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SOME MODIFICATIONS OF THE CORE TOPOLOGY ON THE PLANE

Abstract

In this paper we introduce some modifications of the core topology on the plane. Here the role of the Euclidean topology on the real line on every or almost every direction is played by the density topology or the Hashimoto topology connected with the σ -ideal of null sets and the σ -ideal of meager sets. We demonstrate the proper inclusions between these families, we investigate the separation axioms and functions continuous with respect to these topologies.

1 Introduction

The aim of this paper is to introduce some modifications of the core topology on the plane. The core topology was considered earlier by M. Kuczma and his students ([K], [KK]). The topologies presented here are modifications of this topology, similar to the d -crosswise topology (cf. [LMZ], p. 98).

Generally speaking, a point is an interior point of a set $A \subset \mathbb{R}^2$ if and only if it is an interior point of the set A in a certain topology on the line in every or I -almost every direction, where I is a certain σ -ideal. Investigations of the topologies of this kind ([WBW], [LS]) can lead one to answers to interesting questions connected with the behavior of real-valued functions on \mathbb{R}^2 in different directions. The problem of connections between the partial derivatives and directional derivatives was considered by Bruckner and Rosenfeld in [BR].

The comparison of the topologies defined here and already known density topologies seems to be interesting, especially when one considers the open question of the coarsest topology for which strongly approximately continuous

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functions are continuous; i.e. of the complete regularity of the topologies and an analogue of the Lusin-Menchoff Theorem ([GNN]).

Recall the notion of the core topology, which is denoted here by \mathcal{T}' -core. Let $A \subset \mathbb{R}^2$. A point $x \in A$ is said to be an algebraically interior point of A if and only if, for each $y \in \mathbb{R}^2$, there exists $\varepsilon > 0$ such that $x + ty \in A$ for $t \in (-\varepsilon, \varepsilon)$; i.e., A contains an open segment centered at the point x on each line passing through x .

The set of all points which are algebraically interior to A is denoted by $\text{core } A$. A set $A \subset \mathbb{R}^2$ is called algebraically open if $A = \text{core } A$. The family

$$\mathcal{T}'\text{-core} = \{A \subset \mathbb{R}^2 : A = \text{core } A\}$$

forms a topology in \mathbb{R}^2 , called the core topology.

There are at least two kinds of natural modifications of this definition; changing the set of directions on which we have a neighborhood in some topology, and changing the topology on the line.

This paper consists of two sections. In the first one we introduce several topologies analogous to the core topology on the plane, using the σ -ideal of null sets. Here the role of the Euclidean topology on the real line on every or almost every direction is played by the Hashimoto topology or the density topology on the real line. We demonstrate the proper inclusions between these families and compare them with the density topologies and the Hashimoto topology on the plane. We investigate the separation axioms and prove that functions continuous with respect to these topologies are of Baire class 1.

In the second section we consider analogous problems for Baire category. We obtain a similar diagram for the core topologies connected with the ideal of meager sets. Only the results of the comparison with the ordinary I -density topology on the plane and separation axioms are a little bit different. We prove that functions continuous with respect to these topologies are of Baire class at most 2.

For the terminology and definitions of density point, I -density point, density and I -density topologies on the line and on the plane (d , d_I , d_2 , d_{I_2} , d_2^s , $d_{I_2}^s$), see [CW], [GNN], [PWW], [W], [WB].

Let m_k denote Lebesgue measure on \mathbb{R}^k , $k = 1, 2$, \mathcal{L}_k — the family of all measurable sets on \mathbb{R}^k , $k = 1, 2$, \mathcal{B}_k — the family of all sets having the Baire property on \mathbb{R}^k , $k = 1, 2$.

The ball (circle) centered at a point (x, y) with radius r is denoted by $K((x, y), r)$ ($S((x, y), r)$). Let $\text{sec}(\alpha, \beta)$ denote an angular domain determined by the lines forming angles α and β with the x -axis. Let p_Θ ($p_\Theta(x)$), $\Theta \in \langle 0, \pi \rangle$, denote the line running through the origin (the point x) and forming an angle Θ with the x -axis. Let p_Θ^+ ($p_\Theta^+(x)$), $\Theta \in \langle 0, 2\pi \rangle$, denote the analogous half-line.

We denote by χ_A the characteristic function of a set A and by \mathbb{B}_1 , Baire class 1.

2 Core Topologies Defined with Null Sets

Definition 2.1. A set $A \subset \mathbb{R}^2$ is Hash-core (ap-core) open if and only if, for each point $x \in A$ and for each $\Theta \in \langle 0, \pi \rangle$, the set $p_\Theta(x) \cap A$ contains a neighborhood of x in the Hashimoto topology (density topology) on the line $p_\Theta(x)$; i.e., a set of the form $G \setminus N$ where G is open in the Euclidean topology, N is a null set and $x \in G \setminus N$. (In other words, $p_\Theta(x) \cap A$ contains a linearly measurable set B such that $d_1(B, x) = 1$ and $x \in B$.)

Let \mathcal{T}' -Hash-core (\mathcal{T}' -ap-core) denote the family of all Hash-core (ap-core) open sets.

Definition 2.2. A set $A \subset \mathbb{R}^2$ is core-a.e. (Hash-core-a.e., ap-core-a.e.) open if and only if, for each point $x \in A$, there exists a linear set $H \subset \langle 0, \pi \rangle$ such that $m_1(H) = \pi$ and, for all $\Theta \in H$, the set $p_\Theta(x) \cap A$ contains a neighborhood of x in the Euclidean topology (Hashimoto topology, density topology) on the line $p_\Theta(x)$.

Let \mathcal{T}' -core-a.e. (\mathcal{T}' -Hash-core-a.e., \mathcal{T}' -ap-core-a.e.) denote the family of all core-a.e. (Hash-core-a.e., ap-core-a.e.) open sets.

It is easy to see that the families of sets we have just defined form topologies. They also contain non-measurable sets (see [WBW]).

Theorem 2.3. *The following proper inclusions hold.*

$$\begin{array}{ccccc} \mathcal{T}'\text{-core-a.e.} & \underset{2)}{\subset} & \mathcal{T}'\text{-Hash-core-a.e.} & \underset{3)}{\subset} & \mathcal{T}'\text{-ap-core-a.e.} \\ \cup 1) & & \cup 1) & & \cup 1) \\ \mathcal{T}'\text{-core} & \underset{2)}{\subset} & \mathcal{T}'\text{-Hash-core} & \underset{3)}{\subset} & \mathcal{T}'\text{-ap-core} \end{array}$$

PROOF. It suffices to construct three sets. Let

$$A = \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 : x \neq 0, y = 0\}.$$

Then $A \in \mathcal{T}'\text{-core-a.e.} \setminus \mathcal{T}'\text{-ap-core}$. This shows that the inclusions denoted by 1) are proper. Let $B = \mathbb{R}^2 \setminus \left(\bigcup_{n \in \mathbb{N}} S\left((0, 0), \frac{1}{n}\right) \right)$. Then $B \in \mathcal{T}'\text{-Hash-core} \setminus \mathcal{T}'\text{-core-a.e.}$

This shows that the inclusions denoted by 2) are proper.

Let $E = \bigcup_{n \in \mathbb{N}} (a_n, b_n)$ where $\lim_{n \rightarrow \infty} b_n = 0$, $0 < a_{n+1} < b_{n+1} < a_n$ and 0 is a point of right-hand density of E . Let

$$C = \{(r, \Theta) : r \in E, \Theta \in \langle 0, 2\pi \rangle\}$$

where (r, θ) denotes the polar coordinates. Then $C \in \mathcal{T}'\text{-ap-core} \setminus \mathcal{T}'\text{-Hash-core-a.e.}$. This shows that the inclusions denoted by 3) are proper. \square

To compare the topologies considered with the density topologies on the plane, we assume open sets to be measurable. Thus we define the following families of sets.

$$\begin{aligned}\mathcal{T}\text{-core} &= \mathcal{T}'\text{-core} \cap \mathcal{L}_2, \\ \mathcal{T}\text{-core-a.e.} &= \mathcal{T}'\text{-core-a.e.} \cap \mathcal{L}_2, \\ \mathcal{T}\text{-Hash-core} &= \mathcal{T}'\text{-Hash-core} \cap \mathcal{L}_2, \\ \mathcal{T}\text{-Hash-core-a.e.} &= \mathcal{T}'\text{-Hash-core-a.e.} \cap \mathcal{L}_2, \\ \mathcal{T}\text{-ap-core} &= \mathcal{T}'\text{-ap-core} \cap \mathcal{L}_2, \\ \mathcal{T}\text{-ap-core-a.e.} &= \mathcal{T}'\text{-ap-core-a.e.} \cap \mathcal{L}_2.\end{aligned}$$

It is easily seen that there are proper inclusions between these families because the sets A, B, C from Theorem 2.3 are measurable.

