\in K_0} f(x_2) = g(x_2)$$

and

$$h(x_1) = \inf_{f\in K_0} f(x_1) \leq f_2(x_1) < f_2(x_2) = h(x_2).$$

Thus $g(x)$ and $h(x)$ are strictly increasing functions. Since K is equicontinuous from (ii), for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $f \in K,\ |f(x) -$

$f(y)| < \epsilon/2$ if $|x-y| < \delta$. So for all $x, y \in [0,1]$, $|x-y| < \delta$ and $f \in K_0$,

$$\begin{aligned} f(x) &\leq f(y) + |f(x) - f(y)| \\ &\leq \sup_{k \in K_0} k(y) + \sup_{k \in K_0} |k(x) - k(y)|. \end{aligned}$$

Thus

$$\sup_{k \in K_0} k(x) - \sup_{k \in K_0} k(y) \leq \sup_{k \in K_0} |k(x) - k(y)| \leq \frac{\epsilon}{2}.$$

That is, $g(x) - g(y) < \epsilon$. Changing the positions of x and y yields $g(y) - g(x) < \epsilon$. Therefore $|g(x) - g(y)| < \epsilon$ and so g is continuous. For the continuity of h, note that for all $x, y \in [0,1]$, $|x-y| < \delta$ and $f \in K_0$,

$$f(x) \leq f(y) + \sup_{k \in K_0} |k(x) - k(y)| \leq f(y) + \frac{\epsilon}{2}.$$

So $\inf_{k \in K_0} k(x) \leq f(y) + \frac{\epsilon}{2}$, and hence by the arbitrariness of $f \in K_0$, $\inf_{k \in K_0} k(x) \leq \inf_{k \in K_0} k(y) + \frac{\epsilon}{2}$. That is, $h(x) - h(y) \leq \frac{1}{2}\epsilon < \epsilon$. Changing the positions of x and y yields $h(y) - h(x) < \epsilon$. Thus $|h(x) - h(y)| < \epsilon$ and hence $h(x)$ is continuous. Therefore (iii) is true.

To show the sufficiency of the conditions (i), (ii) and (iii), let $\{f_n\}$ be any infinite sequence in K. Since K is equicontinuous and uniformly bounded, $\{f_n\}$ has a uniformly convergent subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ by the Arzelá-Ascoli theorem. We claim $f \in \mathcal{H}[0,1]$ and the theorem is proved. Suppose $f \notin \mathcal{H}[0,1]$. Then f must be constant on some subinterval $[a,b] \subseteq [0,1]$. Fix $c \in (a,b)$, and consider the set

$$P_1 = \{f_{n_k} : \ f_{n_k}(c) \leq f(c)\}.$$

This set is then closed. If it is non-empty and infinite, then for all $x \in (a,c)$, we have $\sup_{f_{n_k} \in P_1} f_{n_k}(x) = f(c)$, which violates the condition (iii). Thus P_1 is finite. Similarly the set

$$P_2 = \{f_{n_k} : \ f_{n_k}(c) \geq f(c)\}$$

is also closed. If it is non-empty and infinite, then for all $x \in (c,b)$, we have $\inf_{f_{n_k} \in P_2} f_{n_k}(x) = f(c)$, which also violates the condition (iii). Thus P_2 is finite. But both P_1 and P_2 cannot be finite, and so we have a contradiction. Thus $f \in \mathcal{H}[0,1]$ and so K is compact. □

3 Left and Right Transverse Does Not Imply Transverse

We begin with an elementary example that illustrates the distinctness of left and right transverse notions in $\mathcal{H}[0,1]$.

Theorem 3.1. *Given an interval I with the closure contained in $(0,1)$, let $G(I)$ denote the set of functions in the space $\mathcal{H}[0,1]$ that are linear on I. Then the probability measure μ defined by (1) is left-and-right transverse to $G(I)$, but is not transverse to $G(I)$.*

PROOF. First we show that $G(I)$ is a Borel set. It is easy to see that

$$G(I) = \{f \in \mathcal{H}[0,1] : \ f \text{ is linear and slop} f > 0 \text{ on } I\}$$

where slop$f$ is the slope of f on I. Let

$$F_n = \left\{f \in \mathcal{H}[0,1] : \ f \text{ is linear and slop} f \geq \frac{1}{n} \text{ on } I\right\}.$$

Then

$$G(I) = \bigcup_{n=1}^{\infty} F_n.$$

It is easy to see that all sets F_n are closed sets and so $G(I)$ is a Borel set.