So we have

$$\begin{array}{ccccc} \mathcal{T}\text{-core-a.e.} & \subset & \mathcal{T}\text{-Hash-core-a.e.} & \subset & \mathcal{T}\text{-ap-core-a.e.} \\ \cup & & \cup & & \cup & (1) \\ \mathcal{T}\text{-core} & \subset & \mathcal{T}\text{-Hash-core} & \subset & \mathcal{T}\text{-ap-core.} \end{array}$$

Theorem 2.4. *All the families of sets from (1) are contained in the density topology on the plane.*

The proof is analogous to that of Theorem 2.4 in [WBW].

Corollary 2.5. *Each of the families from (1) forms a topology.*

The families from (1) are called core topologies connected with the σ -ideal of null sets.

Now, we compare them with the Euclidean topology, the Hashimoto topology and the density topologies on the plane.

It is easy to see ([S]) that the Euclidean topology is essentially weaker than each of the core topologies connected with the σ -ideal of null sets.

Theorem 2.6. *The strong density topology on the plane is not contained in any of the core topologies connected with the σ -ideal of null sets.*

PROOF. It suffices to prove that $d_2^s \not\subset \mathcal{T}$ -ap-core-a.e. Let $\{a_n\}_{n \in \mathbb{N}}$ be a decreasing sequence of positive numbers tending to 0 such that $\frac{a_{n+1}}{a_n} \xrightarrow{n \rightarrow \infty} 0$. We construct a set A which is a union of parts of ring-shaped domains with radii a_n and $2a_n$. Let $\alpha = \frac{\pi}{4}$. Put

$$A_n^{(k,l)} = \left\{ (r, \Theta) : r \in \langle a_{(2^n - n - 1) + (2^{k-1} - 1) + l}, 2a_{(2^n - n - 1) + (2^{k-1} - 1) + l} \rangle, \right. \\ \left. \Theta \in \left\langle \frac{1}{n+2-k} \alpha + \frac{(l-1) \left(\frac{1}{n+1-k} - \frac{1}{n+2-k} \right) \alpha}{2^{k-1}}, \right. \right. \\ \left. \left. \frac{1}{n+2-k} \alpha + \frac{l \left(\frac{1}{n+1-k} - \frac{1}{n+2-k} \right) \alpha}{2^{k-1}} \right\rangle \right\}$$

for $n \in \mathbb{N}$, $k \in \{1, \dots, n\}$, $l \in \{1, \dots, 2^{k-1}\}$ and

$$A^0 = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^n \bigcup_{l=1}^{2^{k-1}} A_n^{(k,l)}.$$

We arrange the sets $A_n^{(k,l)}$ in a sequence $\{C_m\}_{m \in \mathbb{N}}$ in the following way. $A_{n_1}^{(k_1, l_1)}$ precedes $A_{n_2}^{(k_2, l_2)}$ in this sequence if and only if $n_1 < n_2$ or $n_1 = n_2$ and $k_1 < k_2$ or $n_1 = n_2$, $k_1 = k_2$ and $l_1 < l_2$. So the set C_m is a part of the ring with radii a_m and $2a_m$. Angles which form the lines marking the set C_m with the x -axis are denoted by α_m and β_m , $0 < \alpha_m < \beta_m < \frac{\pi}{4}$. Let $\omega_m = \beta_m - \alpha_m$, $m \in \mathbb{N}$. From the construction it follows that the sequence $\{\omega_m\}_{m \in \mathbb{N}}$ tends to zero.

Put

$$A^1 = \{(x, y) : (y, x) \in A^0\} \\ A = \{(x, y) : (x, y) \in A^0 \cup A^1 \text{ or } (-x, y) \in A^0 \cup A^1 \\ \text{or } (x, -y) \in A^0 \cup A^1 \text{ or } (-x, -y) \in A^0 \cup A^1\}$$

and $B = \mathbb{R}^2 \setminus A$.

Observe that B is not open in the \mathcal{T} -ap-core-a.e. topology. Indeed, for each $\Theta \in \langle 0, \pi \rangle$, the set $p_\Theta \cap B$ does not contain a neighborhood of the point $(0, 0)$ in the density topology on the line because, on the line p_Θ , there are intervals of the form $(a_m, 2a_m)$ arbitrarily close to $(0, 0)$ and contained in A .

We shall show that B is open in the strong density topology on the plane. It suffices to prove that $d_2^s(B, (0, 0)) = 1$; i.e., for each pair of sequences of positive numbers $\{t_n\}_{n \in \mathbb{N}} \searrow 0$ and $\{h_n\}_{n \in \mathbb{N}} \searrow 0$, we have

$$d_n = \frac{m_2(B \cap (\langle -t_n, t_n \rangle \times \langle -h_n, h_n \rangle))}{4t_n h_n} \xrightarrow{n \rightarrow \infty} 1.$$

We need only show that any subsequence of $\{d_n\}_{n \in \mathbb{N}}$ includes a subsequence tending to 1. Let us take an arbitrary subsequence $\{n_m\}_{m \in \mathbb{N}}$ of the sequence of positive integers. There are three cases.

1⁰ The sequence $\{\frac{t_{n_m}}{h_{n_m}}\}_{m \in \mathbb{N}}$ includes a subsequence $\{\frac{t_{n_{m_p}}}{h_{n_{m_p}}}\}_{p \in \mathbb{N}}$ converging to a number a , $0 < a < \infty$. Then there exists a number α such that $0 < \alpha < \frac{t_{n_{m_p}}}{h_{n_{m_p}}} < \frac{1}{\alpha} < \infty$ for sufficiently large $p \in \mathbb{N}$. Hence the sequence of rectangles $E_{n_{m_p}} = \langle -t_{n_{m_p}} t_{n_{m_p}} \rangle \times \langle -h_{n_{m_p}} h_{n_{m_p}} \rangle$, $p \in \mathbb{N}$, is regular. Consequently, we only need to consider the limit

$$\lim_{r \rightarrow 0^+} \frac{m_2(A \cap K((0, 0), r))}{\pi r^2}.$$

Fix $r \in \langle a_n, a_{n-1} \rangle$. Then $m_2(C_n) = \frac{3}{2} a_n^2 \omega_n$ and

$$\begin{aligned} \frac{m_2(A \cap K((0, 0), r))}{\pi r^2} &\leq \frac{8 \frac{3}{2} a_n^2 \omega_n + \pi (2a_{n+1})^2}{\pi r^2} \\ &\leq \frac{12}{\pi} \omega_n + \frac{4a_{n+1}^2}{a_n^2}. \end{aligned}$$

So

$$\frac{m_2(A \cap K((0, 0), r))}{\pi r^2} \xrightarrow{r \rightarrow 0^+} 0.$$

2⁰ The sequence $\{\frac{t_{n_m}}{h_{n_m}}\}_{m \in \mathbb{N}}$ includes a subsequence $\{\frac{t_{n_{m_p}}}{h_{n_{m_p}}}\}_{p \in \mathbb{N}}$ tending to ∞ and 1⁰ does not hold.

Fix $t_{n_{m_p}} \in \langle a_{k_p}, a_{k_p-1} \rangle$. Then, after elementary calculations, we obtain

$$\begin{aligned} &\frac{m_2(A \cap (\langle -t_{n_{m_p}} t_{n_{m_p}} \rangle \times \langle -h_{n_{m_p}} h_{n_{m_p}} \rangle))}{4t_{n_{m_p}} h_{n_{m_p}}} \\ &\leq \frac{4 \cdot 2a_{k_p+1} h_{n_{m_p}} + 4m_2(C_{k_p} \cap (\langle -t_{n_{m_p}} t_{n_{m_p}} \rangle \times \langle -h_{n_{m_p}} h_{n_{m_p}} \rangle))}{4t_{n_{m_p}} h_{n_{m_p}}} \\ &= \frac{2a_{k_p+1}}{t_{n_{m_p}}} + \frac{m_2(C_{k_p} \cap (\langle -t_{n_{m_p}} t_{n_{m_p}} \rangle \times \langle -h_{n_{m_p}} h_{n_{m_p}} \rangle))}{t_{n_{m_p}} h_{n_{m_p}}}; \end{aligned}$$

so,

$$\frac{m_2(A \cap (\langle -t_{n_{m_p}} t_{n_{m_p}} \rangle \times \langle -h_{n_{m_p}} h_{n_{m_p}} \rangle))}{4t_{n_{m_p}} h_{n_{m_p}}} \xrightarrow{p \rightarrow \infty} 0.$$

3⁰ The sequence $\{\frac{t_{n_m}}{h_{n_m}}\}_{m \in \mathbb{N}}$ includes a subsequence $\{\frac{t_{n_{m_p}}}{h_{n_{m_p}}}\}_{p \in \mathbb{N}}$ converging to 0 and neither 1⁰ nor 2⁰ holds.

Similar to 2^0 , we have

$$\frac{m_2(A \cap (\langle -t_{n_{m_p}} t_{n_{m_p}} \rangle \times \langle -h_{n_{m_p}} h_{n_{m_p}} \rangle))}{4t_{n_{m_p}} h_{n_{m_p}}} \xrightarrow{p \rightarrow \infty} 0.$$

So, we have proved that $B \in d_2^s \setminus \mathcal{T}\text{-ap-core-a.e.}$ \square

Corollary 2.7. *The density topology on the plane is not contained in any of the core topologies connected with the σ -ideal of null sets.*

Theorem 2.8. *None of the core topologies connected with the σ -ideal of null sets is contained in the strong density topology on the plane.*

PROOF. It is sufficient to show that $\mathcal{T}\text{-core} \not\subset d_2^s$. Let

$$A = \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 : x > 0, x^4 \leq y \leq x^2\}.$$

We have

$$\frac{m_2\left(A' \cap \left(\langle -\frac{1}{n}, \frac{1}{n} \rangle \times \langle -\frac{1}{n^4}, \frac{1}{n^4} \rangle\right)\right)}{m_2\left(\langle -\frac{1}{n}, \frac{1}{n} \rangle \times \langle -\frac{1}{n^4}, \frac{1}{n^4} \rangle\right)} = \frac{12n - 10}{15n \cdot 4}.$$

Consequently, $\overline{d}_2^s(A', (0, 0)) > 0$; so $A \notin d_2^s$. Obviously, $A \in \mathcal{T}\text{-core}$. Hence $A \in \mathcal{T}\text{-core} \setminus d_2^s$. \square

Corollary 2.9. *None of the core topologies connected with the σ -ideal of null sets are contained in $d \times d$.*

PROOF. We have $d \times d \subset d_2^s$ (see [WB]); so the proof is obvious. \square

Corollary 2.10. *The core topologies connected with the σ -ideal of null sets and the strong density topology are incomparable.*

It is known ([WBW], Th. 2.4) that the topology $d \times d$ is weaker than the $\mathcal{T}\text{-ap-core}$ topology.

Corollary 2.11. *The topology $d \times d$ is weaker than the $\mathcal{T}\text{-ap-core-a.e.}$ topology.*

Theorem 2.12. *The topology $d \times d$ is contained in neither the $\mathcal{T}\text{-Hash-core-a.e.}$ topology nor the $\mathcal{T}\text{-core}$, $\mathcal{T}\text{-core-a.e.}$, $\mathcal{T}\text{-Hash-core}$.*

PROOF. It is sufficient to show that $d \times d \not\subset \mathcal{T}\text{-Hash-core-a.e.}$ Let $E = \bigcup_{n \in \mathbb{N}} (a_n, b_n) \cup (-\infty; 0)$ where $\bigcup_{n \in \mathbb{N}} (a_n, b_n)$ is an interval set such that $d(E, 0) = 1$. Put $A = E \times \mathbb{R}$. Then $A \in d \times d \setminus \mathcal{T}\text{-Hash-core-a.e.}$ \square

Theorem 2.13. *None of the core topologies connected with the σ -ideal of null sets is contained in the Hashimoto topology on the plane (\mathcal{O}^*).*

PROOF. It follows from Theorem 2.8 because $\mathcal{O}^* \subset d_2^s$. □

Theorem 2.14. *The Hashimoto topology on the plane is weaker than the \mathcal{T} -Hash-core-a.e. and the \mathcal{T} -ap-core-a.e. topologies.*

PROOF. It is sufficient to show that $\mathcal{O}^* \subset \mathcal{T}$ -Hash-core-a.e. Let $A \in \mathcal{O}^*$ and $(x_0, y_0) \in A$. We may assume that $x_0 = y_0 = 0$. We prove that $(0, 0)$ is an interior point of A in the \mathcal{T} -Hash-core-a.e. topology. Since $A \in \mathcal{O}^*$, there exist sets $G, N \subset \mathbb{R}^2$ such that $A = G \setminus N$, G is open in the Euclidean topology on the plane and $m_2(N) = 0$. By Fubini's theorem for polar coordinates, there exists a set $H \subset \langle 0, \pi \rangle$ such that $m_1(H) = 0$ and, for $\theta \in \langle 0, \pi \rangle \setminus H$, the set $N_\theta = N \cap p_\theta$ is a null set on the line p_θ . Consequently, we can choose an open interval $(a_\theta, b_\theta) \subset G \cap p_\theta$ such that $(0, 0) \in (a_\theta, b_\theta) \setminus N_\theta \subset p_\theta \cap A$ and $m_1(N_\theta) = 0$ for almost all $\theta \in \langle 0, \pi \rangle$. Thus $(0, 0)$ belongs to the interior of A in the \mathcal{T} -Hash-core-a.e. topology. □

Theorem 2.15. *The Hashimoto topology on the plane is not contained in any of the topologies: \mathcal{T} -core, \mathcal{T} -core-a.e., \mathcal{T} -Hash-core, \mathcal{T} -ap-core.*

PROOF. It is sufficient to show that $\mathcal{O}^* \not\subset \mathcal{T}$ -core-a.e. and $\mathcal{O}^* \not\subset \mathcal{T}$ -ap-core. The set B from Theorem 2.3 is open in \mathcal{O}^* and is not open in the \mathcal{T} -core-a.e. topology. The set A from the same theorem is open in \mathcal{O}^* and is not open in the \mathcal{T} -ap-core topology. □

The results of Theorems and Corollaries 2.4-2.15 are presented in the following scheme.

$$\begin{array}{ccccc}
 \begin{array}{c} \mathcal{O}^* \\ \uparrow \\ \mathcal{T}\text{-core-a.e.} \end{array} & \subset & \begin{array}{c} d \times d \\ \uparrow \\ \mathcal{T}\text{-Hash-core-a.e.} \end{array} & \subset & \begin{array}{c} d_2^s \\ \uparrow \\ \mathcal{T}\text{-ap-core-a.e.} \end{array} \subset d_2 \\
 \cup & & \cup & & \cup \\
 \mathcal{O} \subset \mathcal{T}\text{-core} & \subset & \mathcal{T}\text{-Hash-core} & \subset & \mathcal{T}\text{-ap-core.} \\
 \begin{array}{c} \uparrow \\ d_2^s \\ \uparrow \\ \mathcal{O}^* \end{array} & & & & \begin{array}{c} \cup \\ d \times d \\ \cup \\ \mathcal{O}^* \end{array}
 \end{array} \tag{2}$$

Theorem 2.16. *The \mathcal{T} -core topology is not regular.*

PROOF. We show that there exists a set F closed in the \mathcal{T} -core topology, which cannot be separated from the point $(0, 0)$.