For any $g \in \mathcal{H}[0,1]$, we use the Borel probability measure μ defined by (1) and consider the set

$$T_1 = \{t \in [1/2, 1] : \ g \circ F(t) \in G(I)\}.$$

If $t \in T_1$, then on I, we have $g(x^t) = \alpha(t)x + \beta(t)$ where $\alpha(t) > 0$. Let $y = x^t$, then $g(y) = \alpha(t)y^{1/t} + \beta(t)$ for y on a subinterval of $[0,1]$. Suppose $\lambda_1(T_1) > 0$. Then the functions $y = x^t$ $(t \in T_1)$ map I into uncountably many subintervals of $[0,1]$. Thus there must exist $t_1, t_2 \in T_1$, $t_1 < t_2$ such that for y on some interval J,

$$\alpha(t_1)y^{\frac{1}{t_1}} + \beta(t_1) = \alpha(t_2)y^{\frac{1}{t_2}} + \beta(t_2).$$

Then by differentiating both sides of the equality above with respect to $y \in J$ we have $y^{\frac{1}{t_1} - \frac{1}{t_2}} = \frac{t_1\alpha(t_2)}{t_2\alpha(t_1)}$. This is impossible for all $y \in J$. Thus $\lambda_1(T_1) = 0$ and hence μ is left transverse to $G(I)$.

We now show that μ is also right transverse to $G(I)$. For any $h \in \mathcal{H}[0,1]$, consider the set

$$T_2 = \{t \in [1/2, 1] : \ F(t) \circ h \in G(I)\}.$$

If $t \in T_2$, then on I, we have $[h(x)]^t = \alpha(t)x + \beta(t)$ where $\alpha(t) > 0$. Differentiating both sides with respect to x almost everywhere on I we have $t[h(x)]^{t-1}h'(x) = \alpha(t)$ almost everywhere on I. Since $\alpha(t) > 0$, we have that

$h'(x) \neq 0$ almost everywhere on I, and therefore the set T_2 has at most two elements. If not, there exist $t_1, t_2 \in T_2,\ t_1 < t_2$ such that on I

$$t_1[h(x)]^{t_1-1}h'(x) = \alpha(t_1) \text{ and } t_2[h(x)]^{t_2-1}h'(x) = \alpha(t_2)$$

almost everywhere on I. Then $h(x)^{t_2-t_1} = \frac{t_1\alpha(t_2)}{t_2\alpha(t_1)}$ almost everywhere on I. This is impossible since the function $h(x)$ is strictly increasing. Thus $\lambda_1(T_2) = 0$ and hence μ is right transverse to $G(I)$.

We now show that the probability measure μ is not transverse to $G(I)$. Choose $g, h \in \mathcal{H}[0,1]$ such that $g(x) = 1 + \alpha \ln x$ and $h(x) = e^{x-1}$ on I where α is a positive constant depending on I. Then on $[1/2, 1]$ the function

$$g \circ F(t) \circ h(x) = g(h^t(x)) = \alpha t x + 1 - \alpha t$$

is linear for any $t \in [1/2, 1]$. Thus

$$\mu(g^{-1}G(I)h^{-1}) = 2\lambda_1(\{t \in [1/2,1] :\ g \circ F(t) \circ h \in G(I)\}) = 1 \neq 0.$$

Therefore μ is not transverse to $G(I)$. □

4 Right Transverse Does Not Imply Left Transverse

In this section we will show that a Borel probability measure right transverse to a set need not be left transverse to the set Y. We will use again the compact curve $F(t)$ and the Borel probability measure μ defined by (1) to verify this in the following theorem.

Theorem 4.1. *Given an interval I with the closure contained in $(0,1)$, let $G(I)$ be the set of functions in the space $\mathcal{H}[0,1]$ that are of the form $\alpha \ln x + \beta$ on I where $\alpha > 0$ and β are constants depending on the corresponding functions. Then the probability measure μ defined by (1) is right transverse to $G(I)$ but is not left transverse to $G(I)$.*

Proof. Similar arguments as in the proof of Theorem 3.1 show that $G(I)$ is a Borel set and is of the first category. For any $h \in \mathcal{H}[0,1]$ consider the set

$$T_3 = \{t \in [1/2,1] :\ F(t) \circ h \in G(I)\}.$$

We claim that T_3 contains at most one element. If not, there exist $t_1, t_2 \in T_3,\ t_1 < t_2$ such that $F(t_1) \circ h,\ F(t_2) \circ h \in G(I)$. Then on I

$$[h(x)]^{t_1} = \alpha(t_1) \ln x + \beta(t_1), \text{ and } [h(x)]^{t_2} = \alpha(t_2) \ln x + \beta(t_2).$$