Consider a sequence of rectangles $Q_{k,l}$ where $k \in \mathbb{N} \cup \{0\}$, $l \in \{1, \dots, 2^k\}$ and $Q_{k,l} = \left(\frac{l-1}{2^k}, \frac{l}{2^k}\right) \times \left(\frac{1}{2^{k+1}}, \frac{1}{2^k}\right)$. Form a sequence $\{P_n\}_{n \in \mathbb{N}}$ by putting $P_n = Q_{k,l}$ where $n = \sum_{i=0}^{k-1} 2^i + l$. Choose points $p_n \in P_n$, $n \in \mathbb{N}$, in such a way that any three of them are not colinear. Let $F = \{p_n : n \in \mathbb{N}\}$. Then $F' = \mathbb{R}^2 \setminus F \in \mathcal{T}$ -core and $(0, 0) \notin F$.

Let $G, G_0 \in \mathcal{T}$ -core, $(0, 0) \in G_0$ and $F \subset G$. We shall show that $G \cap G_0 \neq \emptyset$. The set G_0 is a neighborhood of $(0, 0)$ in the \mathcal{T} -core topology; so there exists a number $d \in (0, 1)$ such that $\langle 0, d \rangle \times \{0\} \subset G_0$. Similarly, G is a neighborhood of the points $p_n = (x_n, y_n)$ in the \mathcal{T} -core topology; so there exist $\delta_n \in (0, 1)$, $n \in \mathbb{N}$, such that

$$\bigcup_{n \in \mathbb{N}} (x_n - \delta_n, x_n + \delta_n) \times \{y_n\} \subset G.$$

Let

$$E = \limsup_n (x_n - \delta_n, x_n + \delta_n) = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} (x_m - \delta_m, x_m + \delta_m).$$

From the construction of F it follows that the set $\{x_m, m \geq n\}$ is dense on the interval $(0, 1)$ for each $n \in \mathbb{N}$. Consequently, E is residual on this interval. Let $x \in E \cap (0, d)$. Since $(x, 0) \in \langle 0, d \rangle \times \{0\} \subset G_0$ and $G_0 \in \mathcal{T}$ -core, there exists a number $h > 0$ such that $\{x\} \times \langle 0, h \rangle \subset G_0$. Then

$$(\{x\} \times \langle 0, h \rangle) \cap ((x_n - \delta_n, x_n + \delta_n) \times \{y_n\}) \neq \emptyset$$

for infinitely many $n \in \mathbb{N}$ (since $x \in (x_n - \delta_n, x_n + \delta_n)$ and $y_n < h$ for almost all $n \in \mathbb{N}$ because $y_n \xrightarrow{n \rightarrow \infty} 0$). Hence $G_0 \cap G \neq \emptyset$. \square

Theorem 2.17. *The \mathcal{T} -Hash-core topology is not regular.*

PROOF. Let $F = \bigcup_{w \in \mathbb{Q}_+} S((0, 0), w)$. Obviously, $(0, 0) \notin F$ and F is closed in the \mathcal{T} -Hash-core topology.

Let $G, G_0 \in \mathcal{T}$ -Hash-core, $(0, 0) \in G_0$ and $F \subset G$. We shall show that, for each $\Theta \in \langle 0, \pi \rangle$, the intersection $G_0 \cap G \cap p_\Theta$ is non-empty.

Indeed, for each $\Theta \in \langle 0, \pi \rangle$, the set $G \cap p_\Theta$ includes, with each point of the set $F \cap p_\Theta$, a set of full measure on some interval. Since there are points of the set $F \cap p_\Theta$ arbitrarily close to $(0, 0)$, the set $G_0 \cap p_\Theta$, as a set containing a neighborhood of $(0, 0)$ in the Hashimoto topology on the line p_Θ , is not disjoint from $G \cap p_\Theta$. \square

We shall show that the \mathcal{T} -Hash-core-a.e. topology is not regular. We shall use the following lemmas.

Lemma 2.18. *If $G \in \mathcal{T}$ -Hash-core-a.e. and $(0, 0) \in G$, then, for each $r > 0$,*

$$A_r = \{\Theta \in \langle 0, \pi \rangle : m_1(G \cap K((0, 0), r) \cap p_\Theta) = 2r\}$$

is measurable.

The proof follows immediately from the theorem on integration by substitution and Fubini's theorem.

Lemma 2.19. *If $G \in \mathcal{T}$ -Hash-core-a.e. and $(0,0) \in G$, then there exists a number $n_0 \in \mathbb{N}$ such that $m_1(A_{\frac{1}{n_0}}) > 0$, where the sets A_r are as above.*

Theorem 2.20. *The \mathcal{T} -Hash-core-a.e. topology is not regular.*

PROOF. Put $F = \bigcup_{n \in \mathbb{N}} S((0,0), \frac{1}{n})$. Obviously, $(0,0) \notin F$ and F is closed in the \mathcal{T} -Hash-core-a.e. topology. Let $G, G_0 \in \mathcal{T}$ -Hash-core-a.e., $(0,0) \in G_0$ and $F \subset G$. We assume that $G \cap G_0 = \emptyset$. By Lemma 2.19, there exists a number $n_0 \in \mathbb{N}$ such that

$$A_{\frac{1}{n_0}} = \left\{ \Theta \in (0, \pi) : m_1\left(G_0 \cap K\left((0,0), \frac{1}{n_0}\right) \cap p_\Theta\right) = \frac{2}{n_0} \right\}$$

has positive linear Lebesgue measure. Let Θ_0 be a density point of $A_{\frac{1}{n_0}}$. Let us consider $S((0,0), \frac{1}{n_1})$ where $n_1 > n_0$. Let (x_0, y_0) be one of the points of the intersection $S((0,0), \frac{1}{n_1}) \cap p_{\Theta_0}$. We can assume that $x_0, y_0 > 0$. In the other cases the proof is analogous. Obviously, $(x_0, y_0) \in G$. We shall show that (x_0, y_0) is not a dispersion point of G_0 , which is a contradiction.

We shall show that there exists $\alpha > 0$ such that $\frac{m_2(K \cap G_0)}{m_2(K)} > \alpha$ for all the balls K with center at the point (x_0, y_0) and radius sufficiently small, which means that $\overline{d}_2(G_0, (x_0, y_0)) > 0$.

Let $0 < a < \frac{1}{n_0 n_1}$. We denote by (x'', y'') the point of intersection of the circle $S((0,0), \frac{1}{n_1} + \frac{a}{2})$ and the half-line $p_{\Theta_0}^+$, by (x_2, y_2) , the point of intersection of the circle $S((x_0, y_0), a)$ and the half-line $p_{\frac{\pi}{2} + \Theta_0}^+(x'', y'')$. Let δ denote an angle between the half-line $p_{\Theta_0}^+$ and the half-line running from the point $(0,0)$ through the point (x_2, y_2) . Then

$$\tan \delta = \frac{\frac{\sqrt{3}}{2}a}{\frac{1}{n_1} + \frac{a}{2}}. \quad (3)$$

Thus $\lim_{a \rightarrow 0} \delta = 0$. Put $r_2 = \sqrt{x_2^2 + y_2^2}$. Denote by (x', y') the point of intersection of the circle $S((0,0), \frac{1}{n_1} - \frac{a}{2})$ and the half-line $p_{\Theta_0}^+$, by (x_1, y_1) , the point of intersection of the half-lines $p_{\Theta_0 + \delta}^+$ and $p_{\Theta_0 + \frac{\pi}{2}}^+(x', y')$. Put $r_1 = \sqrt{x_1^2 + y_1^2}$.

Then $\cos \delta = \frac{\frac{1}{n_1} - \frac{a}{2}}{r_1} = \frac{\frac{1}{n_1} + \frac{a}{2}}{r_2}$. Let

$$A = (r_1, r_2) \times \left(A_{\frac{1}{n_0}} \cap (\Theta_0 - \delta, \Theta_0 + \delta) \right),$$

$$U = (r_1, r_2) \times (\Theta_0 - \delta, \Theta_0 + \delta).$$

Obviously, $A \subset U$.

Let $f(r, \Theta) = (r \cos \Theta, r \sin \Theta)$, $(r, \Theta) \in U$. Put $K = K((x_0, y_0), a)$. Then $f(U) \subset K$ and

$$\frac{m_2(f(U))}{m_2(K)} = \frac{(r_2^2 - r_1^2)\delta}{\pi a^2} = \frac{\frac{2}{n_1} \delta}{\pi a \cos^2 \delta}.$$

Since $f(U) \supset f(A)$ and $m_2(f(A)) = m_2(f(A) \cap G_0) + m_2(f(A) \setminus G_0)$, it follows that

$$\frac{m_2(K \cap G_0)}{m_2(K)} \geq \frac{m_2(f(A)) - m_2(f(A) \setminus G_0)}{m_2(K)}.$$

We have

$$\begin{aligned} m_2(f(A)) &= \int_{f(U)} \chi_{f(A)}(p) dp = \int_U \chi_{f(A)}(f(r, \Theta)) r dr d\Theta \\ &= \int_{\Theta_0 - \delta}^{\Theta_0 + \delta} \int_{r_1}^{r_2} \chi_{f(A)}(f(r, \Theta)) r dr d\Theta \geq \int_{A_{\frac{1}{n_0}} \cap (\Theta_0 - \delta, \Theta_0 + \delta)} \frac{r_2^2 - r_1^2}{2} d\Theta \\ &= \frac{r_2^2 - r_1^2}{2} m_1\left(A_{\frac{1}{n_0}} \cap (\Theta_0 - \delta, \Theta_0 + \delta)\right) \end{aligned}$$

and

$$\begin{aligned} m_2(f(A) \setminus G_0) &= \int_{f(U)} \chi_{f(A) \setminus G_0}(p) dp = \int_U \chi_{f(A) \setminus G_0}(f(r, \Theta)) r dr d\Theta \\ &= \int_{\Theta_0 - \delta}^{\Theta_0 + \delta} \int_{r_1}^{r_2} \chi_{f(A) \setminus G_0}(f(r, \Theta)) r dr d\Theta \\ &= \int_{(\Theta_0 - \delta, \Theta_0 + \delta) \cap A_{\frac{1}{n_0}}} \int_{r_1}^{r_2} 0 r dr d\Theta \\ &\quad + \int_{(\Theta_0 - \delta, \Theta_0 + \delta) \setminus A_{\frac{1}{n_0}}} \int_{r_1}^{r_2} r dr d\Theta \\ &\leq \frac{r_2^2 - r_1^2}{2} m_1\left((\Theta_0 - \delta, \Theta_0 + \delta) \setminus A_{\frac{1}{n_0}}\right). \end{aligned}$$

Thus

$$\begin{aligned}
\frac{m_2(K \cap G_0)}{m_2(K)} &\geq \frac{m_2(f(A)) - m_2(f(A) \setminus G_0)}{m_2(f(U))} \frac{m_2(f(U))}{m_2(K)} \\
&\geq \frac{\left(m_1((\Theta_0 - \delta, \Theta_0 + \delta) \cap A_{\frac{1}{n_0}}) - m_1((\Theta_0 - \delta, \Theta_0 + \delta) \setminus A_{\frac{1}{n_0}}) \right)}{(r_2^2 - r_1^2)\delta} \\
&\quad \times \frac{\frac{r_2^2 - r_1^2}{2} \frac{2}{n_1} \delta}{\pi a \cos^2 \delta} \\
&= \left[\frac{m_1((\Theta_0 - \delta, \Theta_0 + \delta) \cap A_{\frac{1}{n_0}})}{2\delta} - \frac{m_1((\Theta_0 - \delta, \Theta_0 + \delta) \setminus A_{\frac{1}{n_0}})}{2\delta} \right] \\
&\quad \times \frac{\frac{2}{n_1}}{\pi \cos^2 \delta} \frac{\delta}{\operatorname{tg} \delta} \frac{\sqrt{3}}{\frac{2}{n_1} + a}
\end{aligned}$$

by (3).

Since $d(A_{\frac{1}{n_0}}, \Theta_0) = 1$, it follows that

$$\begin{aligned}
\lim_{a \rightarrow 0} &\left[\frac{m_1((\Theta_0 - \delta, \Theta_0 + \delta) \cap A_{\frac{1}{n_0}})}{2\delta} - \frac{m_1((\Theta_0 - \delta, \Theta_0 + \delta) \setminus A_{\frac{1}{n_0}})}{2\delta} \right] \\
&\quad \times \frac{\frac{2}{n_1}}{\pi \cos^2 \delta} \frac{\delta}{\operatorname{tg} \delta} \frac{\sqrt{3}}{\frac{2}{n_1} + a} = \frac{\sqrt{3}}{\pi}
\end{aligned}$$

so $\frac{m_2(G_0 \cap K)}{m_2(K)} \geq \frac{\sqrt{3}}{2\pi}$ for sufficiently small a . \square

Denote by $C_{\mathcal{T}}$ the family of all continuous functions from the plane with a topology \mathcal{T} to the real line with the Euclidean topology. The following inclusions hold.

$$\begin{array}{ccccccc}
C_{\mathcal{T}\text{-core-a.e.}} & \subset & C_{\mathcal{T}\text{-Hash-core-a.e.}} & \subset & C_{\mathcal{T}\text{-ap-core-a.e.}} & \subset & C_{d_2} \\
\cup & & \cup & & \cup & & \\
C_{\mathcal{O}} & \subset & C_{\mathcal{T}\text{-core}} & \subset & C_{\mathcal{T}\text{-Hash-core}} & \subset & C_{\mathcal{T}\text{-ap-core}}
\end{array}$$

Since approximately continuous functions are of Baire class 1, all the functions from the last diagram are also of this class.

Theorem 2.21. *The \mathcal{T} -core-a.e., \mathcal{T} -ap-core and \mathcal{T} -ap-core-a.e. topologies are not normal.*

PROOF. Let A, B be two disjoint countable subsets of $\langle 0, 1 \rangle \times \{0\}$ which are dense in the Euclidean topology on $\langle 0, 1 \rangle \times \{0\}$. Then A and B are closed in the \mathcal{T} -core-a.e. (\mathcal{T} -ap-core, \mathcal{T} -ap-core-a.e.) topology.

We suppose the space to be normal. Then there exists $f \in C_{\mathcal{T}\text{-core-a.e.}}$ ($C_{\mathcal{T}\text{-ap-core}}$, $C_{\mathcal{T}\text{-ap-core-a.e.}}$) such that $f(A) = \{1\}$ and $f(B) = \{0\}$. Consequently, f is discontinuous everywhere on $\langle 0, 1 \rangle \times \{0\}$. but this is impossible because f is of Baire class 1. \square

3 Core Topologies Defined with Meager Sets

Analogous to the topologies defined in the previous section we can define the topologies connected with the σ -ideal of meager sets.

Definition 3.1. A set $A \subset \mathbb{R}^2$ is I -Hash-core (I -ap-core) open if and only if, for each point $x \in A$ and for each $\Theta \in \langle 0, \pi \rangle$, the set $p_\Theta(x) \cap A$ contains a neighborhood of x in the I -Hashimoto topology (I -density topology) on the line $p_\Theta(x)$; i.e. a set of the form $G \setminus P$ where G is open in the Euclidean topology, P is a linear meager set and $x \in G \setminus P$. (In other words $p_\Theta(x) \cap A$ contains a set B having the Baire property, such that $d_I(B, x) = 1$ and $x \in B$.)

Let \mathcal{T}' - I -Hash-core (\mathcal{T}' - I -ap-core) denote the family of all I -Hash-core (I -ap-core) open sets.

Definition 3.2. A set $A \subset \mathbb{R}^2$ is core- I -a.e. (I -Hash-core- I -a.e., I -ap-core- I -a.e.) open if and only if, for each $x \in A$, there exists a linear set $H \subset \langle 0, \pi \rangle$ residual on $\langle 0, \pi \rangle$ such that, for all $\Theta \in H$, the set $p_\Theta(x) \cap A$ contains a neighborhood of x in the Euclidean topology (I -Hashimoto topology, I -density topology) on the line $p_\Theta(x)$.