By differentiating both sides of the above two equalities with respect to x almost everywhere on I, it follows that almost everywhere on I,

$$[h(x)]^{t_2-t_1} = \frac{t_1\alpha(t_2)}{t_2\alpha(t_1)}.$$

This is impossible since $h(x)$ is strictly increasing on I. Thus $\lambda_1(T_3) = 0$ and the probability measure μ defined by (1) is right transverse to $G(I)$.

We now show that μ is not left transverse to $G(I)$. Choose a function $g \in \mathcal{H}[0,1]$ such that on I, $g(x) = 1 + \alpha \ln x$. Then for any $t \in [1/2, 1]$,

$$g \circ F(t)(x) = g(x^t) = 1 + \alpha t \ln x.$$

Thus $g \circ F(t) \in G(I)$ and so

$$\mu(g^{-1}G(I)) = 2\lambda_1(\{t \in [1/2,1] : \; g \circ F(t) \in G(I)\}) = 1 \neq 0.$$

Hence μ is not left transverse to $G(I)$. □

Note that if a Borel probability measure μ_1 is right transverse but not left transverse to a set S contained in a Polish group G, then the measure μ_2, defined by $\mu_2(X) \equiv \mu_1(X^{-1})$, is left transverse but not right transverse to the set S^{-1}. Thus from Theorem 4.1 we can obtain a Borel probability measure that is left transverse to a set but not right transverse to this set.

5 A Non-shy Set That Is Left-and-right Shy

Jan Mycielski [16] posed a problem, denoted there by (P_0), whether the existence of a Borel probability measure left transverse to a set Y implies that Y is shy in a non-locally compact, completely metrizable group. In this section we give examples of non-shy sets that are left-and-right shy in $\mathcal{H}[0,1]$ and so answer the problem (P_0) negatively.

We begin by showing that that the sets $\{h \in \mathcal{H}[0,1] : \; h'(0) = \alpha\}$ and $\{h \in \mathcal{H}[0,1] : \; h'(1) = \alpha\}$ are left-and-right shy sets for any $0 < \alpha < \infty$. We then show that these two sets are non-shy sets in $\mathcal{H}[0,1]$. Indeed we show that the set

$$\{h : 0 < h'(0) < \infty\} \tag{2}$$

is left-and-right shy and yet can be decomposed into continuum many non-shy sets, thus giving a dramatic counterexample for Mycielski's problem.

That the set (2) is left-and-right shy is a special case of the following theorem.

Theorem 5.1. *For any function $q(x) \in \mathcal{H}[0,1]$, let*

$$S = \left\{ h \in \mathcal{H}[0,1] : \ 0 < \liminf_{x \to 0^+} \frac{h(x)}{q(x)} \le \limsup_{x \to 0^+} \frac{h(x)}{q(x)} < +\infty \right\}.$$

Then S is a Borel set that is left-and-right shy in $\mathcal{H}[0,1]$.

PROOF. We first show that S is a Borel set. Note

$$S = \bigcup_{p=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ h \in \mathcal{H}[0,1] : \ \frac{1}{p} < \frac{h(2^{-n})}{q(2^{-n})} < p \right\}.$$

Since all $h \in \mathcal{H}[0,1]$ are continuous and the metric on $\mathcal{H}[0,1]$ is equivalent to the uniform metric, all sets

$$\left\{ h \in \mathcal{H}[0,1] : \ \frac{1}{p} < \frac{h(2^{-n})}{q(2^{-n})} < p \right\}$$

are open, and so the set S is a Borel set.