Let \mathcal{T}' -core- I -a.e. (\mathcal{T}' - I -Hash-core- I -a.e., \mathcal{T}' - I -ap-core- I -a.e.) denote the family of all core- I -a.e. (I -Hash-core- I -a.e., I -ap-core- I -a.e.) open sets.

It is easy to see that the families of sets we have just defined form topologies. They also contain some sets without the Baire property (see [WBW]).

Theorem 3.3. *The following proper inclusions hold:*

$$\begin{array}{ccccc} \mathcal{T}'\text{-core-}I\text{-a.e.} & \subset & \mathcal{T}'\text{-}I\text{-Hash-core-}I\text{-a.e.} & \subset & \mathcal{T}'\text{-}I\text{-ap-core-}I\text{-a.e.} \\ \cup 1) & 2) & \cup 1) & 3) & \cup 1) \\ \mathcal{T}'\text{-core} & \subset & \mathcal{T}'\text{-}I\text{-Hash-core} & \subset & \mathcal{T}'\text{-}I\text{-ap-core} \\ & 2) & & 3) & \end{array}$$

The proof is analogous to that of Theorem 2.3. Only in the last part we take the set $E = \bigcup_{n=\mathbb{N}}^\infty (a_n, b_n)$ where $0 < a_{n+1} < b_{n+1} < a_n$, $\lim_{n \rightarrow \infty} b_n = 0$ and 0 is a point of right I -density of E .

To compare the topologies considered here with the I -density topologies on the plane, we assume open sets have the Baire property. Thus we define the following families of sets.

$$\begin{aligned}
\mathcal{T}_I\text{-core} &= \mathcal{T}'\text{-core} \cap \mathcal{B}_2, \\
\mathcal{T}\text{-core-}I\text{-a.e.} &= \mathcal{T}'\text{-core-}I\text{-a.e.} \cap \mathcal{B}_2, \\
\mathcal{T}\text{-}I\text{-Hash-core} &= \mathcal{T}'\text{-}I\text{-Hash-core} \cap \mathcal{B}_2, \\
\mathcal{T}\text{-}I\text{-Hash-core-}I\text{-a.e.} &= \mathcal{T}'\text{-}I\text{-Hash-core-}I\text{-a.e.} \cap \mathcal{B}_2, \\
\mathcal{T}\text{-}I\text{-ap-core} &= \mathcal{T}'\text{-}I\text{-ap-core} \cap \mathcal{B}_2, \\
\mathcal{T}\text{-}I\text{-ap-core-}I\text{-a.e.} &= \mathcal{T}'\text{-}I\text{-ap-core-}I\text{-a.e.} \cap \mathcal{B}_2.
\end{aligned}$$

It is easily seen that there are proper inclusions between these families because the sets A, B, C from Theorem 3.3 have the Baire property.

$$\begin{array}{ccccc}
\mathcal{T}\text{-core-}I\text{-a.e.} & \subset & \mathcal{T}\text{-}I\text{-Hash-core-}I\text{-a.e.} & \subset & \mathcal{T}\text{-}I\text{-ap-core-}I\text{-a.e.} \\
\cup & & \cup & & \cup \\
\mathcal{T}_I\text{-core} & \subset & \mathcal{T}\text{-}I\text{-Hash-core} & \subset & \mathcal{T}\text{-}I\text{-ap-core.}
\end{array} \quad (4)$$

Remark 3.4. Each of the families from (4) forms a topology.

The proof is analogous to those of Lemma 2.2 and Theorem 2.3 in [WBW] for the $\mathcal{T}\text{-}I\text{-ap-core}$ family.

The families from (4) will be called core topologies connected with the σ -ideal of meager sets. Now, we compare them with the Euclidean topology, the I -Hashimoto topology and the I -density topologies on the plane. It is easy to see that the Euclidean topology is essentially weaker than each of the topologies considered here.

Theorem 3.5. *The strong I -density topology is not contained in any of the core topologies connected with the σ -ideal of meager sets.*

PROOF. It suffices to prove that $d_{I_2}^s \not\subset \mathcal{T}\text{-}I\text{-ap-core-}I\text{-a.e.}$ Consider the set B from Theorem 2.6. It is not open in the $\mathcal{T}\text{-}I\text{-ap-core-}I\text{-a.e.}$ topology. We shall show that B is open in the strong I -density topology on the plane. It suffices to prove that $d_{I_2}^s(A^0, (0, 0)) = 0$; i.e., for each pair of sequences of positive integers $\{n'_m\}_{m \in \mathbb{N}}^{\nearrow}$, $\{n''_m\}_{m \in \mathbb{N}}^{\nearrow}$, there exists a subsequence $\{m_p\}_{p \in \mathbb{N}}^{\nearrow}$ such that

$$\limsup_p ((n'_{m_p}, n''_{m_p})A^0) \cap \langle 0, 1 \rangle^2 \in I_2.$$

Let us take any sequence of positive integers $\{n'_m\}_{m \in \mathbb{N}}^{\nearrow}$ and $\{n''_m\}_{m \in \mathbb{N}}^{\nearrow}$ and fix $m \in \mathbb{N}$. Let j_m denote the smallest positive integer for which

$$C_{j_m} \cap \left(\langle 0, \frac{1}{n'_m} \rangle \times \langle 0, \frac{1}{n''_m} \rangle \right) \neq \emptyset.$$

Then $((n'_m, n''_m)C_{j_m}) \cap \langle 0, 1 \rangle^2 \neq \emptyset$ and $a_{j_m} \leq \frac{\sqrt{2}}{n'_m}$. Let

$$\gamma_m = \arctg \left(\frac{n''_m}{n'_m} \operatorname{tg} \alpha_{j_m} \right), \quad \delta_m = \arctg \left(\frac{n''_m}{n'_m} \operatorname{tg} \beta_{j_m} \right).$$

Then $(n'_m, n''_m)C_{j_m} \subset \sec(\gamma_m, \delta_m)$. We shall show that $\delta_m - \gamma_m \xrightarrow{m \rightarrow \infty} 0$.

For an arbitrarily fixed $m \in \mathbb{N}$, let a denote the length of the segment joining the points of intersection of the line $x = n'_m$ with the x -axis and the line p_{γ_m} . Then

$$a_m = n''_m (\operatorname{tg} \beta_{j_m} - \operatorname{tg} \alpha_{j_m}), \quad b_m = n''_m \operatorname{tg} \alpha_{j_m}.$$

By the law of sines, $\sin(\delta_m - \gamma_m) = \frac{a \cos \gamma_m \sin \delta_m}{a + b}$. Since

$$\frac{\beta_m}{\alpha_m} \xrightarrow{m \rightarrow \infty} 1, \quad \alpha_m \xrightarrow{m \rightarrow \infty} 0, \quad \beta_m \xrightarrow{m \rightarrow \infty} 0,$$

we have $\frac{a_m}{b_m} \xrightarrow{m \rightarrow \infty} 0$; so $\sin(\delta_m - \gamma_m) \xrightarrow{m \rightarrow \infty} 0$.

Since $0 < \delta_m < \frac{\pi}{2}$ for each $m \in \mathbb{N}$, we can choose a convergent subsequence $\{\delta_{m_p}\}_{p \in \mathbb{N}}$. Let $\delta_0 = \lim_{p \rightarrow \infty} \delta_{m_p}$. Thus $\delta_0 \in \langle 0, \frac{\pi}{2} \rangle$ and

$$\limsup_p \sec(\gamma_{m_p} \delta_{m_p}) \subset p_{\delta_0}. \quad (5)$$

Since

$$(n'_{m_p}, n''_{m_p})A^0 \cap \langle 0, 1 \rangle^2 \subset (\langle 0, 2a_{j_{m_{p+1}}} n'_{m_p} \rangle \times \langle 0, 1 \rangle) \cup \sec(\gamma_{m_p} \delta_{m_p}),$$

it follows that

$$\begin{aligned} \limsup_p (n'_{m_p}, n''_{m_p})A^0 \cap \langle 0, 1 \rangle^2 &\subset \limsup_p (\langle 0, 2a_{j_{m_{p+1}}} n'_{m_p} \rangle \times \langle 0, 1 \rangle) \\ &\cup \limsup_p \sec(\gamma_{m_p} \delta_{m_p}). \end{aligned}$$

From $a_{j_{m_p}} n'_{m_p} \leq \sqrt{2}$ and $\frac{a_{j_{m_{p+1}}}}{a_{j_{m_p}}} \xrightarrow{p \rightarrow \infty} 0$ we have $2a_{j_{m_{p+1}}} n'_{m_p} \xrightarrow{p \rightarrow \infty} 0$; so the set $\limsup_p (\langle 0, 2a_{j_{m_{p+1}}} n'_{m_p} \rangle \times \langle 0, 1 \rangle)$ is included in the segment $\{0\} \times \langle 0, 1 \rangle$. Consequently, by (5), $\limsup_p ((n'_{m_p}, n''_{m_p})A^0) \cap \langle 0, 1 \rangle^2 \in I_2$.

Corollary 3.6. *The I -density topology on the plane is not contained in any of the core topologies connected with the σ -ideal of meager sets.*

Theorem 3.7. *None of the core topologies connected with the σ -ideal of meager sets is contained in the strong I -density topology on the plane.*

PROOF. It is sufficient to show that $\mathcal{T}_I\text{-core} \not\subset d_{I_2}^s$. The set A from Theorem 2.8 is not open in the strong I -density topology, more precisely, $(0, 0)$ is not a strong I -density point of A . For the sequences $\{n\}_{n \in \mathbb{N}}$ and $\{n^3\}_{n \in \mathbb{N}}$ and any subsequences $\{n_k\}_{k \in \mathbb{N}}$ and $\{n_k^3\}_{k \in \mathbb{N}}$, we have $\bigcap_{k \in \mathbb{N}} \bigcup_{l > k} [(n_l, n_l^3)A'] \cap \langle -1, 1 \rangle^2 \supset (0, 1)^2$. This means that $\limsup_k [(n_k, n_k^3)A'] \cap \langle -1, 1 \rangle^2 \notin I_2$. Hence $A \in \mathcal{T}_I\text{-core} \setminus d_{I_2}^s$. \square

Corollary 3.8. *None of the core topologies connected with the σ -ideal of meager sets is contained in $d_I \times d_I$.*

PROOF. We have $d_I \times d_I \subset d_{I_2}^s$ (see [WB]); so the proof is obvious. \square

Corollary 3.9. *The core topologies connected with the σ -ideal of meager sets and the strong I -density topology are incomparable.*

It is known (see [WBW], Th. 2.3) that the topology $d_I \times d_I$ is weaker than the \mathcal{T} - I -ap-core topology. Consequently the topology $d_I \times d_I$ is weaker than the \mathcal{T} - I -ap-core- I -a.e. topology.

Theorem 3.10. *The topology $d_I \times d_I$ is contained in neither the \mathcal{T} - I -Hash-core- I -a.e. nor the \mathcal{T}_I -core, \mathcal{T} -core- I -a.e., \mathcal{T} - I -Hash-core.*

PROOF. It is sufficient to show that $d_I \times d_I \not\subset \mathcal{T}$ - I -Hash-core- I -a.e. Change the set from Theorem 2.13. Let $E = \bigcup_{n \in \mathbb{N}} (a_n, b_n) \cup (-\infty, 0)$ where $\bigcup_{n \in \mathbb{N}} (a_n, b_n)$ is an interval set such that $d_I(E, 0) = 1$. Put $A = E \times \mathbb{R}$. Then

$$A \in d_I \times d_I \setminus \mathcal{T}\text{-}I\text{-Hash-core-}I\text{-a.e.} \quad \square$$

It is also known (see [WBW], Cor. 2.5) that the \mathcal{T} - I -ap-core topology is not contained in the I -density topology on the plane. Consequently, the \mathcal{T} - I -ap-core- I -a.e. is not contained in the I -density topology on the plane.

Theorem 3.11. *The I -density topology on the plane is stronger than the \mathcal{T} - I -Hash-core- I -a.e., \mathcal{T} -core- I -a.e. and \mathcal{T}_I -core topologies.*

PROOF. It is sufficient to show that \mathcal{T} - I -Hash-core- I -a.e. $\subset d_{I_2}$. Assume there exists a set A such that $A \in \mathcal{T}$ - I -Hash-core- I -a.e. and there exists a point $(x_0, y_0) \in A$ such that $d_{I_2}(A, (x_0, y_0)) \neq 1$. We can assume that $x_0 = y_0 = 0$. Then there exists a sequence $\{n_k\}_{k \in \mathbb{N}}$ of positive integers such that, for any of its subsequences $\{n_{k_m}\}_{m \in \mathbb{N}}$, we have $B = \bigcap_{l=1}^{\infty} \bigcup_{m=l}^{\infty} (n_{k_m}, n_{k_m})A' \cap$

$\langle -1, 1 \rangle^2 \notin I_2$. Obviously, B has the Baire property; so it is residual on some ball $K((x, y), \delta)$. We can assume that $(0, 0) \notin K((x, y), \delta)$ and $y > 0$.

Let $p_{\Theta_1}, p_{\Theta_2}$, where $\Theta_1 < \Theta_2$, are lines tangent to the ball $K((x, y), \delta)$ passing through the point $(0, 0)$. By the Kuratowski-Ulam theorem ([O], p.56), there exists a set $H_1 \subset (\Theta_1, \Theta_2)$ residual on this interval and such that, for $\Theta \in H_1$, the set $B_\Theta = K((x, y), \delta) \cap B \cap p_\Theta$ is residual on the interval $K((x, y), \delta) \cap p_\Theta$. We shall denote this interval by (a_Θ, b_Θ) . Since $A \in \mathcal{T}$ -I-Hash-core-I-a.e. and $(0, 0) \in A$, there exists a set $H_2 \in (0, \pi)$ residual on this interval and such that, for $\Theta \in H_2$, there exist $\varepsilon_\Theta > 0$ and a set $K_\Theta \subset p_\Theta$ which is meager on this line and such that $(-\varepsilon_\Theta, \varepsilon_\Theta) \setminus K_\Theta \subset A \cap p_\Theta$.

Let now $\Theta \in H_1 \cap H_2$. We notice that there exists $n_0 \in \mathbb{N}$ such that $n\varepsilon_\Theta > b_\Theta$ for any $n > n_0$. Since $(-\varepsilon_\Theta, \varepsilon_\Theta) \setminus K_\Theta \subset A$, we have $(a_\Theta, b_\Theta) \setminus nK_\Theta \subset nA$ for $n > n_0$. Let $p \in (a_\Theta, b_\Theta) \cap B \setminus nK_\Theta$. Then $p \in (a_\Theta, b_\Theta) \cap B = B_\Theta$; so $p \in (n_{k_m}, n_{k_m})A'$ for infinitely many $m \in \mathbb{N}$, which means that $(\frac{1}{n_{k_m}}, \frac{1}{n_{k_m}})p \notin A$ for infinitely many $m \in \mathbb{N}$.

On the other hand, $p \in (a_\Theta, b_\Theta) \setminus nK_\Theta \subset nA$ for $n > n_0$. Thus $(\frac{1}{n}, \frac{1}{n})p \in A$ for each $n > n_0$. This contradiction ends the proof. \square

Theorem 3.12. *None of the core topologies connected with the σ -ideal of meager sets is contained in the I-Hashimoto topology on the plane (\mathcal{O}_I^*).*

The proof follows from Theorem 3.7 because $\mathcal{O}_I^* \subset d_{I_2}^s$.