We now consider the set

$$S_{\alpha,\beta} = \left\{ h \in \mathcal{H}[0,1] : \ \alpha < \liminf_{x \to 0^+} \frac{h(x)}{q(x)} \le \limsup_{x \to 0^+} \frac{h(x)}{q(x)} < \beta \right\}$$

for all $\alpha, \beta > 0, \ \alpha < \beta$. Once we show that the set $S_{\alpha,\beta}$ is left shy and right shy, then $S = \bigcup_{n=1}^{\infty} S_{1/n,n}$ is also left shy and right shy. We now show that the Borel probability measure μ defined by (1) is right transverse to $S_{\alpha,\beta}$ for any $\alpha, \beta > 0, \ \alpha < \beta$. For any $h \in \mathcal{H}[0,1]$, consider the set

$$R = \{ t \in [1/2, 1] : \ F(t) \circ h \in S_{\alpha,\beta} \}.$$

If $t_1 \in R$, then for sufficiently small $x \in [0,1]$,

$$\alpha < \frac{h^{t_1}(x)}{q(x)} < \beta.$$

This means that, for any $t_2 \neq t_1$,

$$\frac{h^{t_2}(x)}{q(x)} = \frac{h^{t_1}(x)}{q(x)} h^{t_2 - t_1}(x) \to 0 \text{ or } \infty$$

as $x \to 0^+$. Consequently $t_2 \notin R$. Thus R contains at most one element and μ is right transverse to $S_{\alpha,\beta}$.

To prove that S is left shy we choose a mapping $F_1 : [1/2, 1] \to \mathcal{H}[0, 1]$ by $F_1(t) = q^t(x)$ and define a Borel probability measure μ_1 by

$$\mu_1(X) = 2\lambda_1(\{t \in [1/2, 1] : \ F_1(t) \in X\}).$$

For any $h \in \mathcal{H}[0, 1]$, consider the set

$$L = \{t \in [1/2, 1] : \ h \circ F(t) \in S_{\alpha,\beta}\}.$$

If $s \in L$, then for sufficiently small $x \in [0, 1]$,

$$\alpha < \frac{h(q^s(x))}{q(x)} < \beta.$$

That is, $\alpha q(x) < h(q^s(x)) < \beta q(x)$ for sufficiently small $x \in (0, 1)$. Let $y = q^s(x)$. Then $\alpha y^{\frac{1}{s}} \le h(y) \le \beta y^{\frac{1}{s}}$ for sufficiently small $y \in (0, 1)$. Suppose that L contains two or more elements. Then there exist $s_1, s_2 \in [1/2, 1]$, $s_1 < s_2$ such that $\alpha y^{\frac{1}{s_1}} \le h(y) \le \beta y^{\frac{1}{s_1}}$ and

$$\alpha y^{\frac{1}{s_2}} \le h(y) \le \beta y^{\frac{1}{s_2}}$$

for sufficiently small $y \in (0, 1)$. Thus

$$\frac{\alpha y^{\frac{1}{s_2}}}{\beta y^{\frac{1}{s_1}}} = \frac{\alpha}{\beta} y^{\frac{1}{s_2} - \frac{1}{s_1}} \le 1$$

for sufficiently small $y \in (0, 1)$. This is impossible since $1/s_2 - 1/s_1 < 0$. Thus the set L contains at most one element and hence the probability μ_1 is left transverse to $S_{\alpha,\beta}$. By Theorem 1.5 $S_{\alpha,\beta}$ is left-and-right shy, and so the set S is also left-and-right shy. □

Let $q(x) = x$. Similarly as in Theorem 5.1 we can show easily that the set in Corollary 5.2 is a Borel set. By Theorem 5.1 we have the following corollary immediately.

Corollary 5.2. *For any* α, $0 < \alpha < \infty$, *the set* $\{h \in \mathcal{H}[0, 1] : \ h'(0) = \alpha\}$ *is left-and-right shy in* $\mathcal{H}[0, 1]$.

A similar assertion can be proved at the right hand endpoint of the interval $[0, 1]$ for the set of h with $h'(1) = \alpha$.

We now address the problem of showing that our sets that have been shown to be left-and-right shy are not shy. Lemma 5.3 is a basic fact about the differentiability properties of monotonic functions.

Lemma 5.3. *For any $0 \leq \alpha \leq +\infty$, if $f \in \mathcal{H}[0,1]$ and satisfies $\frac{f(c_n)}{c_n} \to \alpha$ for a decreasing sequence $\{c_n\}$ satisfying $c_n/c_{n+1} \to 1$ and $c_n \to 0+$ as $n \to \infty$, then $f'(0) = \alpha$.*

Proof. For any $x < c_1$, there exists a c_n such that $c_{n+1} \leq x \leq c_n$. Then $f(c_{n+1}) \leq f(x) \leq f(c_n)$ and so

$$\frac{f(c_{n+1})}{c_n} \leq \frac{f(x)}{x} \leq \frac{f(c_n)}{c_{n+1}}.$$

Note that

$$\frac{f(c_{n+1})}{c_n} = \frac{f(c_{n+1})}{c_{n+1}} \cdot \frac{c_{n+1}}{c_n}$$

and

$$\frac{f(c_n)}{c_{n+1}} = \frac{f(c_n)}{c_n} \cdot \frac{c_n}{c_{n+1}}.$$

Thus by the conditions we have $\lim_{x \to 0^+} \frac{f(x)}{x} = \alpha$. That is, $f'(0) = \alpha$. □

Our main theorem in this section can now be proved.