Theorem 3.13. *The I-Hashimoto topology on the plane is weaker than the \mathcal{T} -I-Hash-core-I-a.e. and the \mathcal{T} -I-ap-core-I-a.e. topologies.*

PROOF. As in the proof of Theorem 2.15, $\mathcal{O}_I^* \subset \mathcal{T}$ -I-Hash-core-I-a.e. by using the Kuratowski-Ulam theorem for the polar coordinates. \square

Theorem 3.14. *The I-Hashimoto topology on the plane is not contained in any of the topologies: \mathcal{T}_I -core, \mathcal{T} -core-I-a.e., \mathcal{T} -I-Hash-core, \mathcal{T} -I-ap-core.*

PROOF. It is enough to show that $\mathcal{O}_I^* \not\subset \mathcal{T}$ -core-I-a.e. and $\mathcal{O}_I^* \not\subset \mathcal{T}$ -I-ap-core. The set B from Theorem 2.3 is open in \mathcal{O}_I^* and is not open in the \mathcal{T} -core-I-a.e. The set A from the same theorem is open in \mathcal{O}_I^* and is not open in the \mathcal{T} -I-ap-core topology. \square

The results of Theorems and Corollaries 3.5–3.14 are presented in the fol-

lowing scheme.

$$\begin{array}{ccccc}
 \mathcal{T}\text{-core-}I\text{-a.e.} & \subset & \mathcal{T}\text{-}I\text{-Hash-core-}I\text{-a.e.} & \subset & \mathcal{T}\text{-}I\text{-ap-core-}I\text{-a.e.} \\
 \cup & & \cup & & \cup \\
 \mathcal{T}_I\text{-core} & \subset & \mathcal{T}\text{-}I\text{-Hash-core} & \subset & \mathcal{T}\text{-}I\text{-ap-core.} \\
 \begin{matrix} \overset{\mathcal{O}_I^*}{d_{I_2}^s} \\ \mathcal{O}_I^* \end{matrix} & & & & \begin{matrix} \overset{d_{I_2}^s}{\cap} \\ \cup \\ \overset{\emptyset}{d_I \times d_I} \quad \overset{\emptyset}{\mathcal{O}_I^*} \quad \overset{\emptyset}{d_{I_2}} \end{matrix}
 \end{array}$$

We can see that only the results of the comparison of the I -density topologies with the core topologies connected with the σ -ideal of meager sets differ from the analogous ones obtained in the previous section.

Now, we present theorems on the regularity of the core topologies connected with the σ -ideal of meager sets.

Theorem 3.15. *The \mathcal{T}_I -core topology is not regular.*

The proof is analogous to that of Theorem 2.17.

Theorem 3.16. *The \mathcal{T} -core- I -a.e. topology is not regular.*

PROOF. Consider a sequence of sets $Q_{k,l}$ where $k \in \mathbb{N} \cup \{0\}$, $l \in \{1, \dots, 2^k\}$ and $Q_{k,l} = \left\{ (r, \theta) : r \in \left(\frac{1}{2^{k+1}}, \frac{1}{2^k} \right), \theta \in \left(\frac{l-1}{2^k} \frac{\pi}{2}, \frac{l}{2^k} \frac{\pi}{2} \right) \right\}$. Let us form a sequence $\{K_n\}_{n \in \mathbb{N}}$ by putting $K_n = Q_{k,l}$ where $n = \sum_{i=0}^{k-1} 2^i + l$. Choose points $p_n \in K_n$, $n \in \mathbb{N}$, in such a way that any three of them are not colinear. Let $F = \{p_n : n \in \mathbb{N}\}$. Then $F' = \mathbb{R}^2 \setminus F \in \mathcal{T}\text{-core-}I\text{-a.e.}$ and $(0, 0) \notin F$.

Let $G, G_0 \in \mathcal{T}\text{-core-}I\text{-a.e.}$, $(0, 0) \in G_0$ and $F \subset G$. We shall show that $G_0 \cap G \neq \emptyset$. Let r_p, θ_p denote the polar coordinates of p . Put $\hat{p}_n = (1, \theta_{p_n})$ for $n \in \mathbb{N}$. From the construction of F it follows that the set $\{\hat{p}_k : k \geq n\}$ is dense on the arc $L = \left\{ (r, \theta) : r = 1, \theta \in \left(0, \frac{\pi}{2} \right) \right\}$. Hence, for each $n \in \mathbb{N}$, the set $\{\theta_{p_k} : k \geq n\}$ is dense on $\left(0, \frac{\pi}{2} \right)$.

Let $\hat{G} = \{(1, \theta_p) : p \in G\}$. Since $G \in \mathcal{T}\text{-core-}I\text{-a.e.}$, it contains, with every point p_n , a neighborhood A_n of p_n in the $\mathcal{T}\text{-core-}I\text{-a.e.}$ topology (we may assume that $A_n \subset K_n$). Then \hat{G} contains, with every point \hat{p}_n , an open arc containing \hat{p}_n . We denote that arc by \hat{A}_n . Then $\bigcup_{k \geq n} \hat{A}_k$ is residual on L for every $n \in \mathbb{N}$ and so is $\hat{A} = \limsup_n \hat{A}_n$. Thus the set $U = \{\theta : (1, \theta) \in \hat{A}\}$ is residual on $\left(0, \frac{\pi}{2} \right)$.

Since G_0 is a neighborhood of $(0, 0)$ in the $\mathcal{T}\text{-core-}I\text{-a.e.}$ topology, G_0 contains neighborhoods of $(0, 0)$ in the Euclidean topology on each direction from $\left(0, \frac{\pi}{2} \right) \setminus P$ where P is some set of the first category.

Let $\theta \in U \setminus P$. The set G_0 contains a segment I_θ with center $(0, 0)$, $I_\theta \subset p_\theta$. Since $\theta \in U$, the line p_θ intersects infinitely many sets A_n . Simultaneously, the segment I_θ is disjoint from a finite quantity of the rings $\left\{ (r, \theta) : r \in \left(\frac{1}{2^{k+1}}, \frac{1}{2^k} \right), \theta \in (0, 2\pi) \right\}$; so I_θ intersects infinitely many sets A_n . Hence $G_0 \cap G \neq \emptyset$ because $I_\theta \subset G_0$ and $A_n \subset G$ for each $n \in \mathbb{N}$. \square

Theorem 3.17. *The \mathcal{T} -I-Hash-core, \mathcal{T} -I-Hash-core-I-a.e., \mathcal{T} -I-ap-core and \mathcal{T} -I-ap-core-I-a.e. topologies are not regular.*

PROOF. First consider the \mathcal{T} -I-Hash-core topology. Let $F = \bigcup_{w \in \mathbb{Q}_+} S((0, 0), w)$. Then F is closed in the \mathcal{T} -I-Hash-core topology, and $(0, 0) \notin F$.

Let $G_0, G \in \mathcal{T}$ -I-Hash-core. Then they have the Baire property and they are of the second category on the plane. We assume that $(0, 0) \in G_0$ and $F \subset G$. We shall show that $G \cap G_0 \neq \emptyset$. The set G_0 is residual on a certain ball K . This ball has common points with F and contains their neighborhoods in the \mathcal{T} -I-Hash-core topology, included in G and of the second category on the plane. Thus $G \cap G_0 \neq \emptyset$.

This proof also shows that the \mathcal{T} -I-Hash-core-I-a.e., \mathcal{T} -I-ap-core and \mathcal{T} -I-ap-core-I-a.e. topologies are not regular. (F is closed in these topologies and non-empty open sets have the Baire property and are of the second category on the plane.) \square

We can consider the families of continuous functions with respect to the core topologies connected with the σ -ideal of meager sets.

The following inclusions hold.

$$\begin{array}{ccccc}
 & & C_{d_{I_2}} & & \\
 & & \cup & & \\
 C_{\mathcal{T}\text{-core-I-a.e.}} & \subset & C_{\mathcal{T}\text{-I-Hash-core-I-a.e.}} & \subset & C_{\mathcal{T}\text{-I-ap-core-I-a.e.}} \\
 \cup & & \cup & & \cup \\
 C_{\mathcal{T}\text{-I-core}} & \subset & C_{\mathcal{T}\text{-I-Hash-core}} & \subset & C_{\mathcal{T}\text{-I-ap-core}}
 \end{array}$$

Since $C_{d_{I_2}} \subset \mathbb{B}_1$ (see [PWW]), the functions belonging to $C_{\mathcal{T}\text{-I-core}}, C_{\mathcal{T}\text{-core-I-a.e.}}, C_{\mathcal{T}\text{-I-Hash-core}}, C_{\mathcal{T}\text{-I-Hash-core-I-a.e.}}$ are of Baire class 1. The functions belonging to $C_{\mathcal{T}\text{-I-ap-core}}$ are separate I -approximately continuous. By Theorem 3.1 [BLW], they are at most of Baire class 2.

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