Theorem 5.4. *Let $0 < \alpha < +\infty$, and*

$$D_\alpha = \{h \in \mathcal{H}[0,1] : \ h'(0) = \alpha\}.$$

Then D_α is a Borel set that is left-and-right shy but not shy.

Proof. To show that D_α is not shy it is enough, in view of Lemma 2.1, to show that for any compact set $K \subseteq \mathcal{H}[0,1]$, there exist functions $g, k \in \mathcal{H}[0,1]$ such that $g \circ h \circ k \in D_\alpha$ for any $h \in K$.

We choose a decreasing sequence $\{c_n\} \subseteq (0,1)$ such that all the intervals $[c_n, c_n + 1/nc_n]$ are pairwise disjoint, $c_n \to 0+$ and $c_{n+1}/c_n \to 1$ as $n \to \infty$. For example, $c_n = 1/n$ $(n > 2)$ satisfy the requirements. By Theorem 2.2 there exist functions $f_1, f_2 \in \mathcal{H}[0,1]$ such that $f_1(x) \leq h(x) \leq f_2(x)$ for all $h \in K$ and all $x \in [0,1]$. Now we choose a sequence of line segments $\{I_n\}$ contained in $(0,1) \times (0,1)$, as in the figure below, such that

(i) for any n, the lower end of I_n is above the upper end of I_{n+1},

(ii) the corresponding point x_n of I_n tends to 0 from right, and

(iii) for any n, the line segment connecting $(x_n, f_1(x_n))$ and $(x_n, f_2(x_n))$ is contained in I_n.

We now construct functions $g, k \in \mathcal{H}[0,1]$ so that $g \circ h \circ k \in D_\alpha$ for any $h \in K$.

Figure 1: The construction of the automorphisms g and k

We require that $2c_1\alpha < 1$ so that, for any n,

$$[\alpha c_n, \alpha(1+1/n)c_n] \subseteq (0,1).$$

From the choice of $\{c_n\}$ it is easy to verify that all these intervals are pairwise disjoint. We construct a function $g \in \mathcal{H}[0,1]$ such that

$$g(I_n) = [\alpha c_n, \alpha(1+1/n)c_n],$$

and a function $k \in \mathcal{H}[0,1]$ such that $k(c_n) = x_n$. Then for any $h \in K$,

$$\frac{g(h(k(c_n)))}{c_n} = \frac{g(h(x_n))}{c_n} \leq \frac{\alpha(1+1/n)c_n}{c_n} = \alpha(1+1/n).$$

and

$$\frac{g(h(x_n))}{c_n} \geq \frac{\alpha c_n}{c_n} = \alpha.$$

Thus $\lim_{n\to\infty} \frac{g(h(k(c_n)))}{c_n} = \alpha$. By Lemma 5.3, $(g \circ h \circ k)'(0) = \alpha$. $\square$

Remark. Our methods work for more general cases. In [19] it is shown that the set of functions $h \in \mathcal{H}[0,1]$ satisfying $h'(0) = \alpha$ and $h'(1) = \beta$ is left-and-right shy but not shy for any $0 < \alpha < +\infty$ and $0 \leq \beta \leq +\infty$ or any $0 \leq \alpha \leq +\infty$ and $0 < \beta < +\infty$.

From Theorem 5.4 we immediately have the following observation for the non-locally compact, non-Abelian Polish group $\mathcal{H}[0,1]$. It is of some interest to know if the class of shy sets behaves in the same way that the class of measure zero sets on the real line does. Any disjoint collection of sets of real numbers that are not measure zero must, certainly, be countable. It is known (see [10, pp. 76]) that in certain non-locally compact, Abelian Polish groups there are uncountable disjoint collections of non-shy sets. It has been proved too (see [20, pp. 208]) that in any non-Abelian Polish group with an invariant metric this same statement is true. The results of this section demonstrate that, in the space $\mathcal{H}[0,1]$, again an uncountable disjoint collection of non-shy Borel sets can be exhibited.

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